Majorization and Entanglement transformations

Gustavo Martín Bosyk

Instituto de Física La Plata, UNLP, CONICET, La Plata, Argentina
Università degli studi di Cagliari, Cagliari, Italia

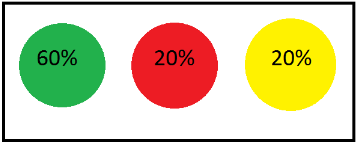
In collaboration with:
GMB, G. Sergioli, H. Freytes, F. Holik and G. Bellomo, arXiv:1608.04818v1 [quant-ph] (2016)

November 3, 2016

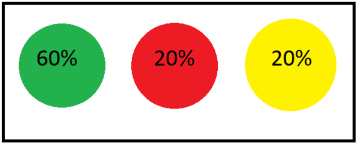
Part I

Majorization

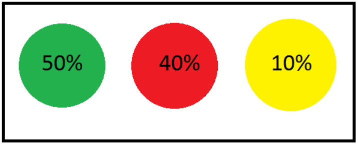
Situation A



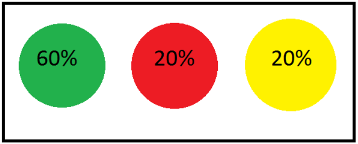
Situation A



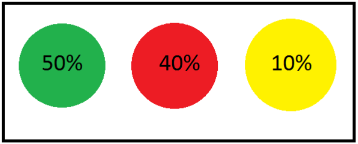
Situation B



Situation A

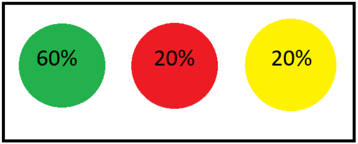


Situation B

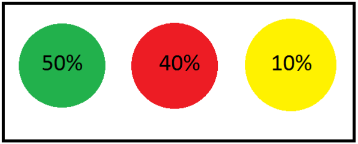


Which situation has less uncertainty?

Situation A



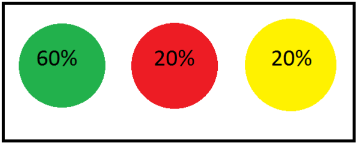
Situation B



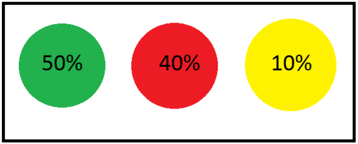
Which situation has less uncertainty?

Game 1: what color is the ball? $\rightarrow$ A

Situation A



Situation B

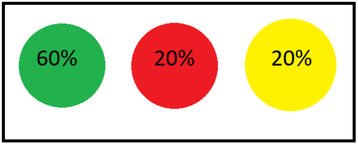


Which situation has less uncertainty?

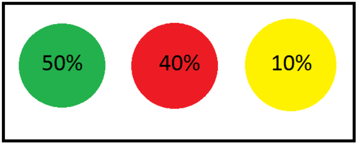
Game 1: what color is the ball? $\rightarrow$ A

Game 2: what color is **NOT** the ball? $\rightarrow$ B

Situation A



Situation B



Which situation has less uncertainty?
Game 1: what color is the ball? $\rightarrow$ A
Game 2: what color is **NOT** the ball? $\rightarrow$ B

How to compare probability vectors

Majorization

Majorization

Definition [Marshall, Olkin y Arnold, *Inequalities: Theory of Majorization and Its Applications*]

Let $p = [p_1, \dots p_N]^t$ and $q = [q_1, \dots q_N]^t$ be probability vectors: $p_i, q_i \geq 0$ and $\sum_{i=1}^N p_i = 1 = \sum_{i=1}^N q_i$.

Majorization

Definition [Marshall, Olkin y Arnold, *Inequalities: Theory of Majorization and Its Applications*]

Let $p = [p_1, \ldots p_N]^t$ and $q = [q_1, \ldots q_N]^t$ be probability vectors: $p_i, q_i \geq 0$ and $\sum_{i=1}^N p_i = 1 = \sum_{i=1}^N q_i$.
p is **majorized** by q, denoted as $p \prec q$, if

$$\sum_{i=1}^{n} p_i^{\downarrow} \leq \sum_{i=1}^{n} q_i^{\downarrow} \; \forall n = 1 \ldots N-1$$

Majorization

Definition [Marshall, Olkin y Arnold, *Inequalities: Theory of Majorization and Its Applications*]

Let $p = [p_1, \dots p_N]^t$ and $q = [q_1, \dots q_N]^t$ be probability vectors: $p_i, q_i \geq 0$ and $\sum_{i=1}^N p_i = 1 = \sum_{i=1}^N q_i$.
p is **majorized** by q, denoted as $p \prec q$, if

$$\sum_{i=1}^{n} p_i^{\downarrow} \leq \sum_{i=1}^{n} q_i^{\downarrow} \; \forall n = 1 \dots N-1$$

Example

$$\left[\frac{1}{N} \dots \frac{1}{N}\right]^t \prec p \prec [1, 0 \dots 0]^t \;\; \forall p$$

Alternative definitions

Definitions

$p \prec q$ sii

Alternative definitions

Definitions

$p \prec q$ sii

1. there exist a double stochastic matrix D such that

$$p = Dq \text{ with } \sum_i D_{ij} = \sum_j D_{ij} = 1$$

Alternative definitions

Definitions

$p \prec q$ sii

1. there exist a double stochastic matrix D such that

$$p = Dq \text{ with } \sum_i D_{ij} = \sum_j D_{ij} = 1$$

2. $\sum_{i=1}^{N} \phi(p_i) \leq \sum_{i=1}^{N} \phi(q_i)$ for all concave function ϕ

