Generalized quantum entropies: a definition and some properties

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Motivations

- Increasing field of investigation on quantum information processing or transmission.
- Necesitate the use of quantum information measures, or of quantum entropies.
- There exist some definitions: von Neumann, quantum versions of Rényi, Tsallis, Kaniadakis types, ...

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Note

In the classical context, there exists a generalized family proposed by Salicrú (Csiszàr); Contains the Shannon entropy, that of Rényi, Havrda-Charvát (Daróczy, Vajda, Tsallis, ...) among others.

Goals

- To define a generalized family of quantum entropies.
- To study their properties (common or specific).
- To apply them in quantum information processing.

Programa

1. Motivations & goals
2. Classical (h,ϕ)-entropies
   - Definition
   - Properties
3. Quantum (h,ϕ)-entropies
   - Definition
   - Basic properties
4. Composite quantum systems
   - Bipartite systems – (sub)additivity, pure state
   - (h,ϕ)-entropy and entanglement
5. Relative (h,ϕ)-entropies
   - Classical context
   - Quantum context
6. Conclusions

Definition

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Let $p = [p_1 \quad \cdots \quad p_N] \in [0\,;\,1]^N, \quad \sum_k p_k = 1$

$$H_{(h,\phi)}(p) = h\left(\sum_k \phi(p_k)\right)$$

$\phi : [0\,;\,1] \to \mathbb{R}$ y $h : \mathbb{R} \to \mathbb{R}$,

- ϕ is concave and h is increasing, or
- ϕ is convex and h is decreasing

Salicru et al., *Asymptotic distribution of (h,ϕ)-entropies*, Comm. in Stat.: Th. Meth. (1993)

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Moreover

- $\phi(0) = 0$ (no elementary uncertainty associated to the probability 0)
- $h(\phi(1)) = 0$ (no uncertainty associated to a deterministic state)

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Famous examples

	ϕ	h	$H_{(h,\phi)}(p)$
Shannon	$-x \ln x$	x	$-\sum_k p_k \ln p_k$
Rényi	x^α	$\frac{\ln x}{1-\alpha}$	$\frac{\ln\left(\sum_k p_k^\alpha\right)}{1-\alpha}$
HCT	x^α	$\frac{x-1}{1-\alpha}$	$\frac{\sum_k p_k^\alpha - 1}{1-\alpha}$
Unified	x^r	$\frac{x^s-1}{(1-r)\,s}$	$\frac{\left(\sum_k p_k^r\right)^s - 1}{(1-r)\,s}$
Kaniadakis	$\frac{x^{1-\kappa}-x^{1+\kappa}}{2\,\kappa}$	x	$\frac{\sum_k \left(p_k^{1-\kappa}-p_k^{1+\kappa}\right)}{2\,\kappa}$

Basic properties

For any pair of entropic functional (h,ϕ),

Basic properties

- Invariance to a permutation of the p_k's
- Expansibility: $H_{(h,\phi)}\left([p_1 \ \cdots \ p_N \quad 0]\right) = H_{(h,\phi)}\left([p_1 \ \cdots \ p_N]\right)$
  (consequence of $\phi(0)=0$)
- Fusion: $H_{(h,\phi)}\left([p_1 \quad p_2 \ \cdots \ p_N]\right) \ge H_{(h,\phi)}\left([p_1+p_2 \quad \cdots \ p_N]\right)$
  (Petković's inequality $\phi(a+b) \le \phi(a)+\phi(b)$ for concave ϕ with $\phi(0)=0$)

Majorization

Definition

p, p' proba. vectors, with components increasingly arranged,

$$p \prec p' \qquad (p \text{ is majorized by } p')$$

if $\displaystyle\sum_{k=1}^{n} p_k \leq \sum_{k=1}^{n} p'_k \quad \forall\, n < \max(N, N') \quad \& \quad \sum_{k=1}^{\max(N,N')} p_k = \sum_{k=1}^{\max(N,N')} p'_k$

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Majorization is a partial order relationship

Examples

For any p, of dimension N,

$$\left[\tfrac{1}{N} \ \cdots \ \tfrac{1}{N}\right] \prec \left[\tfrac{1}{\|p\|_0} \ \cdots \ \tfrac{1}{\|p\|_0} \quad 0 \cdots 0\right] \prec [1 \quad 0 \cdots 0]$$

