Qudit Spaces and a Many-valued Approach to Quantum Computational Logics

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Quantum computational logics are special examples of quantum logic based on the following semantic idea:

- **linguistic formulas** are interpreted as

  pieces of quantum information

  that can be stored and transmitted by some quantum systems.
- **logical connectives** are interpreted as

  quantum logical gates

  .

The simplest piece of quantum information is a **qubit**, a unit-vector of the Hilbert space $\mathbb{C}^2$:

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle .$$

The vectors $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$ (the two elements of the canonical basis of $\mathbb{C}^2$) represent, in this framework, the two **classical bits** or (equivalently) the two **classical truth-values.**

It is interesting to consider a "many-valued generalization" of **qubits**, represented by **qudits**: unit-vectors of a space $\mathbb{C}^d$ (where $d \geq 2$).

The elements of the canonical basis of $\mathbb{C}^d$ can be regarded as different **truth-values**:

$$\left|0\right\rangle = \left|\frac{0}{d-1}\right\rangle = (1, 0, \ldots, 0)$$
$$\left|\frac{1}{d-1}\right\rangle = (0, 1, 0, \ldots, 0)$$
$$\left|\frac{2}{d-1}\right\rangle = (0, 0, 1, 0, \ldots, 0)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\left|1\right\rangle = \left|\frac{d-1}{d-1}\right\rangle = (0, \ldots, 0, 1)$$

The Qutrit-space $\mathbb{C}^3$.

$$|0\rangle = \left|\tfrac{0}{2}\right\rangle = (1, 0, 0)$$
$$\left|\tfrac{1}{2}\right\rangle = (0, 1, 0)$$
$$|1\rangle = \left|\tfrac{2}{2}\right\rangle = (0, 0, 1)$$

In this framework, any piece of quantum information can be identified with a **pure or mixed state** of a quantum system: a **density operator** ρ living in a **qudit-space**

$$\mathcal{H}_d^{(n)} = \underbrace{\mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d}_{n-times}$$

(the n-fold tensor product of $\mathbb{C}^d$).

The canonical basis of a qudit space $\mathcal{H}_d^{(n)}$ is the set:

$\left\{ |v_1, \ldots, v_n\rangle : |v_1\rangle, \ldots, |v_n\rangle \text{ belong the canonical basis of } \mathbb{C}^d \right\}$
(where $|v_1, \ldots, v_n\rangle$ is an abbreviation for $|v_1\rangle \otimes \ldots \otimes |v_n\rangle$.)

The elements of this set, called **registers**, represent **classical** pieces of information.

A **quregister** of $\mathcal{H}_d^{(n)}$ is a pure state, represented by a unit-vector $|\psi\rangle$.
Or, equivalently, by the corresponding density operator $P_{|\psi\rangle}$ (the projection that projects over the closed subspace determined by $|\psi\rangle$).

Quantum information is processed by

(quantum logical) gates,

unitary quantum operations that transform density operators in a reversible way.

Why is **reversibility** so important in quantum computation?
The time-evolution of (pure) quantum systems is described by **unitary operators**.
From a physical point of view, a **quantum computation** can be regarded as the time-evolution of a quantum system that stores and processes pieces of quantum information.

Some many-valued gates that are interesting from a logical point of view.

Logical operations in the Łukasiewicz-semantics

. **The set of truth-values**:

- the real interval $[0, 1]$;
- a finite subset of $[0, 1]$:

$$\left\{ \frac{0}{d-1}, \frac{1}{d-1}, \frac{2}{d-1}, \ldots, \frac{d-1}{d-1} \right\}.$$

The negation

$$v' := 1 - v.$$

The min-conjunction

$$u \sqcap v := min\{u, v\}.$$

The Łukasiewicz-conjunction

$$u \odot v := max\{0, u + v - 1\}.$$

The negation only is a reversible operation!

Of course, the two conjunctions $\sqcap$ and $\odot$ coincide in the two-valued semantics (when $d = 2$).
Generally, $\sqcap$ and $\odot$ have different properties.

The min-conjunction gives rise to possible violations of the non-contradiction principle. We may have:

$$v \sqcap v' \neq 0.$$

The Łukasiewicz-conjunction is generally non-idempotent. We may have:

$$v \odot v \neq v.$$

How to obtain quantum reversible versions of these basic logical operations?
For simplicity, let us refer to the smallest examples of qudit-spaces.