Alternative definitions

Definitions

$p \prec q$ sii

1. there exist a double stochastic matrix D such that

$$p = Dq \text{ with } \sum_i D_{ij} = \sum_j D_{ij} = 1$$

2. $\sum_{i=1}^N \phi(p_i) \leq \sum_{i=1}^N \phi(q_i)$ for all concave function ϕ

Schur-concavity and entropies

$\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ is Schur-concave if $p \prec q \Rightarrow \Phi(p) \geq \Phi(q)$

Alternative definitions

Definitions

$p \prec q$ sii

1. there exist a double stochastic matrix D such that

$$p = Dq \text{ with } \sum_i D_{ij} = \sum_j D_{ij} = 1$$

2. $\sum_{i=1}^N \phi(p_i) \leq \sum_{i=1}^N \phi(q_i)$ for all concave function ϕ

Schur-concavity and entropies

$\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ is Schur-concave if $p \prec q \Rightarrow \Phi(p) \geq \Phi(q)$

- Shannon entropy: $H(p) = -\sum p_i \ln p_i$

Alternative definitions

Definitions

$p \prec q$ sii

1. there exist a double stochastic matrix D such that

$$p = Dq \text{ with } \sum_i D_{ij} = \sum_j D_{ij} = 1$$

2. $\sum_{i=1}^N \phi(p_i) \leq \sum_{i=1}^N \phi(q_i)$ for all concave function ϕ

Schur-concavity and entropies

$\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ is Schur-concave if $p \prec q \Rightarrow \Phi(p) \geq \Phi(q)$

- Shannon entropy: $H(p) = -\sum p_i \ln p_i$
- Tsallis entropy: $T_q(p) = \frac{\sum p_i^q - 1}{1-q}$

Alternative definitions

Definitions

$p \prec q$ sii

1. there exist a double stochastic matrix D such that

$$p = Dq \text{ with } \sum_i D_{ij} = \sum_j D_{ij} = 1$$

2. $\sum_{i=1}^N \phi(p_i) \leq \sum_{i=1}^N \phi(q_i)$ for all concave function ϕ

Schur-concavity and entropies

$\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ is Schur-concave if $p \prec q \Rightarrow \Phi(p) \geq \Phi(q)$

- Shannon entropy: $H(p) = -\sum p_i \ln p_i$
- Tsallis entropy: $T_q(p) = \frac{\sum p_i^q - 1}{1-q}$
- Rényi entropy: $R_q(p) = \frac{\ln \sum p_i^q}{1-q}$

Majorization lattice: *POSET* + infimum and supreumum

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{ [p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^{N} p_i = 1 \geq p_i \right\}$

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{[p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^N p_i = 1 \geq p_i\right\}$

Partially ordered set (POSET)

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{[p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^N p_i = 1 \geq p_i\right\}$

Partially ordered set (POSET)

For all $p, q, r \in \delta_N$ one has

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{ [p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^N p_i = 1 \geq p_i \right\}$

Partially ordered set (POSET)

For all $p, q, r \in \delta_N$ one has

- reflexivity: $p \prec p$

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{ [p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^N p_i = 1 \geq p_i \right\}$

Partially ordered set (POSET)

For all $p, q, r \in \delta_N$ one has

- reflexivity: $p \prec p$
- antisymmetry: if $p \prec q$ and $q \prec p$, then $p = q$

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{ [p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^N p_i = 1 \geq p_i \right\}$

Partially ordered set (POSET)

For all $p, q, r \in \delta_N$ one has

- reflexivity: $p \prec p$
- antisymmetry: if $p \prec q$ and $q \prec p$, then $p = q$
- transitivity: if $p \prec q$ and $q \prec r$, then $p \prec r$

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{[p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^N p_i = 1 \geq p_i\right\}$

Partially ordered set (POSET)

For all $p, q, r \in \delta_N$ one has

- reflexivity: $p \prec p$
- antisymmetry: if $p \prec q$ and $q \prec p$, then $p = q$
- transitivity: if $p \prec q$ and $q \prec r$, then $p \prec r$

Majorization lattice: *POSET* + infimum and supreumum

Set of probability vectors

Let $\delta_N = \left\{ [p_1, \ldots, p_N]^t : p_i \geq p_{i+1} \geq 0, \text{ and } \sum_{i=1}^N p_i = 1 \geq p_i \right\}$

Partially ordered set (POSET)

For all $p, q, r \in \delta_N$ one has

- reflexivity: $p \prec p$
- antisymmetry: if $p \prec q$ and $q \prec p$, then $p = q$
- transitivity: if $p \prec q$ and $q \prec r$, then $p \prec r$

Majorization is **NOT** a total order

If $p = [0.6, 0.2, 0.2]^t$ and $q = [0.5, 0.4, 0.1]^t$, then $p \not\prec q$ and $q \not\prec p$.

Majorization lattice: POSET + *infimum and supremum*

Majorization lattice: POSET + *infimum and supremum*

Majorization lattice [Cicalese y Vaccaro, IEEE Trans. Inf. Theory 48,933 (2002)]

Let $\langle \delta_N, \prec, \wedge, \vee \rangle$, where for all $p, q \in \delta_N$ there exists the *infimum* $p \wedge q$ and the *supremum* $p \vee q$.

