Cordial Volterra integral equations of the first kind

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Consider the equation $V_\varphi u = f$, where $(V_\varphi u)(t) = \int_0^t t^{-1}\varphi(t^{-1}s)u(s)ds$, $0 < t \le T$, $\varphi \in L^1(0,1)$. The spectrum of V_φ as an operator in the space $C^m = C^m[0,T]$ is described in [1]. This enables to establish criteria for the existence and boundedness of the inverse V_φ^{-1} as an operator from C^{m+k} to C^m, $m \ge 0$, $k \ge 1$. In some cases, $\|V_\varphi^{-1}\|$ can be effectively estimated, e.g.:

Theorem 1. *Let $\varphi \in L^1(0,1)$, $\widehat{\varphi}(0) := \int_0^1 \varphi(x)dx > 0$, $x\varphi' \in L^1(0,1)$, and let $x\varphi'(x) + \alpha\varphi(x) \ge 0$ $(0 < x < 1)$ for an $\alpha < 1$. Then $V_\varphi^{-1} \in \mathcal{L}(C^{m+1}, C^m)$ exists, and*

$$\|V_\varphi^{-1} f\|_{C^m} \le \frac{1}{(1-\alpha)\widehat{\varphi}(0)} \|tf' + (1-\alpha)f\|_{C^m} \text{ for } f \in C^{m+1},\ m \ge 0.$$

The claim remains to be true if condition $x\varphi' \in L^1(0,1)$ is relaxed to the form $x\varphi' \in L^1(0,1-\varepsilon)$ for any $\varepsilon > 0$, and $\lim_{x\to 1}\varphi(x) = \infty$, $\lim_{x\to 1}(1-x)\varphi(x) = 0$.

More complete results are obtained using the weighted spaces

$$C_\star^{m,r} = \left\{ u \in C^m(0,T] : \lim_{t\to 0} t^{k-r}u^{(k)}(t) \text{ exists for } k = 0,1,...,m \right\},$$

$$\|u\|_{C_\star^{m,r}} = \max_{0\le k\le m} \sup_{0<t\le T} t^{k-r}|u^{(k)}(t)|, \quad m \ge 0, \quad r \in \mathbf{R}.$$

The power functions t^p, $p \ge r$, are eigenfunctions of V_φ in $C_\star^{m,r}$, $m \ge 0$. This enables to design approximate and exact solvers of equation $V_\varphi u = f$.

References

[1] G. Vainikko. *Cordial Volterra integral equations 1, 2*, Numer. Funct. Anal. Optim., 30 (2009), pp. 1145–1172; 31 (2010), pp. 191–219.