THE NEGATION (on $\mathcal{H}_d^{(1)}$)

The **negation** is the linear operator $\mathrm{NOT}^{(1)}$ defined on $\mathcal{H}_d^{(1)}$ such that, for every element $|v\rangle$ of the canonical basis:

$$\mathrm{NOT}^{(1)}|v\rangle = |1-v\rangle .$$

In order to define a **reversible min-conjunction** and a **reversible Łukasiewicz-conjunction**, we can use:

- **the Toffoli-gate**;
- **the Toffoli-Łukasiewicz gate**.

The Toffoli-gate (which plays a very important role in the case of qubit-spaces) can be naturally generalized to qudit-spaces.

THE TOFFOLI-GATE (on $\mathcal{H}_d^{(3)}$)

The **Toffoli-gate** is the linear operator $\mathtt{T}^{(1,1,1)}$ defined on $\mathcal{H}_d^{(3)}$ such that, for every element $|u, v, w\rangle$ of the canonical basis:

$$\mathtt{T}^{(1,1,1)} |u, v, w\rangle = \begin{cases} |u, v, u \sqcap v\rangle \,, & \text{if } w = 0; \\ |u, v, (u \sqcap v)'\rangle \,, & \text{if } w = 1. \end{cases}$$

THE TOFFOLI- ŁUKASIEWICZ GATE (on $\mathcal{H}_d^{(3)}$)

The **Toffoli-Łukasiewicz gate** is the linear operator ${}^{Ł}\mathbb{T}^{(1,1,1)}$ defined on $\mathcal{H}_d^{(3)}$ such that, for every element $|u, v, w\rangle$ of the canonical basis:

$$
{}^{Ł}\mathbb{T}^{(1,1,1)} \, |u, v, w\rangle = \begin{cases} |u, v, u \odot v\rangle \,, & \text{if } \; w = 0; \\ |u, v, (u \odot v)'\rangle \,, & \text{if } \; w_3 = 1. \end{cases}
$$

The Toffoli-gate and the Toffoli-Łukasiewicz gate allow us to define two different reversible conjunctions, for any quregister $|\psi\rangle$ of the space $\mathcal{H}_d^{(2)}$.

The Toffoli-conjunction

$$\mathtt{AND}^{(1,1)}\,|\psi\rangle := \mathtt{T}^{(1,1,1)}(|\psi\rangle \otimes |0\rangle),$$

where $|0\rangle$ plays the role of an *ancilla*.
In particular:

$$\mathtt{AND}^{(1,1)}\,|u, v\rangle = |u, v, (u \wedge v)\rangle\,.$$

The Toffoli-Łukasiewicz conjunction

$$^{\text{Ł}}\mathtt{AND}^{(1,1)}\,|\psi\rangle := {}^{\text{Ł}}\mathtt{T}^{(1,1,1)}(|\psi\rangle \otimes |0\rangle).$$

In particular:

$$^{\text{Ł}}\mathtt{AND}^{(1,1)}\,|u, v\rangle = |u, v, (u \odot v)\rangle\,.$$

The negation, the Toffoli-gate and the Toffoli-Łukasiewicz gates represent **semiclassical gates**, because they always transform registers (representing classical information) into registers.
Other gates are called **genuine quantum gates**, because they can create quantum superpositions from register-inputs.
An important example of a genuine quantum gate is the **Hadamard-gate**.

THE HADAMARD-GATE (on $\mathcal{H}_2^{(1)}$)

The **Hadamard-gate** is the linear operator $\sqrt{\mathtt{I}}^{(1)}$ defined on $\mathcal{H}_2^{(1)}$ such that:

$$\sqrt{\mathtt{I}}^{(1)}\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle);$$

$$\sqrt{\mathtt{I}}^{(1)}\,|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

We have:

$$\sqrt{\mathtt{I}}^{(1)}\sqrt{\mathtt{I}}^{(1)} = \mathtt{I}^{(1)}.$$

A natural generalization of the Hadamard-gate for the space $\mathcal{H}_d^{(1)} = \mathbb{C}^d$ is the *Vandermonde-operator*

THE VANDERMONDE-GATE (on $\mathcal{H}_d^{(1)}$)
The **Vandermonde-gate** is the linear operator $\mathrm{V}^{(1)}$ defined on $\mathcal{H}_d^{(1)}$ such that for every basis-element $\left|\frac{k}{d-1}\right\rangle$:

$$\mathrm{V}^{(1)}\left(\left|\frac{k}{d-1}\right\rangle\right) = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega^{j\,k}\left|\frac{j}{d-1}\right\rangle,$$

where $\omega = e^{\frac{2\pi i}{d}}$.