Majorization lattice: POSET + *infimum and supremum*

Majorization lattice [Cicalese y Vaccaro, IEEE Trans. Inf. Theory 48,933 (2002)]

Let $\langle \delta_N, \prec, \wedge, \vee \rangle$, where for all $p, q \in \delta_N$ there exists the *infimum* $p \wedge q$ and the *supremum* $p \vee q$.

By defintion, one has

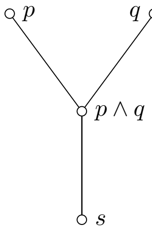
Majorization lattice: POSET + *infimum and supremum*

Majorization lattice [Cicalese y Vaccaro, IEEE Trans. Inf. Theory 48,933 (2002)]

Let $\langle \delta_N, \prec, \wedge, \vee \rangle$, where for all $p, q \in \delta_N$ there exists the *infimum* $p \wedge q$ and the *supremum* $p \vee q$.

By defintion, one has

infimum: $p \wedge q$ iff
$p \wedge q \prec p$ and $p \wedge q \prec q$ and $s \prec p \wedge q$
for all s such that $s \prec p$ and $s \prec q$



Infimum

Majorization lattice: POSET + *infimum and supremum*

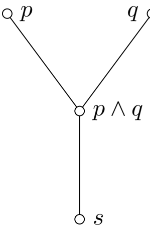
Majorization lattice [Cicalese y Vaccaro, IEEE Trans. Inf. Theory 48,933 (2002)]

Let $\langle \delta_N, \prec, \wedge, \vee \rangle$, where for all $p, q \in \delta_N$ there exists the *infimum* $p \wedge q$ and the *supremum* $p \vee q$.

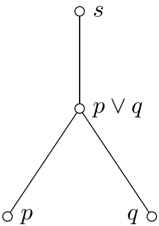
By defintion, one has

infimum: $p \wedge q$ iff
$p \wedge q \prec p$ and $p \wedge q \prec q$ and $s \prec p \wedge q$
for all s such that $s \prec p$ and $s \prec q$

supremum: $p \vee q$ sii
$p \prec p \vee q$ y $q \prec p \wedge q$ and $p \vee q \prec s$
for all s such that $p \prec s$ and $q \prec s$



Infimum



Supremum

Calculating the infimum

Calculating the infimum

Infimum

Let $p, q \in \delta_N$, the infimum $s^{\text{inf}} \equiv p \wedge q$ is such that

$$s_i^{\text{inf}} = \min\left\{\sum_{l=1}^{i} p_l, \sum_{l=1}^{i} q_l\right\} - \sum_{l=1}^{i-1} s_l^{\text{inf}},$$

with $s_0^{\text{inf}} \equiv 0$.

Calculating the supremum

Calculating the supremum

Supremum

Let $p, q \in \delta_N$, the supremum, $s^{\text{sup}} \equiv p \vee q$, is obtained as follows

Calculating the supremum

Supremum

Let $p, q \in \delta_N$, the supremum, $s^{\text{sup}} \equiv p \vee q$, is obtained as follows

1. $s = [s_1, \ldots, s_N]^t$: $s_1 = \max\{p_1, q_1\}$ and $s_i = \max\left\{\sum_{l=1}^{i} p_l, \sum_{l=1}^{i} q_l\right\} - \sum_{l=1}^{i-1} s_l$ *with* $i \in [2, N]$

Calculating the supremum

Supremum

Let $p, q \in \delta_N$, the supremum, $s^{\text{sup}} \equiv p \vee q$, is obtained as follows

1. $s = [s_1, \ldots, s_N]^t$: $s_1 = \max\{p_1, q_1\}$ and
   $s_i = \max\left\{\sum_{l=1}^{i} p_l, \sum_{l=1}^{i} q_l\right\} - \sum_{l=1}^{i-1} s_l$ *with* $i \in [2, N]$
2. $r = [r_1, \ldots, r_N]^t$:

Calculating the supremum

Supremum

Let $p, q \in \delta_N$, the supremum, $s^{\text{sup}} \equiv p \vee q$, is obtained as follows

1. $s = [s_1, \ldots, s_N]^t$: $s_1 = \max\{p_1, q_1\}$ and
   $s_i = \max\left\{\sum_{l=1}^{i} p_l, \sum_{l=1}^{i} q_l\right\} - \sum_{l=1}^{i-1} s_l$ *with* $i \in [2, N]$
2. $r = [r_1, \ldots, r_N]^t$:
   (a) let j be the smallest integer in $[2, N]$ such that $r_j > r_{j-1}$

Calculating the supremum

Supremum

Let $p, q \in \delta_N$, the supremum, $s^{\text{sup}} \equiv p \vee q$, is obtained as follows

1. $s = [s_1, \ldots, s_N]^t$: $s_1 = \max\{p_1, q_1\}$ and
   $s_i = \max\left\{\sum_{l=1}^{i} p_l, \sum_{l=1}^{i} q_l\right\} - \sum_{l=1}^{i-1} s_l$ *with* $i \in [2, N]$
2. $r = [r_1, \ldots, r_N]^t$:
   (a) let j be the smallest integer in $[2, N]$ such that $r_j > r_{j-1}$
   (b) let k be the greatest integer in $[1, j-1]$ such that $r_{k-1} \geq \frac{\sum_{l=k}^{j} r_l}{j-k+1} = a$ with $r_0 > 1$

Calculating the supremum

Supremum

Let $p, q \in \delta_N$, the supremum, $s^{\text{sup}} \equiv p \vee q$, is obtained as follows

