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Universal Analytic Properties of Noise

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Introduction

- noise is unwanted signal (nuisance)
- don’t try to get rid of it, make it your ally!
- “universal” statistical properties of singularities
- useful signal perturbs statistical properties

Z-transform

$s_0, s_1, s_2, \ldots$

$$Z(z) = \sum_{k=0}^{\infty} \frac{s_k}{z^k}$$

zeros, poles
"singularities"

Example 1

One oscillator (damped if $\mathrm{Im}(\omega) > 0$)

$$s_k = A\, e^{i\omega \Delta t\, k}$$

$$Z(z) = \frac{Az}{z - e^{i\omega\Delta t}}$$

pole

Residue relates to amplitude

Phase gives the frequency of the carrier

Magnitude gives the damping coefficient

Example 2

P oscillators

Z-transform is linear, therefore it has P poles (inside the unit circle)

$$s_k = \sum_{j=1}^{P} A_j \, e^{i\omega_j \Delta t\, k} \quad \longrightarrow \quad Z(z) = \sum_{j=1}^{P} \frac{\rho_j}{z - p_j}$$

$$p_j = e^{i\omega_j \Delta t} \;, j = 1, \ldots, N$$

Z transform is a rational function in z

Example 3

A series of random numbers

Steinhaus theorem:

A Taylor series with random coefficients has, *with probability one (that is for a set of measure zero)*, the unit circle as natural boundary.

Damped oscillators + Noise	Z-transform →	Analytic function inside/outside the unit disk + Rational function, one pole for each oscillator

J-operator

Definition: A Hilbert space operator that has Z-transform as the resolvent matrix element.

$$J = \lim_{n\to\infty} J_n$$

Theorem: (Bessis-Perotti) The spectrum of a *J-operator* has two parts:

(1) an essential spectrum with support on the unit circle. Eigenfunctions have infinite norm (open question). At a finite order (*truncated J-matrix*), the spectrum of J_n is made of Froissart doublets.

(2) a discrete spectrum made of a finite number of poles. Eigenfunctions have finite norm.

Finite order Z-transform

In applications we deal with finite length time series: $s_1, s_2, \ldots, s_N$

Poles of a Padé approximant close to the poles of Z(z)

$$\left[\frac{n-1}{n}\right](z) = \frac{N_{n-1}(z)}{D_{n-1}(z)}$$

$N_{n-1}(z)$ → polynomial

$D_{n-1}(z)$ → monic polynomial

3-term recursion: $D_{k+1}(z) = (z - A_k) D_k(z) - R_k(z) \qquad D_{-1}(z) = 1, \quad D_0(z) = 1$

where: $A_k = -(a_{2k} + a_{2k+1}) \qquad R_k = a_{2k-1} a_{2k}$

and: a_k are the coefficients of the Stieltjes continued fraction

Jacobi (J) Matrix

3-term recursion can be written as an eigenvalue problem:

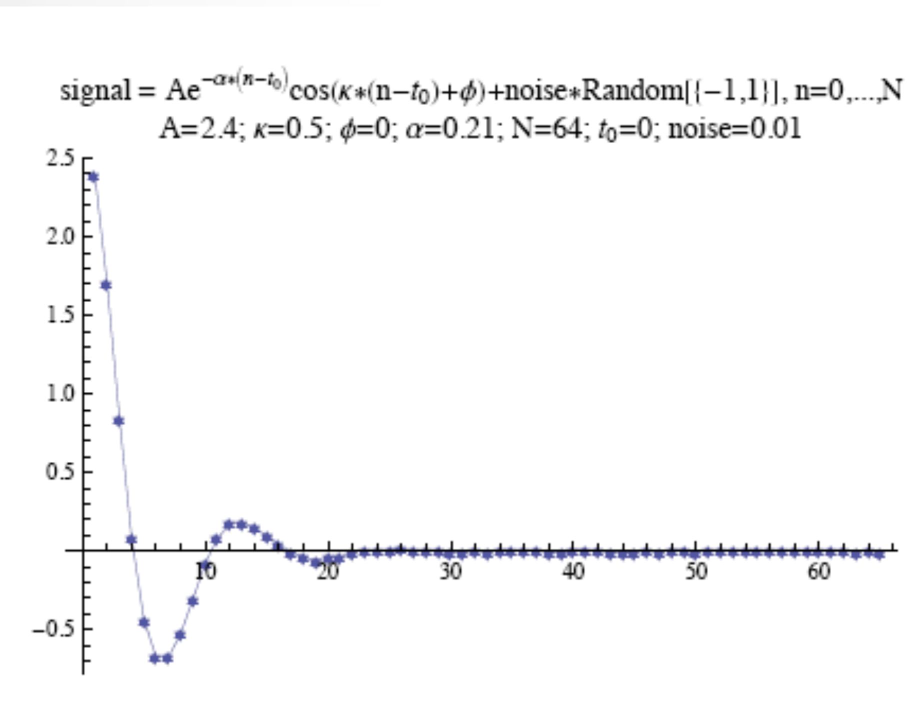
$$J_n V = zV$$

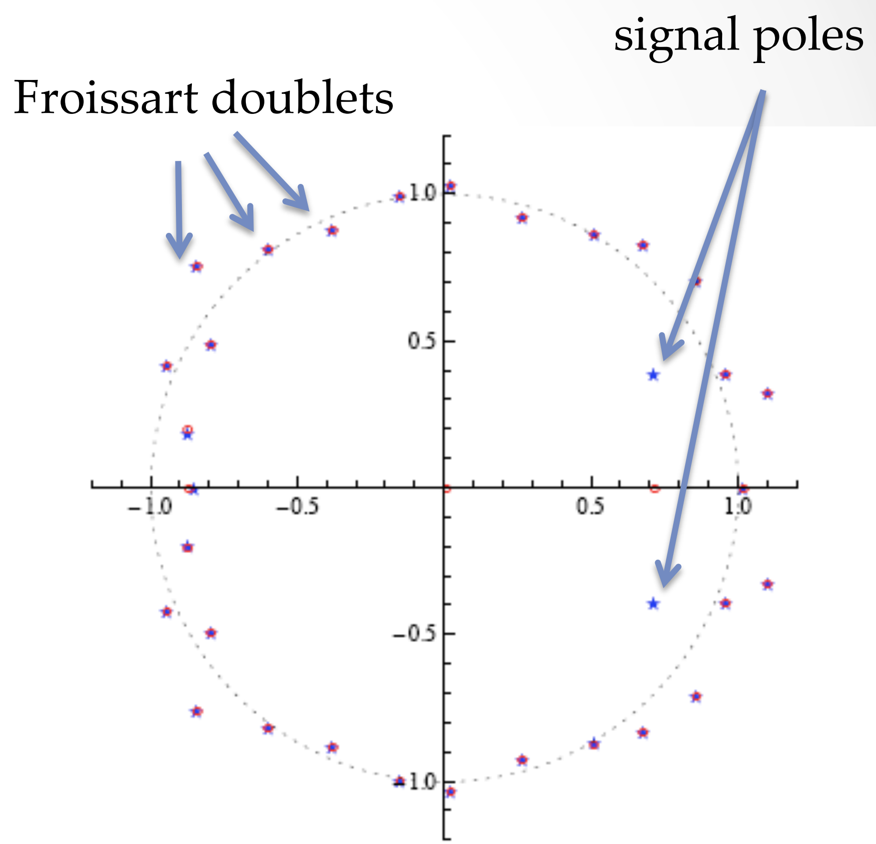
with $$V = [D_0(z), D_1(z), \ldots, D_n(z)]^T$$

and
$$J_n = \begin{bmatrix} A_0 & 1 & 0 & \vdots & 0 & 0 \\ R_1 & A_1 & 1 & \vdots & 0 & 0 \\ 0 & R_2 & A_2 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & \cdots & 0 & R_{n-1} & A_{n-1} & 1 \\ 0 & 0 & \cdots & 0 & R_n & A_n \end{bmatrix}$$

eigenvalues of J_n are poles for the Padé approximant

Example





damped oscillator + (low) noise

Observations

(1) P noiseless damped oscillators are exactly reconstructed from a time series N = 2P long, or longer.

(2) Natural boundary is "approximated" by pairs of poles and zeros (Froissart doublets) surrounding the vicinity of the unit circle.

(3) The position of "true" poles is affected by noise. (open question)

(4) The distribution of Froissart doublets is affected by the "true" poles.

(5) Separation distance in Froissart doublets is proportional to noise magnitude, decreases exponentially with the length of the time series, and is larger for pairs not close to the circle. (open question)

Statistical properties noise-only data

We study the statistical distribution in the complex plane of poles and zeros as a function of noise distribution and length N of the time-series.

