Recovering monomial-exponential sums via matrix-pencil methods

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A non linear approximation problem

Let us consider

$$h(x) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} x^s e^{f_j x}$$

where

- n and $\{m_j\}_{j=1}^{n}$ are positive integers
- $\{f_j\}_{j=1}^{n}$ are complex or real parameters
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- we know h in $2N$ equidistant points with $N > M = \sum_{j=1}^{n} m_j$;
- a reasonable overestimate $\widehat{M}$ of M is given.

Applications

This problem arises in different fields such as

- electromagnetism
- signal processes
- geophysics applications

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It also arises in **the numerical solution of nonlinear partial differential equations of integrable type**

L. Fermo, C. van der Mee and S. Seatzu,
Scattering data computation for the Zakharov-Shabat system,
Calcolo 53(3): 487-520, 2016

L. Fermo, C. van der Mee and S. Seatzu,
Scattering data computation for the Zakharov-Shabat system with non smooth potentials,
Applied Numerical Mathamatics , in press doi:10.1016/j.apnum.2016.09.016

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If $m_j \equiv 1$ for each j

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- Prony's method

B. de Prony.
Essai expérimental et analytique sur les lois de la Dilatabilité des fluides élastiques et sur celles de la Force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures.
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D. Potts and M. Tasche.
Parameter estimation for nonincreasing exponential sums by Prony-like methods.
Linear Algebra and Its Applications, 439(4):1024–1039, 2013.

A new matrix-pencil method

Let us introduce a new matrix-pencil method for the general sum

$$h(x) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} x^s e^{f_j x}$$

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L. Fermo, C. van der Mee and S. Seatzu.
Parameter estimation of monomial-exponential sums
Electronic Transactions on Numerical Analysis, **41**: 249-261, 2014.

The basic idea

Setting $z_j = e^{f_j}$ we write our sum

$$h(x) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} x^s e^{f_j x}$$

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Then we assume to know

$$h(k) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} k^s z_j^k, \quad 0^0 \equiv 1$$

for $k = k_0, k_0+1, \ldots, k_0+2N-1, \quad$ with $\quad k_0 \in \mathbb{N}^+ = \{0, 1, 2, ..., k_0, ...\}$.

We interpret

$$h(k) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} k^s z_j^k, \quad 0^0 \equiv 1, \qquad M = \sum_{j=1}^{n} m_j$$

as the general solution of a homogeneous linear difference equation of order M

$$\sum_{k=0}^{M} p_k h_{k+m} = 0$$

whose characteristic polynomial (Prony polynomial)

$$P(z) = \prod_{j=1}^{n} (z - z_j)^{m_j} = \sum_{k=0}^{M} p_k z^k, \quad p_M \equiv 1$$

is uniquely characterized by the z_j values we are looking for.

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2. Recovering the coefficients $\{c_{js}\}$.

Computation of $\{n, m_j, z_j\}$

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We arrange the $2N > \widehat{M}$ data in the Hankel matrices

$$\mathbf{H}_{N\widehat{M}}^{k_0} = \begin{pmatrix} h(k_0) & \boxed{\begin{matrix} h(k_0+1) & \dots & h(k_0+\widehat{M}-1) \\ h(k_0+2) & \dots & h(k_0+\widehat{M}) \\ \vdots & \vdots & \vdots \\ h(k_0+N) & \dots & h(k_0+N+\widehat{M}-2) \end{matrix}} \\ h(k_0+1) & \\ \vdots & \\ h(k_0+N-1) & \end{pmatrix}$$

$$\mathbf{H}_{N\widehat{M}}^{k_0+1} = \begin{pmatrix} \boxed{\begin{matrix} h(k_0+1) & h(k_0+2) & \dots \\ h(k_0+2) & h(k_0+3) & \dots \\ \vdots & \vdots & \vdots \\ h(k_0+N) & h(k_0+N+1) & \dots \end{matrix}} & \begin{matrix} h(k_0+\widehat{M}) \\ h(k_0+\widehat{M}+1) \\ \vdots \\ h(k_0+N+\widehat{M}-1) \end{matrix} \end{pmatrix}$$

The theory of finite difference equations allows us to prove the following lemma

Lemma

If the data are noiseless

- $rank(\mathbf{H}^{k_0}_{N\widehat{M}}) = rank(\mathbf{H}^{k_0+1}_{N\widehat{M}}) = M$

