Reconstructing the electrical conductivity and the magnetic permeability of the soil by a FDEM data inversion

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Outline

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Goal: Detect or infer inhomogeneities in the ground or the presence of particular conductive substances such as metals, minerals and other geological structures by Electromagnetic Induction.

Applications

- Hydrological and hydrogeological characterizations.
- Hazardous waste characterization studies.
- Archaeological surveys.
- Precision-agriculture applications.
- Unexploded ordnance detection (UXO).

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How does it work?

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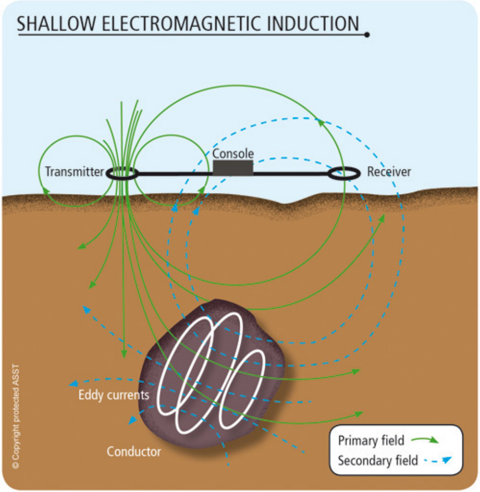
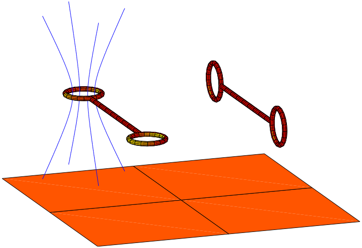
How does it work? $\Rightarrow$



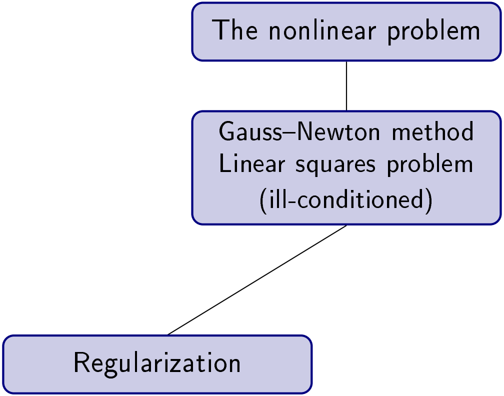


The two coils axes may be aligned either vertically or horizontally giving a different response.

The nonlinear problem

The nonlinear problem

Gauss–Newton method
Linear squares problem
(ill-conditioned)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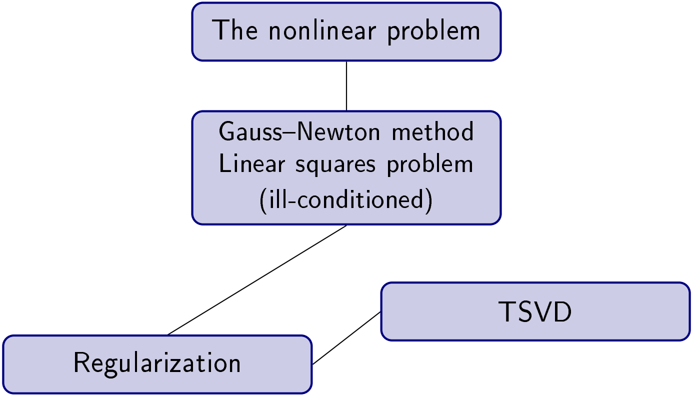
Regularization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The nonlinear problem

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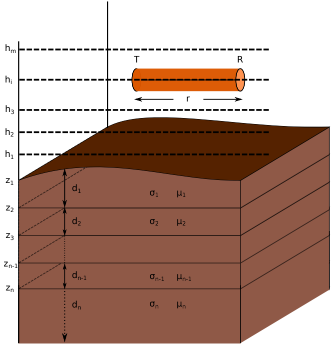
TGSVD

Outline

The nonlinear model is derived from Maxwell's equations keeping into account the cylindrical symmetry of the problem. [J. R. Wait, *Geo-Electromagnetism*]



- d_k are the thickness of each layer.
- σ_k and μ_k are the electrical conductivity and the magnetic permeability of the k-th layer, respectively.
- h_j are the heights above the ground.
- ω is the angular frequency.
- r is the distance between the coils.