1. $s = [s_1, \ldots, s_N]^t$: $s_1 = \max\{p_1, q_1\}$ and
   $s_i = \max\left\{\sum_{l=1}^{i} p_l, \sum_{l=1}^{i} q_l\right\} - \sum_{l=1}^{i-1} s_l$ *with* $i \in [2, N]$
2. $r = [r_1, \ldots, r_N]^t$:
   (a) let j be the smallest integer in $[2, N]$ such that $r_j > r_{j-1}$
   (b) let k be the greatest integer in $[1, j-1]$ such that $r_{k-1} \geq \frac{\sum_{l=k}^{j} r_l}{j-k+1} = a$ with $r_0 > 1$
   (c) let t the probability vector given by

$$t_l \equiv \begin{cases} a & \text{for } l = k, k+1, \ldots, j \\ r_l & \text{otherwise.} \end{cases}$$

Calculating the supremum

Supremum

Let $p, q \in \delta_N$, the supremum, $s^{\text{sup}} \equiv p \vee q$, is obtained as follows

1. $s = [s_1, \dots, s_N]^t$: $s_1 = \max\{p_1, q_1\}$ and
   $s_i = \max\left\{\sum_{l=1}^{i} p_l, \sum_{l=1}^{i} q_l\right\} - \sum_{l=1}^{i-1} s_l$ *with* $i \in [2, N]$
2. $r = [r_1, \dots, r_N]^t$:
   (a) let j be the smallest integer in $[2, N]$ such that $r_j > r_{j-1}$
   (b) let k be the greatest integer in $[1, j-1]$ such that $r_{k-1} \geq \frac{\sum_{l=k}^{j} r_l}{j-k+1} = a$ with $r_0 > 1$
   (c) let t the probability vector given by

$$t_l \equiv \begin{cases} a & \text{for } l = k, k+1, \dots, j \\ r_l & \text{otherwise.} \end{cases}$$

3. Applying transformations $2.(a) - 2.(c)$ with the input probability vector s, one obtains the supremum in no more than $N-1$ iterations.

Continuation

Continuation

Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

Continuation

Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

- $p \wedge q = [0.5, 0.25, 0.15, 0.1]^t$

Continuation

Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

- $p \wedge q = [0.5, 0.25, 0.15, 0.1]^t$
- $p \vee q = [0.6, 0.175, 0.175, 0.05]^t$

Continuation

Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

- $p \wedge q = [0.5, 0.25, 0.15, 0.1]^t$
- $p \vee q = [0.6, 0.175, 0.175, 0.05]^t$

Continuation

Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

- $p \wedge q = [0.5, 0.25, 0.15, 0.1]^t$
- $p \vee q = [0.6, 0.175, 0.175, 0.05]^t$

Properties

Continuation

Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

- $p \wedge q = [0.5, 0.25, 0.15, 0.1]^t$
- $p \vee q = [0.6, 0.175, 0.175, 0.05]^t$

Properties

- bottom element: $s^{\mathbf{0}} \equiv \left[\frac{1}{N} \dots \frac{1}{N}\right]^t$

Continuation

Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

- $p \wedge q = [0.5, 0.25, 0.15, 0.1]^t$
- $p \vee q = [0.6, 0.175, 0.175, 0.05]^t$

Properties

- bottom element: $s^{\mathbf{0}} \equiv \left[\frac{1}{N} \dots \frac{1}{N}\right]^t$
- top element: $s^{\mathbf{1}} \equiv [1, 0 \dots 0]^t$

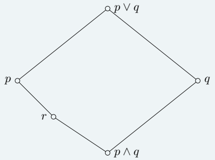
Example

If $p = [0.6, 0.15, 0.15, 0.1]^t$ y $q = [0.5, 0.25, 0.20, 0.05]^t$, then

- $p \wedge q = [0.5, 0.25, 0.15, 0.1]^t$
- $p \vee q = [0.6, 0.175, 0.175, 0.05]^t$

Properties

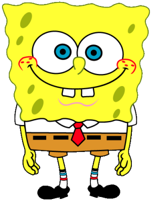
- bottom element: $s^{\mathbf{0}} \equiv \left[\frac{1}{N} \dots \frac{1}{N}\right]^t$
- top element: $s^{\mathbf{1}} \equiv [1, 0 \dots 0]^t$
- majorization lattice is **NOT** modular: if $r \prec q \nRightarrow r \vee (p \wedge q) = (r \vee p) \wedge q$



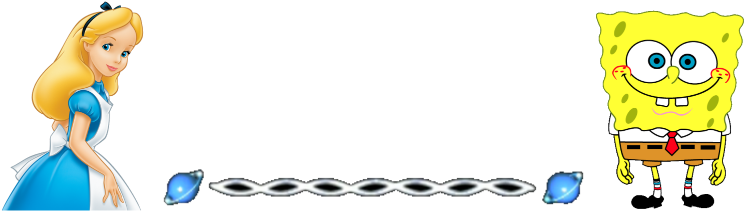
Part II

Entanglement transformations

Local operations and classical communication (LOCC)

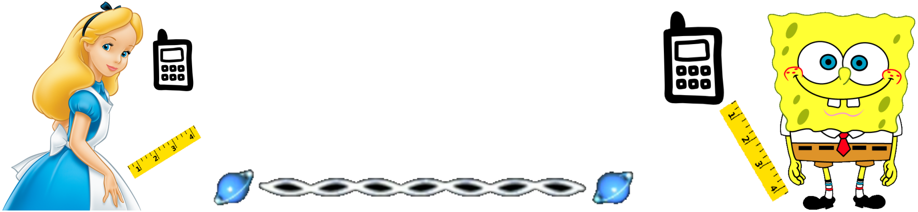


Local operations and classical communication (LOCC)



