Inner-iteration GMRES methods for underdetermined least squares problems

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Consider the underdetermined least squares problem

$$\min_{\boldsymbol{x} \in \mathbf{R}^n} \|\boldsymbol{b} - A\boldsymbol{x}\|_2, \tag{1}$$

where $A \in \mathbf{R}^{m \times n}$, $\boldsymbol{b} \in \mathbf{R}^m$, and $m < n$.

We can precondition (1) from the right as

$$\min_{\boldsymbol{u} \in \mathbf{R}^n} \|AB\boldsymbol{u} - \boldsymbol{b}\|_2, \quad \boldsymbol{x} = B\boldsymbol{u} \tag{2}$$

or from the left as

$$\min_{\boldsymbol{x} \in \mathbf{R}^n} \|B\boldsymbol{b} - BA\boldsymbol{x}\|_2 \tag{3}$$

by using a preconditioner $B \in \mathbf{R}^{n \times m}$ [2].

However, when solving inconsistent systems ($\boldsymbol{b} \notin \mathcal{R}(A)$), the effective condition number becomes dangerously large[1], and GMRES for (2) will breakdown numerically before it determines a least squares solution.

On the other hand, (3) is consistent when $\mathcal{R}(B^T) = \mathcal{R}(A)$. Thus, GMRES can numerically determine a least squares solution for (1) even when $m < n$ and $\boldsymbol{b} \notin \mathcal{R}(A)$. To form such a preconditioner B, we propose using inner-iteration preconditioners, which do not require a preconditioning matrix and can save memory. Numerical experiments illustrate that the methods are efficient and robust for large ill-conditioned and rank-deficient problems.

References

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