Properties linked to the majorization

For any pair of entropic functionals (h, ϕ),

Schur-concavity

- $p \prec p' \quad \Rightarrow \quad H_{(h,\phi)}(p) \geq H_{(h,\phi)}(p')$ (equality iif $p \equiv p'$)

  (consequence of the Karamata's theorem)

- Reciprocal if for all pairs (h, ϕ)

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Bounds

$$0 \leq H_{(h,\phi)}(p) \leq h\left(\|p\|_0 \, \phi\left(\frac{1}{\|p\|_0}\right)\right) \leq h\left(N\phi\left(\frac{1}{N}\right)\right)$$

certainty uniform

(consequence of majorization relationships)

Programa

Quantum (h,ϕ)-entropy: definition

Let ρ be a density operator acting on $\mathcal{H}^N$
($\rho \geq 0$ hermitian, with $\operatorname{Tr}\rho = 1$)

Definition

$$\boldsymbol{H}_{(h,\phi)}(\rho) = h\left(\operatorname{Tr}\phi(\rho)\right)$$

with $\phi : [0\,;\,1] \to \mathbb{R}$, $\phi(0) = 0$ & $h : \mathbb{R} \to \mathbb{R}$, $h(\phi(1)) = 0$,

- ϕ is concave and h is increasing, or
- ϕ is convex and h is decreasing

(for $\rho = \sum_k \lambda_k \, |e_k\rangle\langle e_k|$, $\phi(\rho) = \sum_k \phi(\lambda_k) \, |e_k\rangle\langle e_k|$)

Quantum vs classical (h, ϕ)-entropy

Diagonal form

$$\rho = \sum_k \lambda_k \, |e_k\rangle\langle e_k|$$

where

- $\{|e_k\rangle\}$ is the orthonomal base of $\mathcal{H}^N$ that diagonalizes ρ,
- $\lambda = [\lambda_1 \ \cdots \ \lambda_N] \in [0\,;\,1]^N, \quad \sum_k \lambda_k = 1$ the eigenvalues of ρ

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Quantum vs classical

$$\boldsymbol{H}_{(h,\phi)}(\rho) = H_{(h,\phi)}(\lambda)$$

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equality iif $\rho' = U \rho U^\dagger$ or $\rho = U \rho' U^\dagger$ with U isometry $(U^\dagger U = I)$

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Bounds

$$0 \leq \boldsymbol{H}_{(h,\phi)}(\rho) \leq h\left(\operatorname{rank}\rho\, \phi\left(\frac{1}{\operatorname{rank}\rho}\right)\right) \leq h\left(N \phi\left(\tfrac{1}{N}\right)\right)$$

pure state $|\psi\rangle\langle\psi|$ max. mixed $\frac{I}{N}$

Properties specific to the quantum context

Concavity

If h is concave, then $\boldsymbol{H}_{(h,\phi)}(\cdot)$ is concave,

$$\boldsymbol{H}_{(h,\phi)}\left(\omega\rho + (1-\omega)\rho'\right) \geq \omega \boldsymbol{H}_{(h,\phi)}(\rho) + (1-\omega)\boldsymbol{H}_{(h,\phi)}\left(\rho'\right)$$

(Peierls's inequality, $\operatorname{Tr}(\rho) \leq \sum_k \phi(\langle f_k |\, \rho \,| f_k\rangle)$ & ϕ concave)

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Mixture

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \quad \Rightarrow \quad \boldsymbol{H}_{(h,\phi)}(\rho) \leq H_{(h,\phi)}(p)$$

(Schrödinger's mixture $p = B\lambda$, B bistoch., Hardy-Littlewood-Pólya $p \prec \lambda$)

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Entropy vs diagonal

$p^E(\rho)$ diag. ρ in $E=\{e_k\}$ orth. base: $\quad \boldsymbol{H}_{(h,\phi)}(\rho) \le H_{(h,\phi)}\left(p^E(\rho)\right)$

(Schur-Horn's theorem: $p^E(\rho) \prec \lambda$)