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- $rank(\mathbf{H}^{k_0}_{N\widehat{M}}) = rank(\mathbf{H}^{k_0+1}_{N\widehat{M}}) = M$
- $\mathbf{H}^{k_0+1}_{N\widehat{M}} = \mathbf{H}^{k_0}_{N\widehat{M}}\,\mathbf{C}_{\widehat{M}}(P)$ *where*
  - $\mathbf{C}_{\widehat{M}}(P)$ *is the companion matrix of the Prony polynomial, i.e.*

$$\mathbf{C}_{\widehat{M}}(P) = \begin{pmatrix} 0 & 0 & \dots & 0 & -p_0 \\ 1 & 0 & \dots & 0 & -p_1 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & -p_{\widehat{M}-1} \end{pmatrix}.$$

To the monomial-power sum, we associate the following matrix-pencil

$$\mathbf{H}_{\widehat{M}\widehat{M}}(z) = (\mathbf{H}^{k_0}_{N\widehat{M}})^*(\mathbf{H}^{k_0+1}_{N\widehat{M}} - z\mathbf{H}^{k_0}_{N\widehat{M}})$$

where the asterisk denotes the conjugate transpose.

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The difference equations properties allow us to state the following theorem

Theorem

The zeros z_j of the Prony polynomial, with their multiplicities, are exactly the generalized eigenvalues of the matrix-pencil $\mathbf{H}_{\widehat{M}\widehat{M}}(z)$.

Applying the Generalized Singular Value Decomposition to the matrices $\mathbf{H}^{k_0}_{N\widehat{M}}$ and $\mathbf{H}^{k_0+1}_{N\widehat{M}}$

$$\mathbf{H}^{k_0}_{N\widehat{M}} = \mathbf{V}_{NN} \begin{pmatrix} \mathbf{\Sigma}^{k_0}_{\widehat{M}\widehat{M}} \\ \mathbf{0}_{N-\widehat{M},\widehat{M}} \end{pmatrix} \mathbf{X}_{\widehat{M}\widehat{M}}$$

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We reduce the computation of the generalized eigenvalues of the matrix-pencil $\mathbf{H}_{\widehat{M}\widehat{M}}(z)$ to the computation of the eigenvalues of the $\widehat{M} \times \widehat{M}$ matrix

$$(\mathbf{\Sigma}_{\widehat{M}\widehat{M}}^{k_0})^{-1}(\mathbf{V}_{N\widehat{M}})^{*}\mathbf{U}_{N\widehat{M}}\mathbf{\Sigma}_{\widehat{M}\widehat{M}}^{k_0+1}$$

Computation of $\{c_{js}\}$

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Theorem

The coefficients vector $\mathbf{c} = [c_{1\,0}, \ldots, c_{1\,n_1-1}, \ldots, c_{\widehat{M}\,0}, \ldots, c_{\widehat{M}\,n_M-1}]^T$ *can be computed by solving (in the least square sense) the overdetermined linear system*

$$\mathbf{K}_{N\widehat{M}}^{k_0}\mathbf{c} = \mathbf{h}^{k_0}$$

where

- $\mathbf{h}^{k_0} = [h(k_0),\ h(k_0+1),\ \ldots,\ h(k_0+N-1)]^T$
- $\mathbf{K}_{N\widehat{M}}^{k_0}$ *is the Casorati matrix.*

$$
\mathbf{K}^{k_0}_{N\widehat{M}} = \begin{pmatrix}
z_1^{k_0} & k_0 z_1^{k_0} & \ldots & k_0^{m_1-1}\, z_1^{k_0} & \ldots & z_n^{k_0} & k_0 z_n^{k_0} & \ldots & k_0^{n_n-1} z_n^{k_0} \\
z_1^{k_1} & k_1 z_1^{k_1} & \ldots & k_1^{m_1-1}\, z_1^{k_1} & \ldots & z_n^{k_1} & k_1\, z_n^{k_1} & \ldots & k_1^{n_n-1} z_n^{k_1} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
z_1^{k_{N-2}} & k_{N-2} z_1^{k_{N-2}} & \ldots & k_{N-2}^{m_1-1} z_1^{k_{N-2}} & \ldots & z_n^{k_{N-2}} & k_{N-2} z_n^{k_{N-2}} & \ldots & k_{N-2}^{m_n-1} z_m^{k_{N-2}} \\
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\end{pmatrix}
$$