Conjecture: (1) radial distribution is Lorentzian,

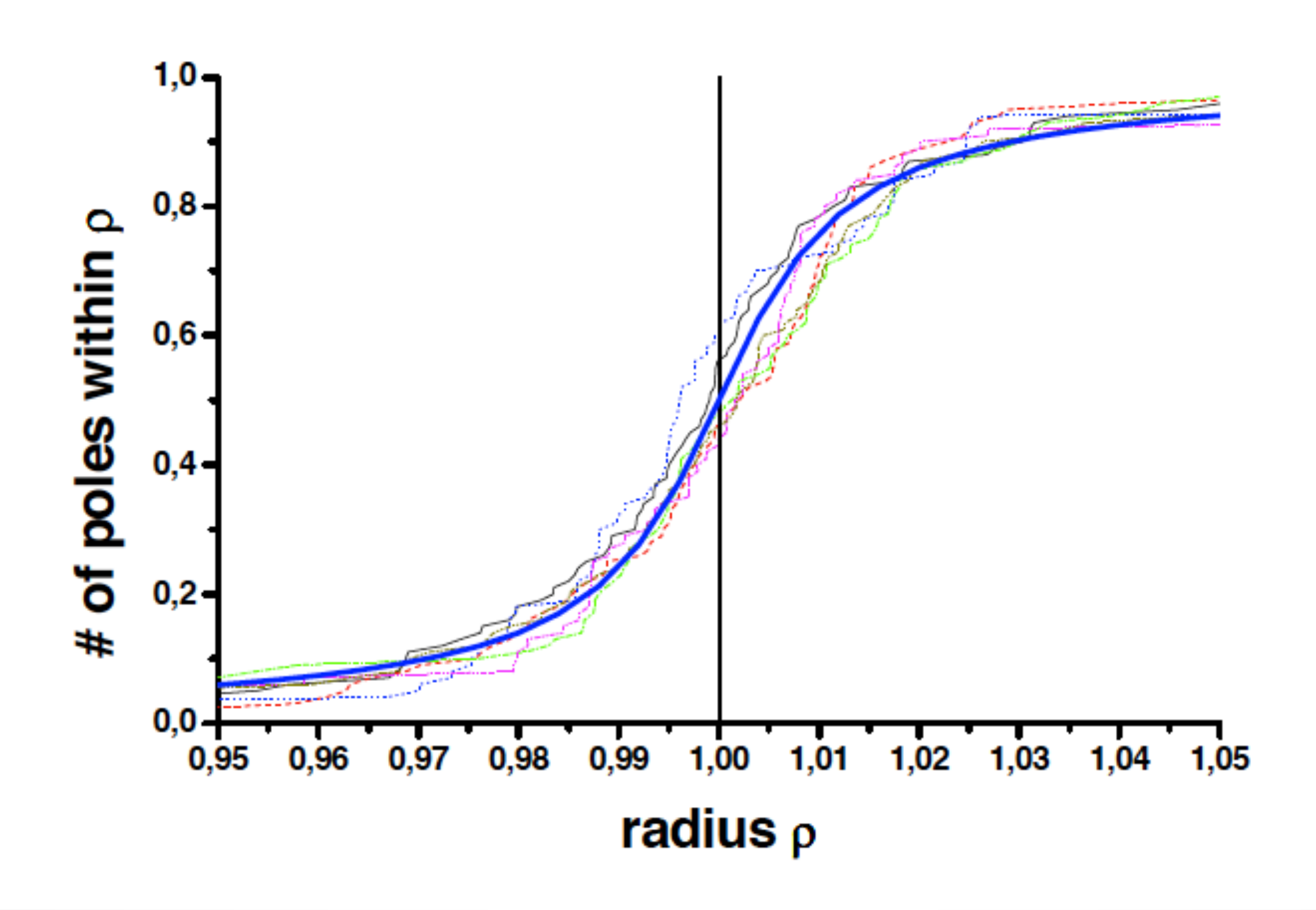
(2) phase distribution is uniform,

UNIVERSALLY, for any kind of noise !