The operator $\mathbb{V}^{(1)}$ represents a good generalization of the Hadamard-gate in the space $\mathbb{C}^2$.
We have:

- $\mathbb{V}^{(1)}$ transforms each element of the basis of $\mathbb{C}^d$ into a superposition of all basis-elements, assigning to each basis-element the same probability-value.
- $\mathbb{V}^{(1)} = \sqrt{\mathtt{I}}$, if $d = 2$.
- $\mathbb{V}^{(1)}\mathbb{V}^{(1)}\mathbb{V}^{(1)}\mathbb{V}^{(1)} = \mathtt{I}$.

- The negation and the Hadamard-gate can be generalized to any space $\mathcal{H}_d^{(n)}$.
- The Toffoli-gate and the Toffoli-Łukasiewicz gate can be generalized to any space $\mathcal{H}_d^{(m+n+p)}$.
- All gates can be generalized to density operators.

Physical implementations

Physical implementations of gates represent the basic issue for the technological realization of quantum computers.
We consider the case of optical devices, where photon-beams (possibly consisting of single photons) move in different directions.
Let us conventionally assume that $|0\rangle$ represents the state of a beam moving along the x-direction, while $|1\rangle$ is the state of a beam moving along the y-direction.

In the framework of this “physical semantics”, one-qubit gates (like $\mathtt{NOT}^{(1)}$, $\sqrt{\mathtt{I}}^{(1)}$) can be easily implemented.
A natural implementation of $\mathtt{NOT}^{(1)}$ can be obtained by a mirror $\mathtt{M}$ that reflects in the *y*-direction any beam moving along the *x*-direction, and viceversa. Hence we have:

$$|0\rangle \quad \rightarrowtail_{\mathtt{M}} \quad |1\rangle \, ; \quad |1\rangle \quad \rightarrowtail_{\mathtt{M}} \quad |0\rangle \, .$$

The mirror transforms the state $|0\rangle$ into the state $|1\rangle$, and viceversa.

An implementation of the Hadamard-gate $\sqrt{\mathtt{I}}^{(1)}$ can be obtained by a beam splitter $\mathtt{BS}$. Any beam that goes through $\mathtt{BS}$ is split into two components: one component moves along the x-direction, while the other component moves along the y-direction. And the probability of both paths (along the x-direction or along the y-direction) is $\frac{1}{2}$.
We have:

$$|0\rangle \quad \rightarrowtail_{\mathtt{BS}} \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \quad |1\rangle \quad \rightarrowtail_{\mathtt{BS}} \quad \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Other apparatuses that may be useful for optical implementations of gates are the *relative phase shifters* along a given direction.

A particular example: the relative phase shifter along the y-direction (on $\mathcal{H}_d^{(1)}$-

The **relative phase shifter along the y-direction** is the linear operator $\mathtt{U}_{\mathtt{PS}}$ that is defined for every element of the canonical basis of $\mathbb{C}^2$ as follows:

$$\mathtt{U}_{\mathtt{PS}}\,|v\rangle = c\,|v\rangle, \text{ where } c = \begin{cases} e^{i\pi}, & \text{if } v = 1; \\ 1, & \text{otherwise.} \end{cases}$$

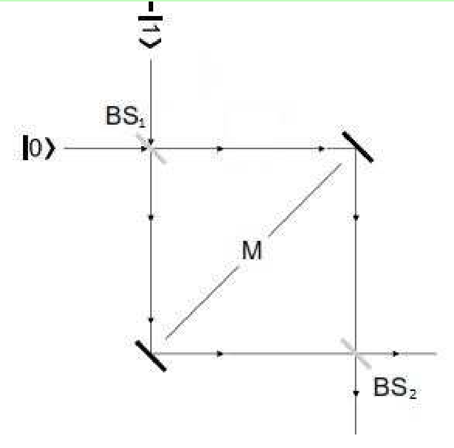
We obtain:

$$\mathtt{U}_{\mathtt{PS}} |0\rangle = |0\rangle ; \; \mathtt{U}_{\mathtt{PS}} |1\rangle = - |1\rangle .$$

Let us indicate by $\mathtt{PS}$ a physical apparatus that realizes the phase shift described by $\mathtt{U}_{\mathtt{PS}}$.