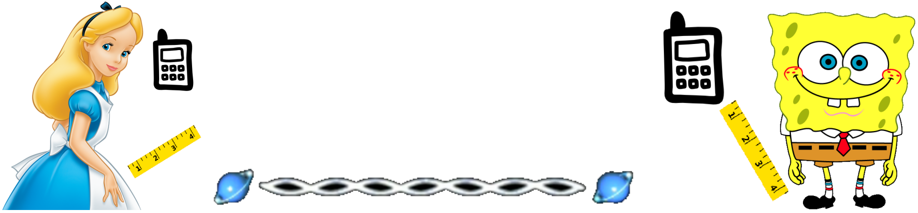
- Alice y Bob share an *initial* entangled pure state $|\psi\rangle$

Local operations and classical communication (LOCC)



- Alice y Bob share an *initial* entangled pure state $|\psi\rangle$
- Goal: obtain the *target* entangled pure state $|\phi\rangle$ by using **local operations and classical communications (LOCC)**

Local operations and classical communication (LOCC)



- Alice y Bob share an *initial* entangled pure state $|\psi\rangle$
- Goal: obtain the *target* entangled pure state $|\phi\rangle$ by using **local operations and classical communications (LOCC)**
- which is the condition for this process of entanglement transformation to be possible?

Entanglement transformation and majorization

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i} \, |i^A\rangle \, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\text{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if}$$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\mathrm{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if } \psi \prec \phi$$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\mathrm{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if } \psi \prec \phi$$

This condition does not depend on the Schmidt basis

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\text{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if } \psi \prec \phi$$

This condition does not depend on the Schmidt basis

As it is expected, the Nielsen condition is not satisfied in general

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i} |i^A\rangle |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j} |j^A\rangle |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\mathrm{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if } \psi \prec \phi$$

This condition does not depend on the Schmidt basis

As it is expected, the Nielsen condition is not satisfied in general

Example

- $|\psi\rangle = \sqrt{0.6}\,|00\rangle + \sqrt{0.15}\,|11\rangle + \sqrt{0.15}\,|22\rangle + \sqrt{0.1}\,|33\rangle$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\mathrm{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if } \psi \prec \phi$$

This condition does not depend on the Schmidt basis

As it is expected, the Nielsen condition is not satisfied in general

Example

- $|\psi\rangle = \sqrt{0.6}\,|00\rangle + \sqrt{0.15}\,|11\rangle + \sqrt{0.15}\,|22\rangle + \sqrt{0.1}\,|33\rangle$
- $|\phi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.2}\,|22\rangle + \sqrt{0.05}\,|33\rangle$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \dots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \dots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\text{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if } \psi \prec \phi$$

This condition does not depend on the Schmidt basis

As it is expected, the Nielsen condition is not satisfied in general

Example

- $|\psi\rangle = \sqrt{0.6}\,|00\rangle + \sqrt{0.15}\,|11\rangle + \sqrt{0.15}\,|22\rangle + \sqrt{0.1}\,|33\rangle$
- $|\phi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.2}\,|22\rangle + \sqrt{0.05}\,|33\rangle$

Entanglement transformation and majorization

Nielsen Theorem [Phys. Rev. Lett. 83, 436 (1999)]

Let consider the Schmidt decomposition of the states:

- initial: $|\psi\rangle = \sum_{i=1}^{N} \sqrt{\psi_i}\, |i^A\rangle\, |i^B\rangle$ con $\psi = [\psi_1, \ldots, \psi_N] \in \delta_N$
- target: $|\phi\rangle = \sum_{j=1}^{N} \sqrt{\phi_j}\, |j^A\rangle\, |j^B\rangle$ con $\phi = [\phi_1, \ldots, \phi_N] \in \delta_N$

$$|\psi\rangle \underset{\mathrm{LOCC}}{\longrightarrow} |\phi\rangle \text{ if and only if } \psi \prec \phi$$

This condition does not depend on the Schmidt basis

As it is expected, the Nielsen condition is not satisfied in general

Example

- $|\psi\rangle = \sqrt{0.6}\,|00\rangle + \sqrt{0.15}\,|11\rangle + \sqrt{0.15}\,|22\rangle + \sqrt{0.1}\,|33\rangle$
- $|\phi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.2}\,|22\rangle + \sqrt{0.05}\,|33\rangle$

$$\text{one has } |\psi\rangle \underset{\mathrm{LOCC}}{\not\leftrightarrow} |\phi\rangle \text{ due to } \psi \not\prec \phi \text{ and } \phi \not\prec \psi$$