- Characteristic admittance: $N_k(\lambda) = \dfrac{u_k(\lambda)}{\mathrm{i}\mu_k \omega}$
- Surface admittance: $Y_k(\lambda) = N_k(\lambda) \dfrac{Y_{k+1}(\lambda) + N_k(\lambda)\tanh(d_k u_k(\lambda))}{N_k(\lambda) + Y_{k+1}(\lambda)\tanh(d_k u_k(\lambda))}$
- Reflection factor: $R_0(\lambda) = \dfrac{N_0(\lambda) - Y_1(\lambda)}{N_0(\lambda) + Y_1(\lambda)}$

Ratio of the secondary to the primary field, $\dfrac{H_S}{H_P}$

- Vertical orientation:
$$M_1(\boldsymbol{\sigma}, \boldsymbol{\mu}; h, \omega) = -r^3 \int_0^\infty \lambda^2 e^{-2h\lambda} R_0(\lambda) J_0(r\lambda)\, d\lambda$$
- Horizontal orientation:
$$M_2(\boldsymbol{\sigma}, \boldsymbol{\mu}; h, \omega) = -r^2 \int_0^\infty \lambda e^{-2h\lambda} R_0(\lambda) J_1(r\lambda)\, d\lambda$$

We let

$$\frac{H_S}{H_P} = M_\nu(\boldsymbol{\sigma}, \boldsymbol{\mu}; h_i, \omega_j), \quad \nu = 1, 2,\ i = 1, \ldots, m_h,\ j = 1, \ldots, m_\omega,$$

where ν indicates the orientation, m_h is the number of heights and m_ω is the number of frequencies.

In our numerical experiements we will reconstruct the electrical conductivity assuming that the magnetic permeability is known, and vice versa, and we will let the inter–coil distance, r, to be constant.

Outline

1. Motivation
2. The forward problem
3. The nonlinear inverse problem
4. Numerical results

The problem of data inversion consists of fitting the model to the data, that is determine the electrical conductivity $\boldsymbol{\sigma}$, and the permeability vector $\boldsymbol{\mu}$ which produce the best approximations

$$M_\nu(\boldsymbol{\sigma}, \boldsymbol{\mu}; h_i, \omega_j) \approx b_{ij} \quad \nu = 1, 2,\ i = 1, \ldots, m_h,\ j = 1, \ldots, m_\omega.$$

Let us consider the error in the model prediction

$$\mathbf{r}_{ij}(\boldsymbol{\sigma}, \boldsymbol{\mu}; h_i, \omega_j) = b_{ij}^\nu - M_\nu(\boldsymbol{\sigma}, \boldsymbol{\mu}; h_i, \omega_j).$$

From now on, we will consider h_i and ω_j fixed and we vectorize the data values b_{ij}^ν in lexicographical order into $\mathbf{b} \in \mathbb{C}^m$, $m = 2m_h m_\omega$.

We proceed similarly for the model predictions, obtaining the vector $M_\nu \in \mathbb{C}^m$, and minimize the Euclidean norm of the (complex) residual between the data and the model, that is

$$(\boldsymbol{\sigma}^*, \boldsymbol{\mu}^*) = \arg \min_{\boldsymbol{\sigma},\boldsymbol{\mu} \in \mathbb{R}^n} \frac{1}{2} \|\mathbf{r}(\boldsymbol{\sigma}, \boldsymbol{\mu})\|^2.$$

Iterative methods

Newton method

Newton's method requires the computation of both the gradient vector and the Hessian matrix of the residual, which have a large computational complexity.