$$\mathbf{K}_{N\widehat{M}}^{k_0} = \begin{pmatrix} z_1^{k_0} & k_0 z_1^{k_0} & \dots & k_0^{m_1-1} z_1^{k_0} & \dots & z_n^{k_0} & k_0 z_n^{k_0} & \dots & k_0^{n_n-1} z_n^{k_0} \\ z_1^{k_1} & k_1 z_1^{k_1} & \dots & k_1^{m_1-1} z_1^{k_1} & \dots & z_n^{k_1} & k_1 z_n^{k_1} & \dots & k_1^{n_n-1} z_n^{k_1} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ z_1^{k_{N-2}} & k_{N-2} z_1^{k_{N-2}} & \dots & k_{N-2}^{m_1-1} z_1^{k_{N-2}} & \dots & z_n^{k_{N-2}} & k_{N-2} z_n^{k_{N-2}} & \dots & k_{N-2}^{m_n-1} z_m^{k_{N-2}} \\ z_1^{k_{N-1}} & k_{N-1} z_1^{k_{N-1}} & \dots & k_{N-1}^{m_1-1} z_1^{k_{N-1}} & \dots & z_n^{k_{N-1}} & k_{N-1} z_n^{k_{N-1}} & \dots & k_{N-1}^{m_n-1} z_m^{k_{N-1}} \end{pmatrix}$$

If $m_j \equiv 1$ then $\mathbf{K}_{Nn}^{k_0}$ reduces to the Vandermonde matrix of order $N \times n$ associated to zeros $z_1, \dots, z_n$.

Numerical Results

For a numerical evidence of the effectiveness of the method we adopt the following error estimates

$$e(\mathbf{f}) = \max_{j=1,\dots,n} \left| 1 - \frac{f_j}{f_j^*} \right|, \quad e(\mathbf{c}) = \max_{\substack{j=1,\dots,n \\ s=0,\dots,m_j-1}} \left| 1 - \frac{c_{js}}{c_{js}^*} \right|$$

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$$e(\mathbf{h}) = \max_{x \in X} \left| 1 - \frac{h(x)}{h^*(x)} \right|$$

where

$$X = \{x_i = i\Delta h, \quad \Delta h = \frac{b}{50}, i = 1, \dots, 50\}$$

and f_j^* and c_{js}^* denote the corresponding exact values of the parameters

We also consider data with noise

$$h(k) = \tilde{h}(k) + \delta e_k, \quad k = k_0, \ldots, k_0 + 2N - 1$$

where

- $\tilde{h}(k)$ is the exact value of the monomial exponential sum
- e_k is a random array
- δ is a parameter modeling the size of the noise

Example 1

$$h(x) = c_1 e^{-\lambda_1 x} + c_2 e^{-\lambda_2 x} + c_3 e^{-\lambda_3 x} + c_4 e^{-\lambda_4 x}$$

where

$$\mathbf{c} = [1,\ 2,\ 3,\ 4] + i, \quad \lambda = 10^{-1}[1 + 7i, 1.2 + 8i, 1.4 + 6i, 3 + 1.6i]$$

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N	δ	$\widehat{M}$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
4	0	4	2.87e-12	4.69e-11	1.58e-15
8	0	4	2.16e-13	3.54e-12	4.47e-15
16	0	4	4.17e-14	6.11e-13	6.95e-15

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4	10^{-8}	4	9.83e-06	1.47e-04	4.23e-08
8	10^{-8}	5	4.12e-07	6.36e-06	9.48e-09
16	10^{-8}	5	9.61e-08	1.40e-06	1.89e-08

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4	10^{-5}	4	1.32e-02	1.80e-01	1.67e-05
8	10^{-5}	5	8.80e-05	1.25e-03	9.44e-06
16	10^{-5}	5	1.49e-04	2.43e-03	1.92e-05

Example 2

$$h(x) = (c_1 x + c_2)e^{-\lambda_1 x} + c_3 e^{-\lambda_3 x} + c_4 e^{-\lambda_4 x}$$