Effect of a transform or a measure

Transform

- Invariance to a unitary transf. U (e.g., time evolution)

$$\boldsymbol{H}_{(h,\phi)}\left(U\rho U^\dagger\right) = \boldsymbol{H}_{(h,\phi)}(\rho)$$

- Decrease s.t. bistochastic operation (e.g., general measure):
  $\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger, \quad \sum_k A_k^\dagger A_k = I = \sum_k A_k A_k^\dagger$ (complete)

$$\boldsymbol{H}_{(h,\phi)}(\mathcal{E}(\rho)) \ge \boldsymbol{H}_{(h,\phi)}(\rho) \qquad \text{(information degradation)}$$

  Equality iif $\mathcal{E}(\rho) = U\rho U^\dagger$, U unitary
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Consequence

$\{E_k\} \in \mathbb{E}$ **rank one** POVM, $p^E(\rho) = \mathrm{Tr}(E_k \rho)$,

$$\boldsymbol{H}_{(h,\phi)}(\rho) = \min_{\mathbb{E}} H_{(h,\phi)}\left(p^E(\rho)\right)$$

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Additivities, pure state

Let $\mathcal{H}^A \otimes \mathcal{H}^B, \quad \rho^{AB}, \quad \rho^A = \mathrm{Tr}_B\, \rho^{AB}, \quad \rho^B = \mathrm{Tr}_A\, \rho^{AB}$

(Sub)additivity

- <u>If</u> (i) $\phi(ab) = \phi(a)b + a\phi(b)$ and $h(x+y) = h(x) + h(y)$, or

  (ii) $\phi(ab) = \phi(a)\phi(b)$ and $h(xy) = h(x) + h(y)$, <u>then</u>

$$\boldsymbol{H}_{(h,\phi)}\left(\rho^A \otimes \rho^B\right) = \boldsymbol{H}_{(h,\phi)}\left(\rho^A\right) + \boldsymbol{H}_{(h,\phi)}\left(\rho^B\right)$$

  (e.g., von Neuman, Rényi)

- $\boldsymbol{H}_{(h,\phi)}\left(\rho^{AB}\right) \le \boldsymbol{H}_{(h,\phi)}\left(\rho^A \otimes \rho^B\right) \Leftrightarrow \phi(x) = -x \ln x$

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Pure states

$$\rho^{AB} = |\psi\rangle\langle\psi| \quad \Rightarrow \quad \boldsymbol{H}_{(h,\phi)}\left(\rho^A\right) = \boldsymbol{H}_{(h,\phi)}\left(\rho^B\right)$$

(Schmidt's decomposition)

Separable states

Separable states:

$$\rho^{AB} = \sum_m \omega_m \left|\Psi_m^A\right\rangle\left\langle\Psi_m^A\right| \otimes \left|\Psi_m^B\right\rangle\left\langle\Psi_m^B\right| \quad \omega_m \geq 0,\ \sum_m \omega_m = 1$$

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Separability inequality

If ρ^{AB} is separable, then

$$\boldsymbol{H}_{(h,\phi)}\left(\rho^{AB}\right) \geq \max\left\{\boldsymbol{H}_{(h,\phi)}\left(\rho^{A}\right) \, , \, \boldsymbol{H}_{(h,\phi)}\left(\rho^{B}\right)\right\}$$

$(\rho^{AB} \prec \rho^A \;\; \& \;\; \rho^{AB} \prec \rho^B)$

Generalizable to multipartite systems and
totally separable states

Entanglement Detection: an example

Werner: $\rho^{AB} = \omega \, |\Psi^-\rangle\langle\Psi^-| + (1-\omega)\frac{I}{4}$, $\qquad |\Psi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

Entangled iif $\omega > \frac{1}{3}$; $\qquad \rho^A = \rho^B = \frac{I}{2}$

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Entangled iif $\omega > \frac{1}{3}$; $\quad \rho^A = \rho^B = \frac{I}{2}$ $\quad \phi(x) = x^\alpha,\ h(x) = \frac{f(x)}{1-\alpha}$

Detection

Criterion: $\dfrac{f\left(3\left(\frac{1-\omega}{4}\right)^\alpha + \left(\frac{1+3\omega}{4}\right)^\alpha\right) - f\left(2^{1-\alpha}\right)}{\alpha - 1} > 0 \ \Rightarrow$ entangled