Gauss–Newton method

When the residuals are small or mildly nonlinear in a neighborhood of the solution, the Gauss–Newton method is expected to behave similarly to Newton's method. We remark that, while the physical problem is obviously consistent, this is not necessarily true in our case, where in the presence of noise in the data the problem will certainly be inconsistent.

Iterative methods

Damped Gauss–Newton method:

$$(\boldsymbol{\sigma}_{k+1}, \boldsymbol{\mu}_{k+1}) = (\boldsymbol{\sigma}_k + \alpha_k \mathbf{s}_k, \boldsymbol{\mu}_k + \alpha_k \mathbf{s}_k),$$

where α_k is parameter to be determined and s_k is the solution of the linear least squares problem

$$\min_{\mathbf{s}\in\mathbb{R}^n} \|\mathbf{r}(\boldsymbol{\sigma}_k, \boldsymbol{\mu}_k) + J_k \mathbf{s}\|,$$

with $J_k = J(\boldsymbol{\sigma}_k \boldsymbol{\mu}_k)$ is the Jacobian of $\mathbf{r}(\boldsymbol{\sigma}, \boldsymbol{\mu})$ or some approximation.

Iterative methods

How to choose α_k?

Iterative methods

How to choose α_k?

Armijo Goldstein principle

α_k is selected as the largest number in the sequence 2^{-i}, $i = 0, 1...$, for which the following inequality holds

$$\|\mathbf{r}(\boldsymbol{\sigma}_k, \boldsymbol{\mu}_k)\|^2 - \|\mathbf{r}(\boldsymbol{\sigma}_k + \alpha_k \mathbf{s}_k, \boldsymbol{\mu}_k + \alpha_k \mathbf{s}_k)\|^2 \geq \frac{1}{2}\alpha_k \|J_k \mathbf{s}_k\|^2$$

This choice of α_k ensures convergence of the method, provided that $(\boldsymbol{\sigma}_k, \boldsymbol{\mu}_k)$ is not a critical point.

Regularization TSVD

The regularization technique we use is the one based on a low-rank approximation of the Jacobian matrix.

The best rank ℓ approximation according to the 2-norm can be obtained by the SVD decomposition, $J = U\Gamma V^T$.

This procedure allows us to replace the ill-conditioned Jacobian matrix with a well-conditioned rank-deficient matrix A_ℓ. The solution is known as the truncated SVD (TSVD) and it can be expressed as

$$\mathbf{s}^{(\ell)} = -A_\ell^\dagger \mathbf{r} = -\sum_{i=1}^{\ell} \frac{\mathbf{u}_i^T \mathbf{r}}{\gamma_i} \mathbf{v}_i,$$

Regularization TGSVD

Let us introduce a regularization matrix $L \in \mathbb{R}^{t \times n}$ ($t \leq n$) replaced by

$$\min_{\mathbf{s} \in \mathcal{S}} \|L\mathbf{s}\|, \qquad \mathcal{S} = \{\mathbf{s} \in \mathbb{R}^n \ : \ J^T J\mathbf{s} = -J^T\mathbf{r}\},$$

and solve the problem under the assumption $\mathcal{N}(J) \cap \mathcal{N}(L) = \{0\}$ and $t > \max(0, n-2m)$.

The generalized singular value decomposition (GSVD) of the matrix pair (J, L) is the factorization

$$J = U\Sigma_J Z^{-1}, \qquad L = V\Sigma_L Z^{-1},$$

By the generalized singular value decomposition (GSVD) of (J, L) it is possible to define the truncated GSVD (TGSVD) solution $\mathbf{s}_\ell$.

Choosing ℓ, regularization parameter

The optimal choice would be the discrepancy principle

$$\|\mathbf{b} - M_\nu(\boldsymbol{\sigma}^{(\ell_{\text{discrepancy}})}, \boldsymbol{\mu}^{(\ell_{\text{discrepancy}})})\| \leq \kappa\|\mathbf{e}\|, \qquad \kappa > 1,$$

but it can seldom be applied to EMI techniques because in applications

- The noise on the data is not necessarily equally distributed.
- An accurate estimate of $\|\mathbf{e}\|$ is often unknown.