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N	δ	$\widehat{M}$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
4	0	4	2.29e-06	2.22e-04	3.96e-08
8	0	4	6.38e-07	6.88e-05	1.11e-07
16	0	4	2.09e-07	2.85e-05	1.61e-07
4	10^{-9}	4	8.79e-04	8.67e-02	1.52e-05
8	10^{-9}	5	2.95e-04	3.19e-02	5.16e-05
16	10^{-9}	5	6.87e-05	9.35e-03	5.30e-05
4	10^{-8}	4	3.90e-03	4.41e-01	6.74e-05
8	10^{-8}	5	5.14e-04	5.50e-02	8.97e-05
16	10^{-8}	5	1.40e-04	1.90e-02	1.08e-04

Example 2

$$h(x) = (c_1 x + c_2)e^{-\lambda_1 x} + c_3 e^{-\lambda_3 x} + c_4 e^{-\lambda_4 x}$$

where

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Example 3 (literature)

$$h(x) = \sum_{j=1}^{40} c_j z_j^x, \quad z_j = e^{f_j}$$

where

- z_j are equidistant nodes on three circles having radius $r = 0.7, 0.8, 0.9$
- the coefficients c_j are random.

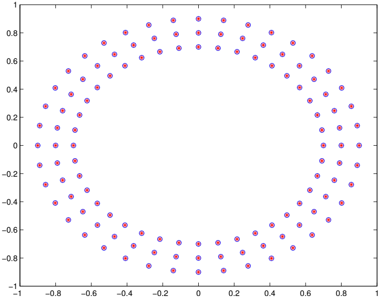

Figure: $\delta = 0$

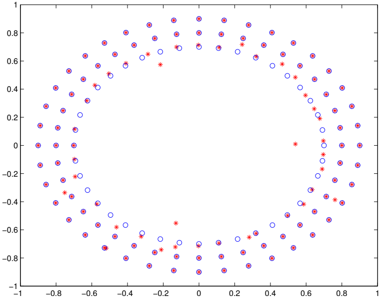

Figure: $\delta = 10^{-11}$

radius	N	$\widehat{M}$	δ	$e(\mathbf{c})$	$e(\mathbf{h})$
0.7	40	40	0	7.01e-09	6.03e-09
0.8	40	40	0	1.21e-10	3.95e-11
0.9	40	40	0	1.00e-11	1.76e-12
0.7	40	40	10^{-11}	2.46e+00	2.32e-02
0.8	40	40	10^{-11}	5.51e-03	1.48e-05
0.9	40	40	10^{-11}	1 .04e-06	1.23e-07

The bivariate case

Let us consider the **bivariate monomial-exponential sum**

$$h(x_1, x_2) = \sum_{j_1=1}^{n_1} \sum_{s_1=0}^{m_{1j_1}-1} \sum_{j_2=1}^{n_2} \sum_{s_2=0}^{m_{2j_2}-1} c_{(j_1,s_1),(j_2,s_2)}\, x_1^{s_1}\, e^{f_{1j_1} x_1}\, x_2^{s_2} e^{f_{2j_2} x_2}$$

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The problem consists of **recovering**

- the **positive integers** n_1, n_2, m_{1j_1} and m_{2j_2}
- the **complex or real parameters** f_{1j_1} and f_{2j_2}
- the **complex or real coefficients** $c_{(j_1,s_1),(j_2,s_2)}$

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under the assumption that we know

- h in 2N points (x_{1k_1}, x_{2k_2}) of a regular grid of $[a_1, b_1] \times [a_2, b_2]$ with $N_1 \geq M_1$, $N_2 \geq M_2$ where $M_1 = m_{11} + \cdots + m_{1n_1}$, $M_2 = m_{21} + \cdots + m_{2n_2}$ knowing $k_1 = 0, 1, ..., 2N_1$, $k_2 = 0, 1, ..., 2N_2$
- a reasonable overestimate of M_1 and M_2

A look at the literature

To our knowledge, the identification of parameters and coefficients

$$h(x_1, x_2) = \sum_{j_1=1}^{n_1} \sum_{s_1=0}^{m_{1j_1}-1} \sum_{j_2=1}^{n_2} \sum_{s_2=0}^{m_{2j_2}-1} c_{(j_1,s_1),(j_2,s_2)}\, x_1^{s_1}\, e^{f_{1j_1} x_1}\, x_2^{s_2} e^{f_{2j_2} x_2}$$

has never been investigated before.