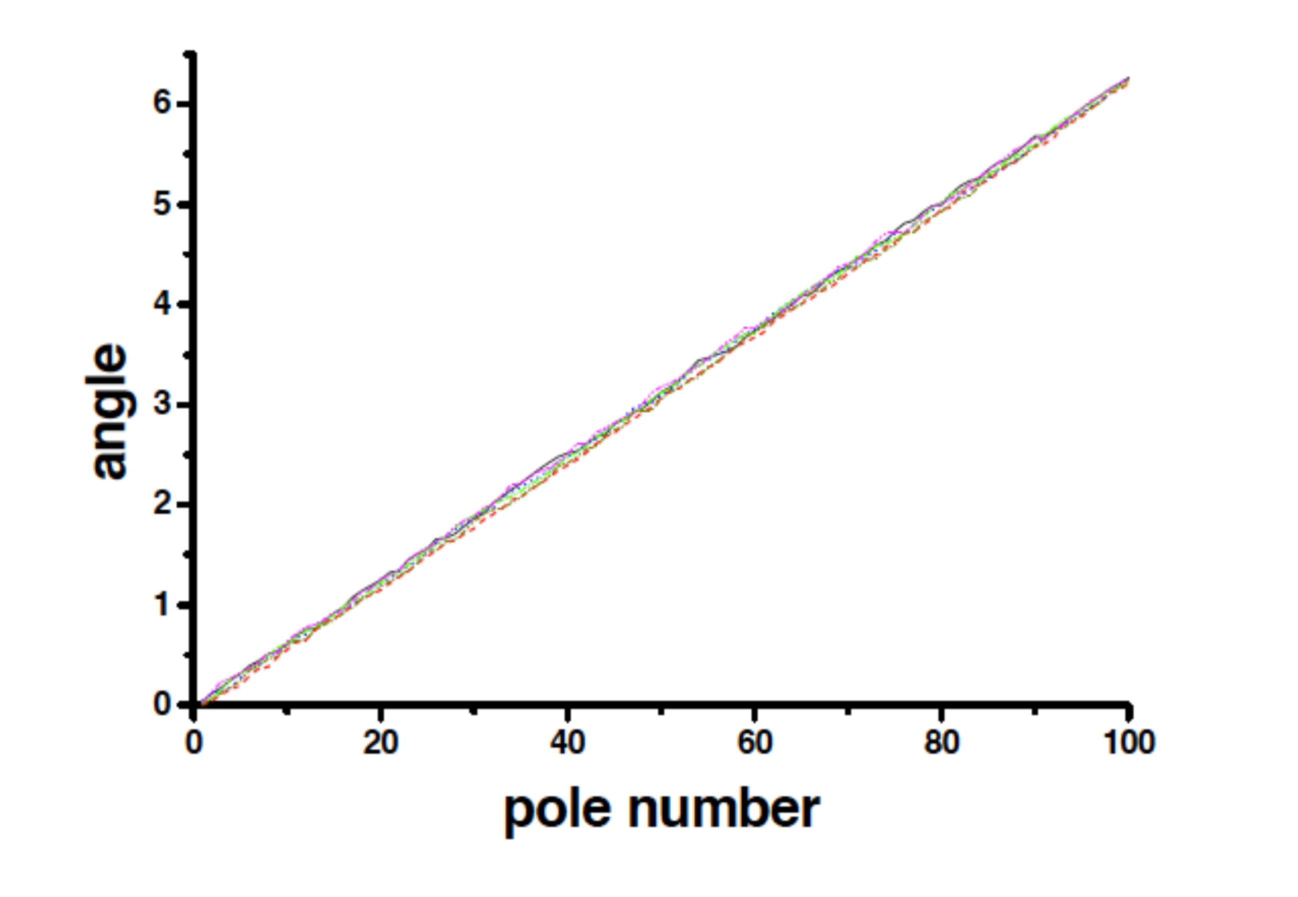
We tested it for various distributions:

(1) complex uniform in magnitude and phase
(2) complex uniform in a square
(3) complex uniform in a circle
(4) standard pink
(5) complex normal
(6) autoregressive moving average (ARMA) model
(7) real normal
(8) Cauchy