Relative phase shifters, beam splitters and mirrors are the basic physical components of the *Mach-Zehnder interferometer* (`MZI`), an apparatus that has played a very important role in the logical and philosophical debates about the foundations of quantum theory.

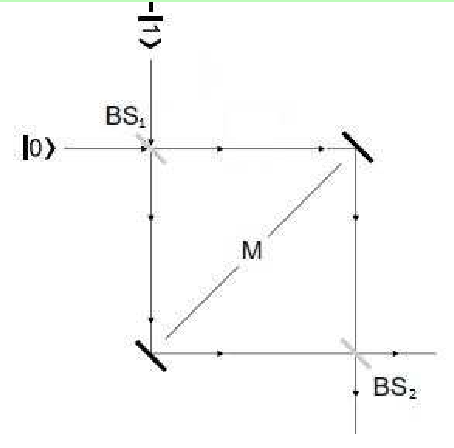
The Mach-Zehnder interferometer



A beam (which may move either along the *x*-direction or along the *y*-direction) goes through the relative phase shifter $\mathtt{PS}$ of $\mathtt{MZI}$:

$$|0\rangle \ \rightarrowtail_{\mathtt{PS}} \ |0\rangle\,; \ |1\rangle \ \rightarrowtail_{\mathtt{PS}} \ -|1\rangle\,.$$

The phase of the beam changes only in the case where the beam is moving along the *y*-direction.
Soon after the beam goes through the first beam splitter $\mathtt{BS}_1$.



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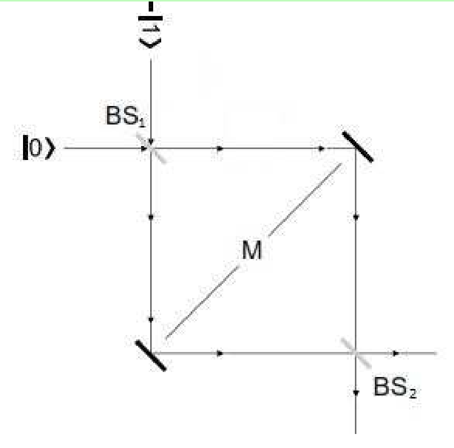
As a consequence, it is split into two components: one component moves along the interferometer's arm in the x-direction, the other component moves along the arm in the y-direction.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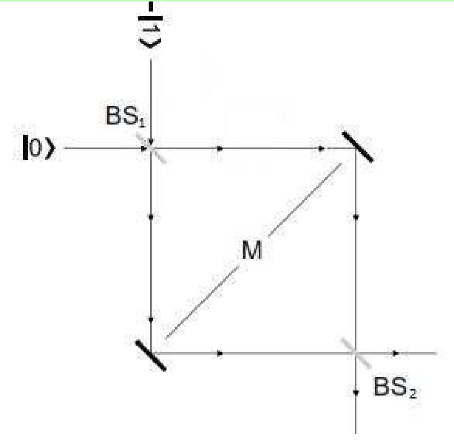
We have:

$$|0\rangle \;\rightarrowtail_{\mathtt{BS}_1}\; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \quad -|1\rangle \;\rightarrowtail_{\mathtt{BS}_1}\; \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle).$$