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Recently the much simpler sums

$$h(x_1, x_2) = \sum_{j=1}^{n} c_j e^{f_{1j} x_1 + f_{2j} x_2}$$

have been studied.

D. Potts and M. Tasche.
Parameter estimation for multivariate exponential sums.
Electronic Transactions on Numerical Analysis, 40:2014-224, 2013.

A new matrix-pencil method

The general **bivariate monomial-exponential sum**

$$h(x_1,x_2)=\sum_{j_1=1}^{n_1}\sum_{s_1=0}^{m_{1j_1}-1}\sum_{j_2=1}^{n_2}\sum_{s_2=0}^{m_{2j_2}-1} c_{(j_1,s_1),(j_2,s_2)}\,x_1^{s_1}\,e^{f_{1j_1}x_1}\,x_2^{s_2}e^{f_{2j_2}x_2}.$$

L. Fermo, C. van der Mee and S. Seatzu
Parameter estimation of monomial-exponential sums in one and two variables,
Applied Mathematics and Computation, 258:576-586, 2015.

The matrix-pencil method

(1) Fixing x_2, consider the univariate monomial-exponential sum

$$h_{x_2}(x_1) = \sum_{j_1=1}^{n_1} \sum_{s_1=0}^{m_{1j_1}-1} a_{j_1,s_1}(x_2)\, x_1^{s_1}\, e^{f_{1j_1} x_1}$$

and apply our matrix-pencil method to recover

$$\{n_1,\ m_{1j_1},\ f_{1j_1}\},$$

given $h_{x_2}(x_{1k_1})$, $k_1 = 0, 1, ..., 2N_1$.

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(2) Fixing x_1, consider the univariate monomial-exponential sum

$$h_{x_1}(x_2) = \sum_{j_1=1}^{n_2} \sum_{s_2=0}^{m_{2j_2}-1} a_{j_2,s_2}(x_1)\, x_1^{s_1}\, e^{f_{2j_2} x_2}$$

and apply our matrix-pencil method to recover

$$\{n_2,\ m_{2j_2},\ f_{2j_2}\},$$

given $h_{x_1}(x_{2k_2})$, $k_2 = 0, 1, \ldots, 2N_2$.

(3) Once recovered the parameters

$$\{n_1,\ n_2,\ m_{1j_1},\ m_{2j_2},\ f_{1j_1},\ f_{2j_2}\}$$

we have to estimate the coefficients $c_{(j_1,s_1),(j_2,s_2)}$ of

$$h(x_1,x_2) = \sum_{j_1=1}^{n_1} \sum_{s_1=0}^{m_{1j_1}-1} \sum_{j_2=1}^{n_2} \sum_{s_2=0}^{m_{2j_2}-1} c_{(j_1,s_1),(j_2,s_2)}\, x_1^{s_1}\, e^{f_{1j_1}x_1}\, x_2^{s_2} e^{f_{2j_2}x_2}$$

that is to solve a linear system

$$\mathcal{F}\,\mathbf{c} = \mathbf{h}$$

where

- the rows of $\mathcal{F}$, as well as the entries of $\mathbf{h}$, depend on the pair (k_1, k_2),
- the columns of $\mathcal{F}$, as well as the entries of $\mathbf{c}$, depend on the pair $(j_1, s_1), (j_2, s_2)$

Example 4

Let us consider

$$h(x_1, x_2) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{3} c_{j_1,j_2} e^{f_{1j_1} x_1 + f_{2j_2} x_2}$$

where

$$\mathbf{c} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{f_1} = [2, 4], \quad \mathbf{f_2} = [1, 3, 5].$$

N	δ	$\widehat{M}_1$	$\widehat{M}_2$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
8	0	5	5	1.68e-13	9.09e-13	1.46e-11
16	0	5	5	3.08e-12	4.35e-12	5.02e-11

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8	10^{-7}	5	5	3.00e-08	3.33e-08	1.08e-07
16	10^{-7}	5	5	2.93e-08	8.18e-09	1.38e-08

THANKS FOR THE ATTENTION !