A family of rules for parameter choice in Tikhonov regularization of ill-posed problems with inexact noise level

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We consider the equation $Au = f, f \in \mathcal{R}(A)$, where $A \in \mathcal{L}(H, F)$ and H, F are real Hilbert spaces. We suppose that instead of f we have an approximation $\tilde{f} \in F$ and inexact noise level δ. We consider the case where δ may be a serious overestimation with $\|\tilde{f} - f\| \le \delta$, but also the case of possible underestimation of the noise level: for example, it may be known only that with high probability $\delta/\|y - y_*\| \in [1/10, 10]$. We consider choice of the regularization parameter α in the Tikhonov method $u_\alpha = (\alpha I + A^*A)^{-1}A^*\tilde{f}$.

To guarantee convergence of u_α, the choice of α must use the noise level. Classical rules for parameter choice as the discrepancy principle, monotone error rule and the balancing principle are unstable with respect to the inaccuracies of the noise level: they fail in case of underestimated noise level and give large error of u_α already at very moderate overestimation of the noise level. We propose for choice of $\alpha = \alpha(\delta)$ the following family of rules.

Define $B_\alpha = \sqrt{\alpha}(\alpha I + AA^*)^{-1/2}$, $D_\alpha = \alpha^{-1}AA^*B_\alpha^2$. Fix the parameters q, k, l such that $3/2 \le q < \infty$, $l \ge 0$, $k \ge l/q$. Choose $\alpha = \alpha(\delta)$ as the largest solution of the equation

$$\frac{(1 + \alpha\|A\|^{-2})^{((k+s_0)q-l)/(2q-2)}\|D_\alpha^k B_\alpha(Au_\alpha - \tilde{f})\|^{q/(q-1)}}{\|D_\alpha^l B_\alpha^{2q-2}(Au_\alpha - \tilde{f})\|^{1/(q-1)}} = b\delta,$$

where b is constant large enough and $s_0 = 0$ if $k = l/q$, $s_0 = 1/2$ if $k > l/q$.

We will analyze the quasi-optimality and the stability of these rules. The advantages of some rules of this family over classical rules in case of the over- or underestimated noise level with $\delta/\|\tilde{f} - f\| \in [1/64, 64]$ are demonstrated on extensive numerical experiments in test problems of P.C. Hansen and from paper C. Brezinski, G. Rodriguez and S. Seatzu.