Then, both components of the superposed beam (on both arms) are reflected by the mirrors $\mathtt{M}$: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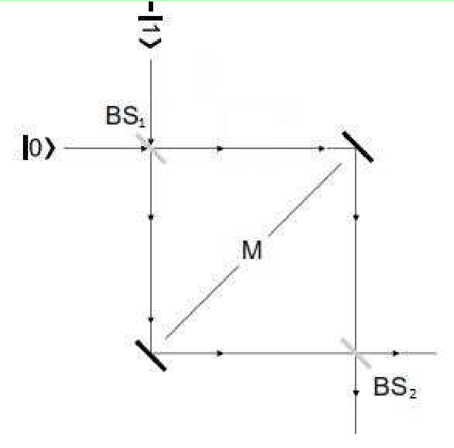
We have:

$$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \quad \rightarrowtail_{\mathtt{M}} \quad \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle); \quad \frac{1}{\sqrt{2}}(-|0\rangle+|1\rangle) \quad \rightarrowtail_{\mathtt{M}} \quad \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle).$$

Finally, the superposed beam goes through the second beam splitter $\mathtt{BS}_2$, which re-composes the two components.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We have:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrowtail_{\mathtt{BS}_2} |0\rangle\,; \quad \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \rightarrowtail_{\mathtt{BS}_2} |1\rangle\,.$$

Accordingly, $\mathtt{MZI}$ transforms the input $|0\rangle$ into the output $|0\rangle$, while the input $|1\rangle$ is transformed into the output $|1\rangle$.

One is dealing with a result that has for a long time been described as deeply counter-intuitive. In fact, according to a “classical way of thinking” we would expect that the outcoming photons from the second beam splitter should be detected with probability $\frac{1}{2}$ either along the x-direction or along the y-direction.

While optical implementations of one-qubit gates are relatively simple, trying to implement many-qubit gates may be rather complicated.
Consider the case of the Toffoli-gate $\mathtt{T}^{(1,1,1)}$.
Mathematically we have:

$$\mathtt{T}^{(1,1,1)} \left| u, v, w \right\rangle = \begin{cases} \left| u, v, u \sqcap v \right\rangle, & \text{if } w = 0; \\ \left| u, v, (u \sqcap v)' \right\rangle, & \text{if } w = 1. \end{cases}$$

The main problem is finding a device that can realize a physical dependence of the target-bit ($u \sqcap v$ or $(u \sqcap v)'$) from the control-bits (u, v).

A possible strategy is based on an appropriate use of the optical "Kerr-effect": a substance with an intensity-dependent refractive index is placed into a given device, giving rise to an intensity-dependent phase shift.

A unitary operator that describes a particular form of *conditional phase shift*.

The relative conditional phase shifter of the space $\mathcal{H}_2^{(3)} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is the unitary operator $\mathtt{U}_{\mathtt{CPS}}$ that is defined for every element of the canonical basis as follows:

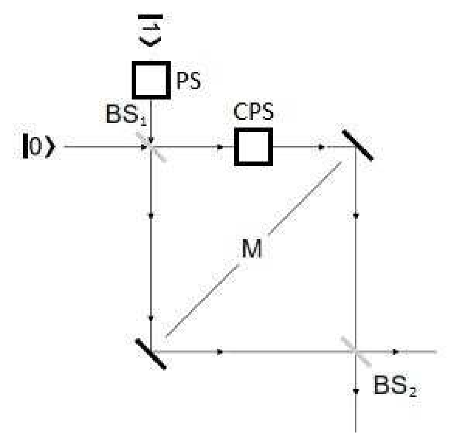
$$\mathtt{U}_{\mathtt{CPS}} \left| u, v, w \right\rangle = \left| u, v \right\rangle \otimes c \left| w \right\rangle ,$$

where $c = \begin{cases} e^{i\pi}, & \text{if } u = 1, v = 1 \text{ and } w = 0; \\ 1, & \text{otherwise.} \end{cases}$.

Let us indicate by $\mathtt{CPS}$ a physical apparatus that realizes the phase shift described by the operator $\mathtt{U}_{\mathtt{CPS}}$.
Clearly, $\mathtt{CPS}$ determines a *conditional* phase shift. For, the phase of a three-beam system in state $|u, v, w\rangle$ is changed only in the case where both control-bits $|u\rangle$ and $|v\rangle$ are the state $|1\rangle$, while the *ancilla*-bit $|w\rangle$ is the state $|0\rangle$.
From a physical point of view, such a result can be obtained by using a convenient substance that produces the Kerr-effect.