Choosing ℓ, L-curve

L-curve principle can be adapted quite naturally to the nonlinear case.

It chooses the value of ℓ which identifies the corner of the curve connecting the points

$$\left\{ \log \|\mathbf{r}(\boldsymbol{\sigma}^{(\ell)}, \boldsymbol{\mu}^{(\ell)})\|, \log \|L(\boldsymbol{\sigma}^{(\ell)} \boldsymbol{\mu}^{(\ell)})\| \right\}.$$

The curve is L-shaped in many discrete ill-posed problems.

To detect the corner of the L-curve we used the L-corner method [P. C. Hansen, T. K. Jensen and G. Rodriguez, *An adaptive pruning algorithm for the discrete L-curve criterion*].

Outline

For fixed n and m, we apply the model previously described to generate the instrument readings

$$\mathbf{b} = M_\nu(\boldsymbol{\sigma}, \boldsymbol{\mu}; h_i, \omega_j),$$

with $i = 1, \ldots, m_h$ and $j = 1, \ldots, m_\omega$, corresponding to frequency $\omega_j = 2\pi f_j$ and height h_i.

Finally, we add Gaussian noise to the synthetic data by the formula

$$\mathbf{b} = \widehat{\mathbf{b}} + \frac{\tau \|\widehat{\mathbf{b}}\|}{\sqrt{m}} \mathbf{w},$$

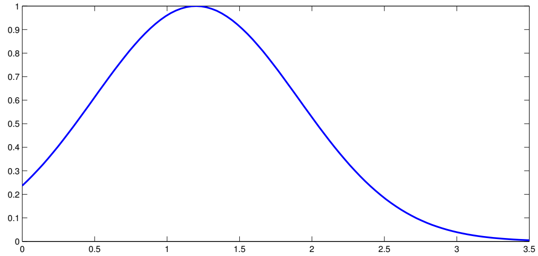
where $\mathbf{w}$ is a vector with normally distributed entries with zero mean and unitary variance, $m = 2m_h m_\omega$, and τ is the noise level.

Electrical conductivity

For the numerical experiments we consider

- The coils to be in both orientations at a fixed distance $r = 1.66$m.
- h is either 1m $(m_h = 1)$ or 0.5m and 1m $(m_h = 2)$.

Each data set is recorded simultaneously with the operating frequencies $f_j = 775, 1175, 3925, 9825, 21725, 47025$, all expressed in Hertz.



	L	m_h	$n = 20$	$n = 30$	$n = 40$
$\mathcal{R}$	I	1	3.6e-01	3.7e-01	3.7e-01
		2	4.5e-01	4.5e-01	4.4e-01
	D_1	1	3.0e-01	3.3e-01	2.9e-01
		2	2.4e-01	2.4e-01	2.3e-01
	D_2	1	2.4e-01	2.1e-01	2.8e-01
		2	2.5e-01	2.5e-01	2.3e-01
$\mathcal{I}$	I	1	3.2e-01	3.9e-01	4.0e-01
		2	3.0e-01	3.4e-01	3.4e-01
	D_1	1	1.9e-01	2.3e-01	1.9e-01
		2	2.1e-01	1.9e-01	1.9e-01
	D_2	1	2.3e-01	2.0e-01	2.4e-01
		2	2.1e-01	2.2e-01	2.1e-01

	μ_0	$\mu_r = 10$	$\mu_r = 10^2$	$\mu_r = 10^3$
optimal - $\mathcal{R}$	2.3e-01	4.3e-01	5.3e-01	5.5e-01
	0	13	9	19
optimal - $\mathcal{I}$	2.4e-01	5.3e-01	4.5e-01	7.1e-01
	0	6	4	12
L–curve - $\mathcal{R}$	2.6e-01	6.3e-01	4.7e-01	5.4e-01
	0	20	18	27
L–curve - $\mathcal{I}$	2.6e-01	4.2e-01	5.5e-01	7.4e-01
	0	23	10	16