Approximate entanglement transformations

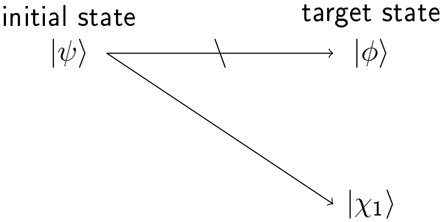
initial state
$|\psi\rangle$

target state
$|\phi\rangle$

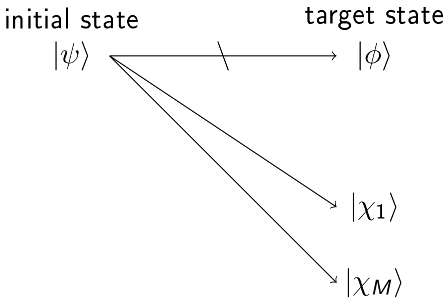
Approximate entanglement transformations

initial state $|\psi\rangle$ $\not\longrightarrow$ target state $|\phi\rangle$

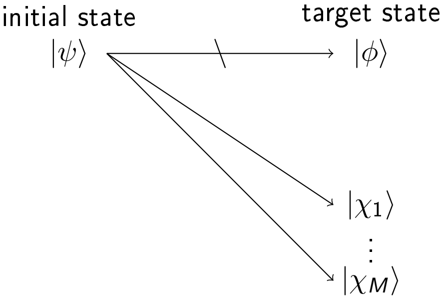
Approximate entanglement transformations



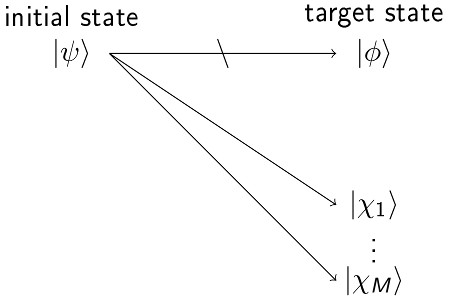
Approximate entanglement transformations



Approximate entanglement transformations



Approximate entanglement transformations



Goal: find $|\chi\rangle$ *closest* to $|\phi\rangle$

Vidal *et. al* Criterion [Phys. Rev. A 62, 012304 (2000)]

Let $|\psi\rangle$ and $|\phi\rangle$ be the initial and target such that $|\psi\rangle \underset{\text{LOCC}}{\nrightarrow} |\phi\rangle$.

Approximate entanglement transformations with Fidelity

Vidal *et. al* Criterion [Phys. Rev. A 62, 012304 (2000)]

Let $|\psi\rangle$ and $|\phi\rangle$ be the initial and target such that $|\psi\rangle \underset{\text{LOCC}}{\nrightarrow} |\phi\rangle$.

They define $|\chi^{\text{opt}}\rangle$ as the closest to the target in the sense of maximal fidelity:

Approximate entanglement transformations with Fidelity

Vidal *et. al* Criterion [Phys. Rev. A 62, 012304 (2000)]

Let $|\psi\rangle$ and $|\phi\rangle$ be the initial and target such that $|\psi\rangle \underset{\text{LOCC}}{\not\rightarrow} |\phi\rangle$.

They define $|\chi^{\rm opt}\rangle$ as the closest to the target in the sense of maximal fidelity:

$$|\chi^{\rm opt}\rangle = \underset{|\chi\rangle : |\psi\rangle \underset{\text{LOCC}}{\rightarrow} |\chi\rangle}{\arg\max} F(|\phi\rangle, |\chi\rangle),$$

where $F(|\phi\rangle, |\chi\rangle) = |\langle\phi|\chi\rangle|^2$ is the fidelity between the states $|\phi\rangle$ and $|\chi\rangle$

Approximate entanglement transformations with Fidelity

Vidal *et. al* Criterion [Phys. Rev. A 62, 012304 (2000)]

Let $|\psi\rangle$ and $|\phi\rangle$ be the initial and target such that $|\psi\rangle \underset{\mathrm{LOCC}}{\not\rightarrow} |\phi\rangle$.

They define $|\chi^{\mathrm{opt}}\rangle$ as the closest to the target in the sense of maximal fidelity:

$$|\chi^{\mathrm{opt}}\rangle = \underset{|\chi\rangle : |\psi\rangle \underset{\mathrm{LOCC}}{\rightarrow} |\chi\rangle}{\arg\max} F(|\phi\rangle, |\chi\rangle),$$

where $F(|\phi\rangle, |\chi\rangle) = |\langle\phi|\chi\rangle|^2$ is the fidelity between the states $|\phi\rangle$ and $|\chi\rangle$

Equivalent problem

$$\chi^{\mathrm{opt}} = \arg \max_{\chi : \psi \prec \chi} F(\phi, \chi)$$

where $F(\phi, \chi) = \left(\sum_i \sqrt{\phi_i \chi_i}\right)^2$ is the fidelity between the vectors ϕ and χ

Expression of the optimum: χ^{opt}

Expression of the optimum: χ^{opt}

$$\chi^{\text{opt}} = \begin{bmatrix} r_k \begin{bmatrix} \phi_{l_k} = \phi_1 \\ \vdots \\ \phi_{l_{k-1}-1} \end{bmatrix} \\ \vdots \\ r_2 \begin{bmatrix} \phi_{l_2} \\ \vdots \\ \phi_{l_{2-1}-1} \end{bmatrix} \\ r_1 \begin{bmatrix} \phi_{l_1} \\ \vdots \\ \phi_{l_{1-1}-1} = \phi_N \end{bmatrix} \end{bmatrix}$$

Expression of the optimum: $\chi^{\rm opt}$

$$\chi^{\rm opt} = \begin{bmatrix} r_k \begin{bmatrix} \phi_{l_k} = \phi_1 \\ \vdots \\ \phi_{l_{k-1}-1} \end{bmatrix} \\ \vdots \\ r_2 \begin{bmatrix} \phi_{l_2} \\ \vdots \\ \phi_{l_{2-1}-1} \end{bmatrix} \\ r_1 \begin{bmatrix} \phi_{l_1} \\ \vdots \\ \phi_{l_{1-1}-1} = \phi_N \end{bmatrix} \end{bmatrix}$$

with l_k the least integer in $[1, l_k - 1]$ such that

$$r_k = \min_{l \in [1, l_{k-1}-1]} \frac{E_l(\psi) - E_{l_{k-1}}(\psi)}{E_l(\phi) - E_{l_{k-1}}(\phi)}$$

where $E_l(\psi) = \sum_{l'=l}^{N} \psi_{l'}$ for all $l = 1, \ldots, N$

Relationship with the supremum

Relationship with the supremum

Does the lattice structure of majorization play some role?