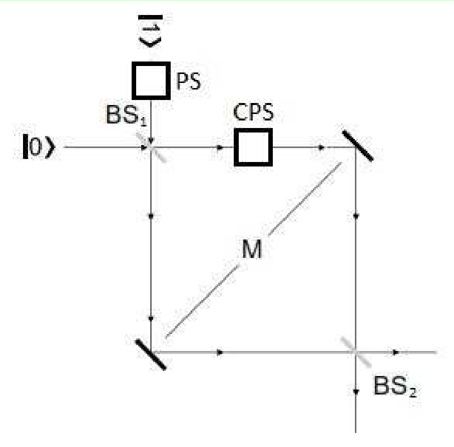
In order to obtain an implementation of the Toffoli-gate $\mathtt{T}^{(1,1,1,)}$ we consider a "more sophisticated" version of the Mach-Zehnder interferometer: the "Kerr-Mach-Zehnder interferometer" ($\mathtt{KMZI}$).
Besides the relative phase shifter ($\mathtt{PS}$), the two beam splitters ($\mathtt{BS}_1$, $\mathtt{BS}_2$) and the mirrors ($\mathtt{M}$), the Kerr-Mach-Zehnder interferometer also contains a relative conditional phase shifter ($\mathtt{CPS}$) that can produce the Kerr-effect.

The Kerr-Mach-Zehnder interferometer



While the inputs of the canonical Mach-Zehnder interferometer are single beams (whose states live in the space $\mathbb{C}^2$), the apparatus $\mathtt{KMZI}$ acts on composite systems consisting of three beams (S_1, S_2, S_3), whose states live in the space $\mathcal{H}_2^{(3)} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

For the sake of simplicity we can assume that S_1, S_2, S_3 are single photons that may enter into the interferometer-box either along the x-direction or along the y-direction.
Let $|u, v, w\rangle$ be the input-state of the composite system $S_1 + S_2 + S_3$. Photons S_1, S_2 (whose states $|u\rangle, |v\rangle$ represent the control-bits) are supposed to enter into the box along the yz-plane, while photon S_3 (whose state $|w\rangle$ is the *ancilla*-bit) will enter through the first beam-splitter $\mathtt{BS}_1$.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Mathematically, the action performed by the apparatus $\mathtt{KMZI}$ is described by the following unitary operator (of the space $\mathcal{H}_2^{(3)}$):

$\mathtt{U}_{\mathtt{KMZ}}$:=

$(\mathtt{I}\otimes\mathtt{I}\otimes\sqrt{\mathtt{I}}^{(1)})\circ(\mathtt{I}\otimes\mathtt{I}\otimes\mathtt{NOT}^{(1)})\circ\mathtt{U}_{\mathtt{CPS}}\circ(\mathtt{I}\otimes\mathtt{I}\otimes\sqrt{\mathtt{I}}^{(1)})\circ(\mathtt{I}\otimes\mathtt{I}\otimes\mathtt{U}_{\mathtt{PS}})$.

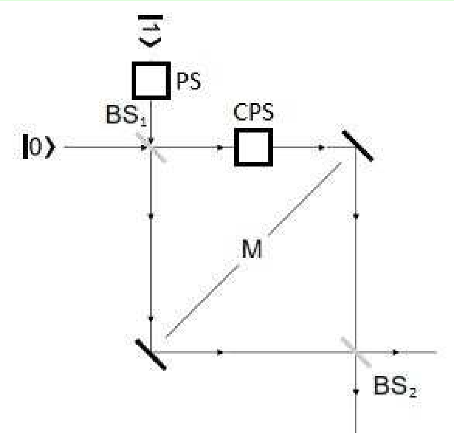
In order to “see” how $\mathtt{KMZI}$ is working from a physical point of view, it is expedient to consider a particular example.
Take the input $|u, v, w\rangle = |1, 1, 0\rangle$ and let us describe the physical evolution determined by the operator $\mathtt{U}_{\mathtt{KMZ}}$ for the system $S_1 + S_2 + S_3$, whose initial state is $|1, 1, 0\rangle$.

$$(\mathtt{I} \otimes \mathtt{I} \otimes \mathtt{U}_{\mathtt{PS}}) \, |1, 1, 0\rangle = |1, 1, 0\rangle \, .$$