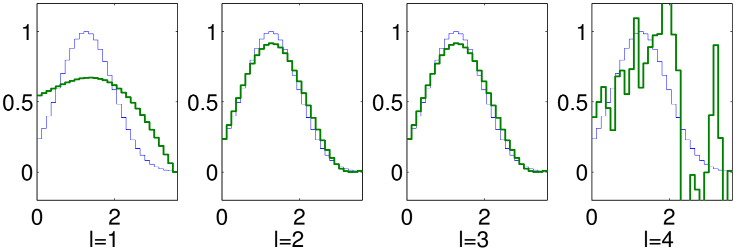


Figure: Plot of the first 4 regularized solutions, computed by minimizing the real part of the signal, compared to the exact solution. The magnetic permeability $\mu = \mu_0$ is constant, the noise level is $\tau = 10^{-3}$.

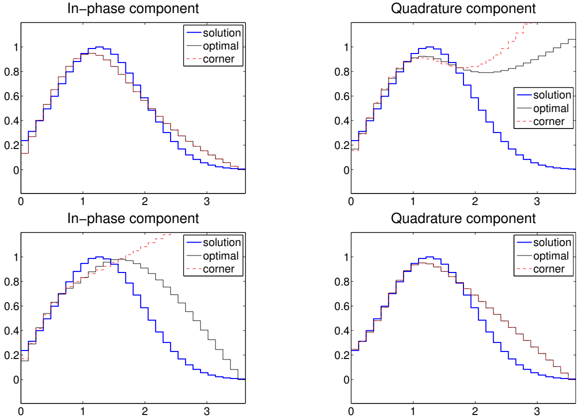


Figure: Solution obtained by minimizing the real part of the data (left) or the imaginary part (right); $\tau = 10^{-3}$, $\mu_r = 10$ in the top row, $\mu_r = 10^2$ in the bottom row. The value of ℓ is chosen either optimally or by the L–curve.

Magnetic permeability

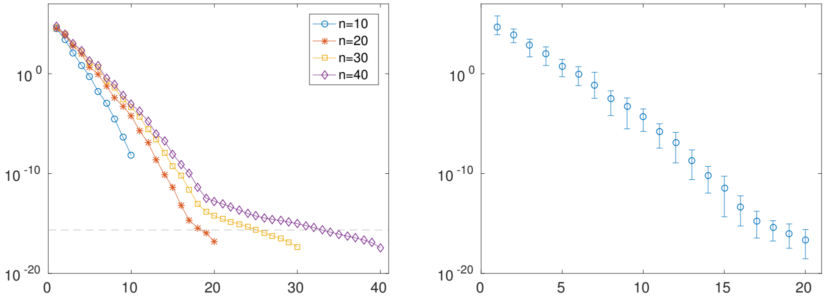


Figure: Average of the singular values of the Jacobian $J(\boldsymbol{\mu})$ computed on 100 random points in $\mathbb{R}^n$, for $m = n = 10, 20, 30, 40$ (left-hand side); each component of $\boldsymbol{\mu}$ is in $[\mu_0, 100\mu_0]$. The right-hand side graph shows the average singular values for $n = 20$ together with their maximum and minimum value across the random tests.

Magnetic permeability

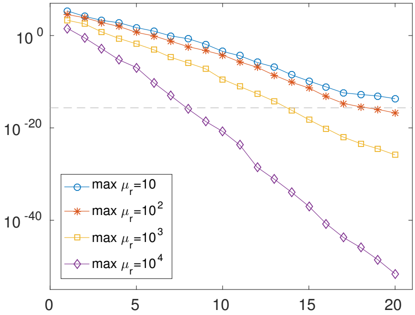


Figure: Average of the singular values of the Jacobian $J(\boldsymbol{\mu})$ computed on 100 random points in $\mathbb{R}^n$, for $m = n = 20$; each component of $\boldsymbol{\mu}$ is in $[\mu_0, \mu_r\mu_0]$, with $\mu_r = 10, 10^2, 10^3, 10^4$.