Relationship with the supremum

Does the lattice structure of majorization play some role?

Theorem [GMB, G. Sergioli, H. Freytes, F. Holik and G. Bellomo]

Let $|\psi\rangle$ and $|\phi\rangle$ the initial and target states, one has

Relationship with the supremum

Does the lattice structure of majorization play some role?

Theorem [GMB, G. Sergioli, H. Freytes, F. Holik and G. Bellomo]

Let $|\psi\rangle$ and $|\phi\rangle$ the initial and target states, one has

$$\phi \prec \chi^{\rm sup} \prec \chi^{\rm opt}$$

where $\chi^{\rm sup} \equiv \psi \vee \phi$

Relationship with the supremum

Theorem [GMB, G. Sergioli, H. Freytes, F. Holik and G. Bellomo]

Let $|\psi\rangle$ and $|\phi\rangle$ the initial and target states, one has

$$\phi \prec \chi^{\mathrm{sup}} \prec \chi^{\mathrm{opt}}$$

where $\chi^{\mathrm{sup}} \equiv \psi \vee \phi$

Relationship with the supremum

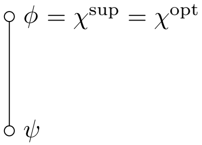
Theorem [GMB, G. Sergioli, H. Freytes, F. Holik and G. Bellomo]

Let $|\psi\rangle$ and $|\phi\rangle$ the initial and target states, one has

$$\phi \prec \chi^{\text{sup}} \prec \chi^{\text{opt}}$$

where $\chi^{\text{sup}} \equiv \psi \vee \phi$

$$\text{Case 1: if } |\psi\rangle \underset{\text{LOCC}}{\rightarrow} |\phi\rangle$$



Relationship with the supremum

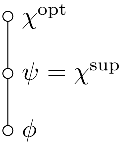
Theorem [GMB, G. Sergioli, H. Freytes, F. Holik and G. Bellomo]

Let $|\psi\rangle$ and $|\phi\rangle$ the initial and target states, one has

$$\phi \prec \chi^{\text{sup}} \prec \chi^{\text{opt}}$$

where $\chi^{\text{sup}} \equiv \psi \vee \phi$

$$\text{Case 2: if } |\psi\rangle \underset{\text{LOCC}}{\not\to} |\phi\rangle \text{ and } |\psi\rangle \underset{\text{LOCC}}{\leftarrow} |\phi\rangle$$



Relationship with the supremum

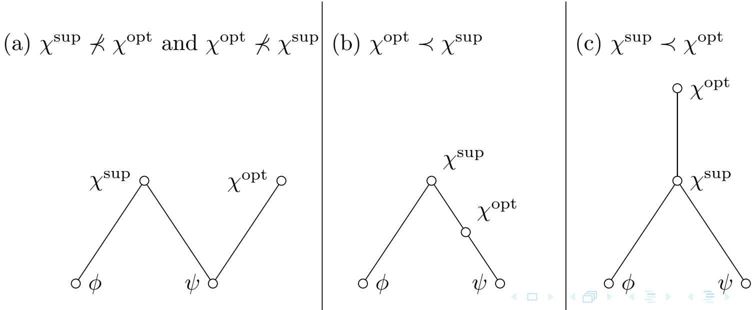
Theorem [GMB, G. Sergioli, H. Freytes, F. Holik and G. Bellomo]

Let $|\psi\rangle$ and $|\phi\rangle$ the initial and target states, one has

$$\phi \prec \chi^{\text{sup}} \prec \chi^{\text{opt}}$$

where $\chi^{\text{sup}} \equiv \psi \vee \phi$

Case 3: $|\psi\rangle \underset{\text{LOCC}}{\not\leftrightarrow} |\phi\rangle$



Fidelity vs. majorization

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$
- $\phi = [0.5, 0.25, 0.2, 0.5]^t$

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$
- $\phi = [0.5, 0.25, 0.2, 0.5]^t$

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$
- $\phi = [0.5, 0.25, 0.2, 0.5]^t$

one has

- $\chi^{\rm opt} = [0.6, 0.2, 0.16, 0.4]^t$ with fidelity $F(\phi, \chi^{\rm opt}) \approx 0.989$

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$
- $\phi = [0.5, 0.25, 0.2, 0.5]^t$

one has

- $\chi^{\mathrm{opt}} = [0.6, 0.2, 0.16, 0.4]^t$ with fidelity $F(\phi, \chi^{\mathrm{opt}}) \approx 0.989$
- $\chi^{\mathrm{sup}} = [0.6, 0.175, 0.175, 0.05]^t$ with fidelity $F(\phi, \chi^{\mathrm{sup}}) \approx 0.987$