The relative phase shifter along the y-direction ($\mathtt{PS}$) does not change the state of photon S_3, which is moving along the x-direction.

$$(\mathtt{I} \otimes \mathtt{I} \otimes \sqrt{\mathtt{I}}^{(1)})\,|1,1,0\rangle = |1,1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Photon S_3 goes through the first beam splitter $\mathtt{BS}_1$ splitting into two components: one component moves along the interferometer's arm along the x-direction, the other component moves along the arm in the y-direction (like in the case of the canonical Mach-Zehnder interferometer). At the same time, photons S_1 and S_2 (both in state $|1\rangle$) enter into the interferometer-box along the yz-plane.



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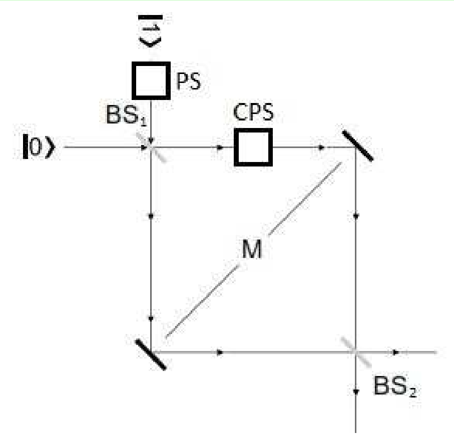
$$\mathrm{U}_{\mathtt{CPS}}(|1,1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) = |1,1\rangle \otimes \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle).$$

The conditional phase shifter $\mathtt{CPS}$ determines a phase shift for the component of S_3 that is moving along the x-direction; because both photons S_1 and S_2 (in state $|1\rangle$) have gone through the substance (contained in $\mathtt{CPS}$) that produces the Kerr-effect.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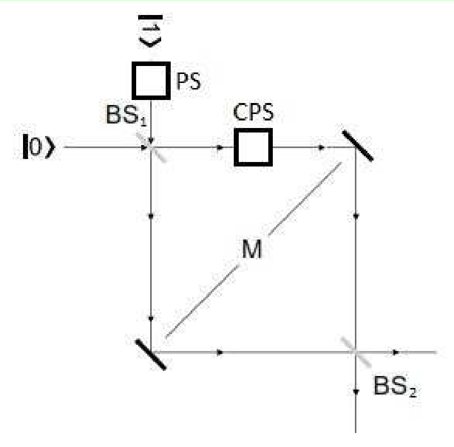
$$(\mathtt{I} \otimes \mathtt{I} \otimes \mathtt{NOT}^{(1)})(|1,1\rangle \otimes \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)) = |1,1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Both components of S_3 (on both arms) are reflected by the mirrors.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$$(\mathtt{I} \otimes \mathtt{I} \otimes \sqrt{\mathtt{I}}^{(1)})(|1,1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) = |1,1,1\rangle \, .$$

Finally, the second beam splitter $\mathtt{BS}_2$ re-composes the two components of the superposed photon S_3.

Consequently, we obtain:

$$\mathrm{U}_{\mathrm{KMZ}} \left|1,1,0\right\rangle = \left|1,1,1\right\rangle = \mathrm{T}^{(1,1,1)} \left|1,1,0\right\rangle .$$

In general, one can easily prove that:

$$\mathtt{U}_{\mathtt{KMZ}} = \mathtt{T}^{(1,1,1)}.$$

Although, from a mathematical point of view, $\mathtt{U}_{\mathtt{KMZ}}$ and $\mathtt{T}^{(1,1,1)}$ represent the same gate, physically it is not guaranteed that the apparatus KMZI always realizes its “expected job”.

All difficulties are due to the behavior of the conditional phase shifter. In fact, the substances used to produce the Kerr-effect generally determine only stochastic results.
As a consequence one shall conclude that the Kerr-Mach-Zehnder interferometer allows us to obtain an *approximate* implementation of the Toffoli-gate with an accuracy that is, in some cases, very good.

So far we have considered possible optical implementations of gates in the case of qubit-spaces. The techniques we have illustrated can be also generalized to qudit-spaces.
The main idea is using, instead of single beams, systems consisting of many beams (corresponding to different truth-values) that may move either along the x-direction or along the y-direction.

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