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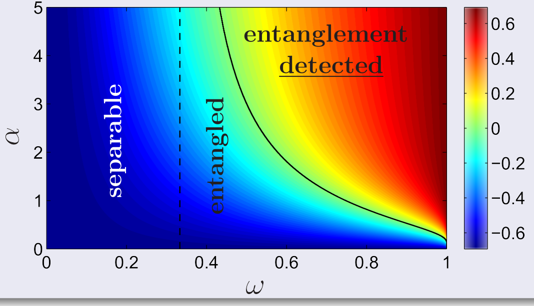

Programa

Relative entropy and mutual information

Conditional probability: $p^{A|B=b} = \frac{p^{AB}_{a,b}}{p^B_b}$

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From the conditional probability

Relative entropy: $H^{\mathcal{J}}_{(h,\phi)}(A|B) = \sum_b p_b^B H_{(h,\phi)}\left(p^{A|B=b}\right)$

Mutual information: $\mathcal{J}_{(h,\phi)}(A;B) = H_{(h,\phi)}(A) - H^{\mathcal{J}}_{(h,\phi)}(A|B)$

h concave guarantees that $\mathcal{J}_{(h,\phi)} \geq 0\ldots$ $\mathcal{J}_{(h,\phi)}$ not symmetrical...

Relative entropy and mutual information

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From the chain rule

Relative entropy: $H^{\mathcal{I}}_{(h,\phi)}(A|B) = H_{(h,\phi)}(A,B) - H_{(h,\phi)}(B)$

Mutual information: $\mathcal{I}_{(h,\phi)}(A;B) = H_{(h,\phi)}(A) - H^{\mathcal{I}}_{(h,\phi)}(A|B)$

$\mathcal{I}_{(h,\phi)}$ symmetrical, but with no guarantee that $\mathcal{I}_{(h,\phi)} \ge 0 \ldots$

Relative entropy and mutual information

$\left\{\Pi^B\right\}$ local projective measurement:

$$p_j^B = \mathrm{Tr}\left(I \otimes \Pi_j^B \, \rho^{AB}\right), \qquad \rho^{A|\Pi_j^B} = \frac{I \otimes \Pi_j^B \, \rho^{AB} \, I \otimes \Pi_j^B}{p_j^B}$$

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From the conditional state

Relative entropy vs Π^B: $\boldsymbol{H}_{(h,\phi)}^{\mathcal{J}}\left(A|\Pi^B\right) = \sum_j p_j^B \boldsymbol{H}_{(h,\phi)}\left(\rho^{A|\Pi_j^B}\right)$

Relative entropy vs B: $\boldsymbol{H}_{(h,\phi)}^{\mathcal{J}}\left(A|B\right) = \min_{\{\Pi^B\}} \boldsymbol{H}_{(h,\phi)}^{\mathcal{J}}\left(A|\Pi^B\right)$

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- We proposed an extension of the (h, ϕ)-entropies for the quantum systems (that extends the trace-entropies).

- These extensions are based on two entropic functionals ϕ & h, and encompass various famous entropies such that the von Neuman's, Tsallis's, Rényi's (thanks to h), unified, trace entropies or not.

- We proposed possibles associated measures such that relative entropies and mutual informations; a unified point of view is still missing; there properties remain to be investigated.

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- We proposed an extension of the (h, ϕ)-entropies for the quantum systems (that extends the trace-entropies).
- These extensions are based on two entropic functionals ϕ & h, and encompass various famous entropies such that the von Neuman's, Tsallis's, Rényi's (thanks to h), unified, trace entropies or not.
- We proposed possibles associated measures such that relative entropies and mutual informations; a unified point of view is still missing; there properties remain to be investigated.

Summary

- We studied various properties shared by the whole family; the main ones rely on the notion of majorization.

- In particular, the Schur-concavity appears to be crucial in the quantum context.

- We studied the effect of quantum operations (unitary transform, measures) on these entropies.

- We studied their properties for composite systems: they allow to propose entanglement detection criteria.

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Grazie
¡Gracias!
Thank you!
DANKE!
Merci!
你很