For the numerical experiments we consider

- The coils to be in both orientations.
- $m = 20$, $n = 40$, and $f = 1460$ Hertz.
- $\sigma(z) = e^{-(z-1.2)^2}$.

We considere the following model for the magnetic permeability as a function of depth

$$\mu_\theta(z) = \mu_0(\theta \mathrm{e}^{-(z-1.2)^2} + 1),$$

where θ is a parameter to be chosen, which takes values in $[\mu_0, (\theta + 1)\mu_0]$ and has a maximum at $z = 1.2\text{m}$.

Magnetic permeability

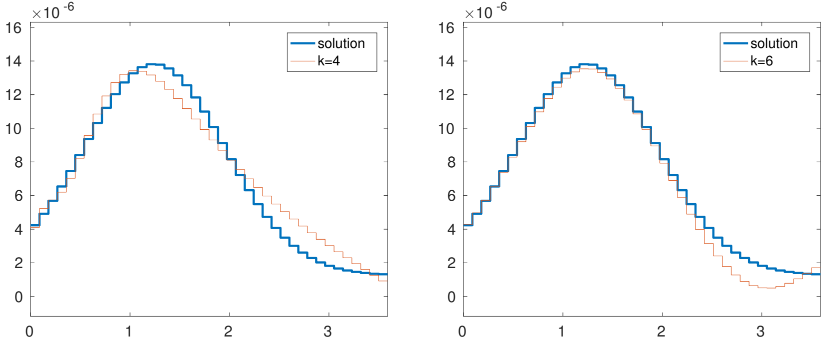


Figure: $\theta = 10$, in-phase component (left), quadrature component (right)

Magnetic permeability

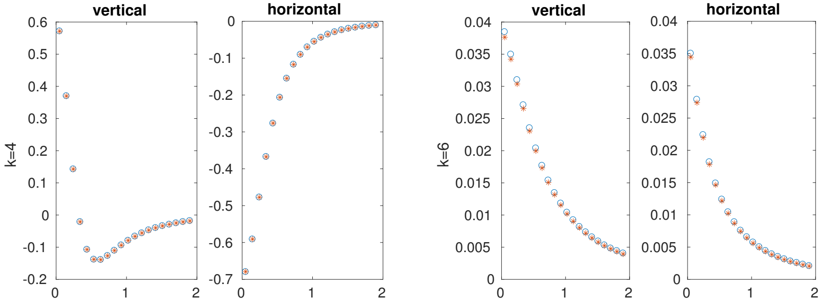
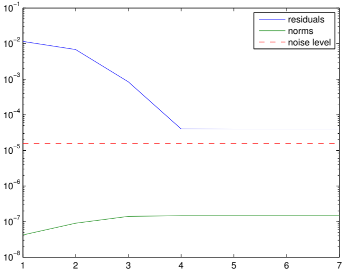


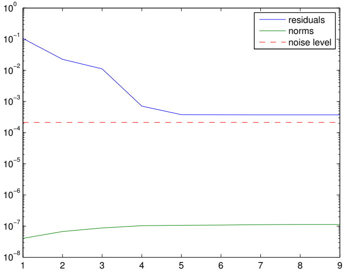
Figure: $\theta = 10$, in-phase component (left), quadrature component (right)


THANK YOU!

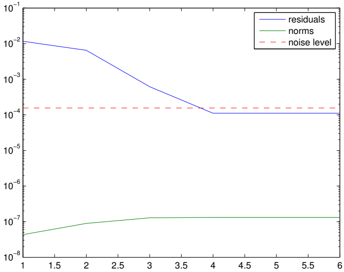
Quadrature component of the signal



In-phase component of the signal



Quadrature component of the signal



In-phase component of the signal