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$
- $\phi = [0.5, 0.25, 0.2, 0.5]^t$

one has

- $\chi^{\rm opt} = [0.6, 0.2, 0.16, 0.4]^t$ with fidelity $F(\phi, \chi^{\rm opt}) \approx 0.989$
- $\chi^{\rm sup} = [0.6, 0.175, 0.175, 0.05]^t$ with fidelity $F(\phi, \chi^{\rm sup}) \approx 0.987$

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$
- $\phi = [0.5, 0.25, 0.2, 0.5]^t$

one has

- $\chi^{\mathrm{opt}} = [0.6, 0.2, 0.16, 0.4]^t$ with fidelity $F(\phi, \chi^{\mathrm{opt}}) \approx 0.989$
- $\chi^{\mathrm{sup}} = [0.6, 0.175, 0.175, 0.05]^t$ with fidelity $F(\phi, \chi^{\mathrm{sup}}) \approx 0.987$

But, we have seen that:

$$\phi \prec \chi^{\mathrm{sup}} \prec \chi^{\mathrm{opt}}$$

Fidelity vs. majorization

Example

Let:

- $\psi = [0.6, 0.15, 0.15, 0.1]^t$
- $\phi = [0.5, 0.25, 0.2, 0.5]^t$

one has

- $\chi^{\text{opt}} = [0.6, 0.2, 0.16, 0.4]^t$ with fidelity $F(\phi, \chi^{\text{opt}}) \approx 0.989$
- $\chi^{\text{sup}} = [0.6, 0.175, 0.175, 0.05]^t$ with fidelity $F(\phi, \chi^{\text{sup}}) \approx 0.987$

But, we have seen that:

$$\phi \prec \chi^{\text{sup}} \prec \chi^{\text{opt}}$$

Fidelity does not respect the majorization order in general

Intepreting the supreum

Intepreting the supreum

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

Intepreting the supreum

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$

Intepreting the supreum

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$
- symmetry: $d(p,q) = d(q,p)$

Intepreting the supreum

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$
- symmetry: $d(p,q) = d(q,p)$
- triangle inequality: $d(p,r) + d(r,q) \geq d(p,q)$

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$
- symmetry: $d(p,q) = d(q,p)$
- triangle inequality: $d(p,r) + d(r,q) \geq d(p,q)$
- compatible with the lattice: if $p \prec q \prec r \Rightarrow d(p,r) = d(p,q) + d(q,r)$

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$
- symmetry: $d(p,q) = d(q,p)$
- triangle inequality: $d(p,r) + d(r,q) \geq d(p,q)$
- compatible with the lattice: if $p \prec q \prec r \Rightarrow d(p,r) = d(p,q) + d(q,r)$

Intepreting the supreum

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$
- symmetry: $d(p,q) = d(q,p)$
- triangle inequality: $d(p,r) + d(r,q) \geq d(p,q)$
- compatible with the lattice: if $p \prec q \prec r \Rightarrow d(p,r) = d(p,q) + d(q,r)$

Supremum state

$$|\psi\rangle \underset{\text{LOCC}}{\rightarrow} |\chi^{\text{sup}}\rangle \equiv \sum_j \sqrt{\chi_j^{\text{sup}}} \, |j^A\rangle \, |j^B\rangle \text{ with } \chi^{\text{sup}} \equiv \psi \vee \phi$$

Intepreting the supreum

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$
- symmetry: $d(p,q) = d(q,p)$
- triangle inequality: $d(p,r) + d(r,q) \geq d(p,q)$
- compatible with the lattice: if $p \prec q \prec r \Rightarrow d(p,r) = d(p,q) + d(q,r)$

Supremum state

$$|\psi\rangle \underset{\mathrm{LOCC}}{\rightarrow} |\chi^{\mathrm{sup}}\rangle \equiv \sum_j \sqrt{\chi_j^{\mathrm{sup}}} \, |j^A\rangle \, |j^B\rangle \text{ with } \chi^{\mathrm{sup}} \equiv \psi \vee \phi$$

- is the closest to target in the sense of minimal distance d:

$$|\chi^{\mathrm{sup}}\rangle = \underset{|\chi\rangle : |\psi\rangle \underset{\mathrm{LOCC}}{\rightarrow} |\chi\rangle}{\mathrm{argmin}} d(|\phi\rangle, |\chi\rangle)$$

Intepreting the supreum

Distance on the majorization lattice

Let two proability vectors $p, q \in \delta_N$. A distance d is defined as:

$$d(p,q) = H(p) + H(q) - 2H(p \vee q) \text{ con } H(p) = -\sum_i p_i \ln p_i$$

- positivity: $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$
- symmetry: $d(p,q) = d(q,p)$
- triangle inequality: $d(p,r) + d(r,q) \geq d(p,q)$
- compatible with the lattice: if $p \prec q \prec r \Rightarrow d(p,r) = d(p,q) + d(q,r)$

Supremum state

$$|\psi\rangle \underset{\text{LOCC}}{\rightarrow} |\chi^{\text{sup}}\rangle \equiv \sum_j \sqrt{\chi_j^{\text{sup}}} \, |j^A\rangle \, |j^B\rangle \text{ with } \chi^{\text{sup}} \equiv \psi \vee \phi$$

- is the closest to target in the sense of minimal distance d:

$$|\chi^{\text{sup}}\rangle = \underset{|\chi\rangle : |\psi\rangle \underset{\text{LOCC}}{\rightarrow} |\chi\rangle}{\text{argmin}} d(|\phi\rangle, |\chi\rangle)$$

- it has more entanglement entropy than the optimum
  Recall: Entropy of Schmidt coefficients is the entanglement entropy

Grazie mille!!!

Questions, comments...