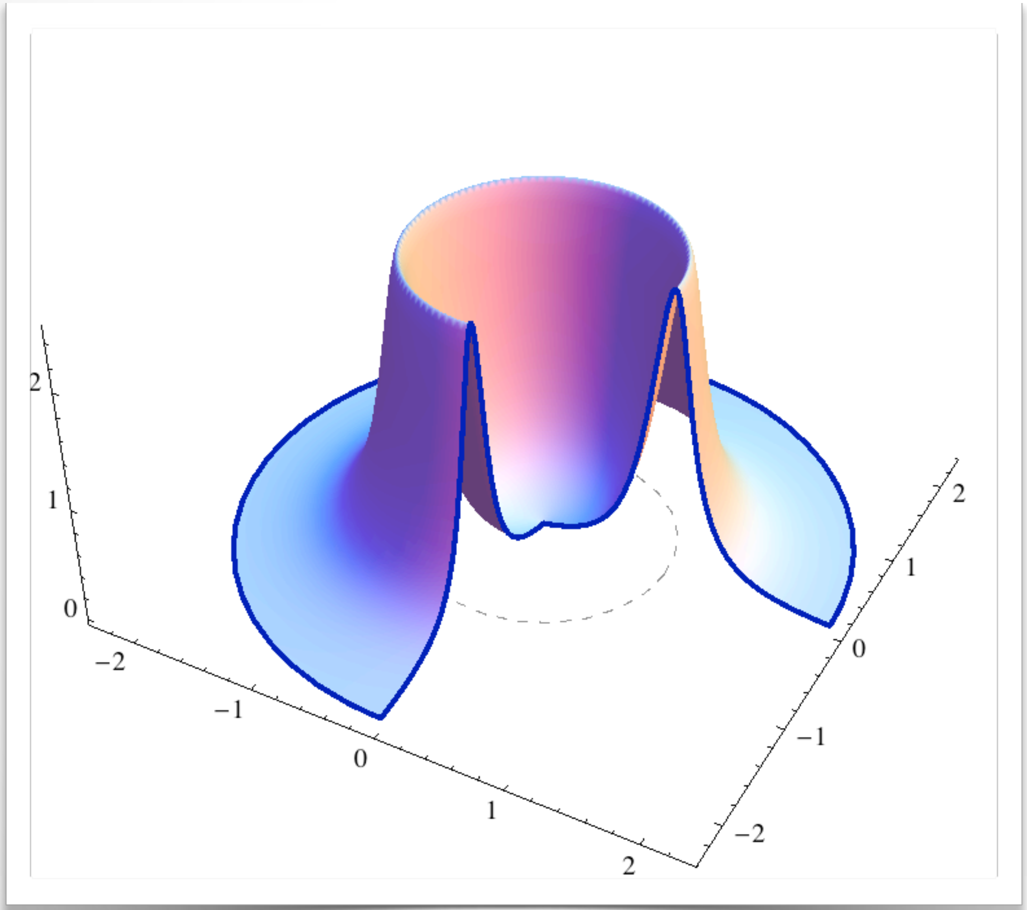
Radial distribution



Phase distribution

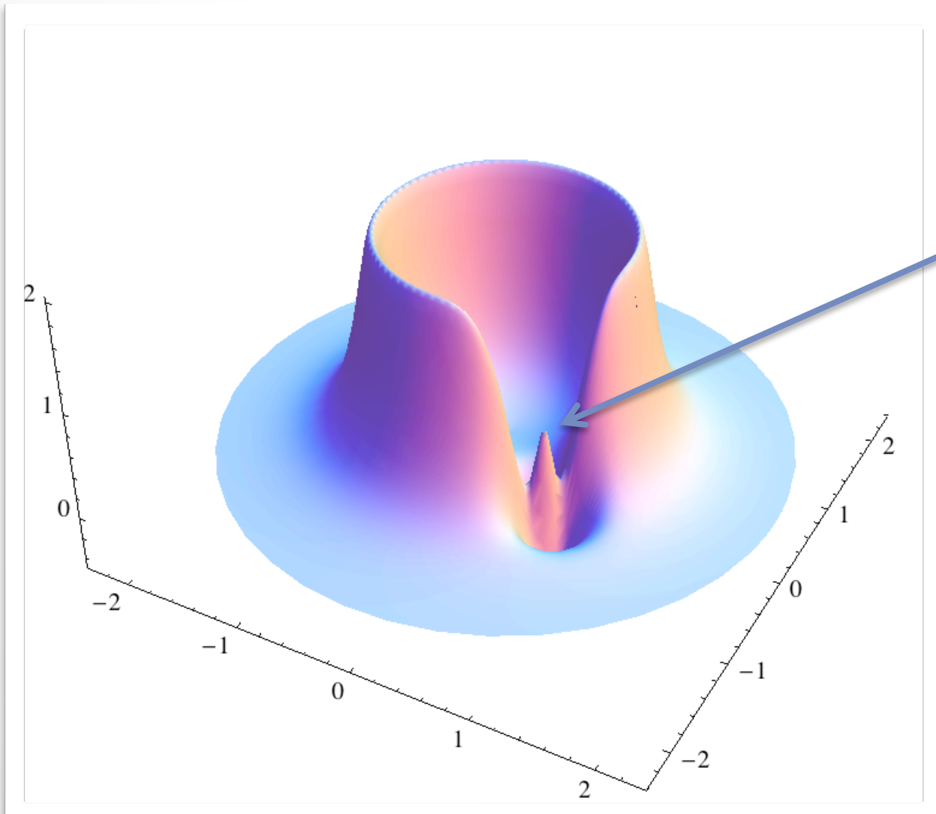


Density distribution



- Rotationally invariant
- Narrow at r=1
- Sharper for longer series width → 0 as N→ ∞
- Slow tail (power-like)
- The same distribution for zeros, slightly shifted outwards shift → 0 as N→∞

Non-linear effect



(weak) signal pole

A signal added to noise changes the distribution: "symmetry breaking"

Conclusions

(1) Alternative to Fourier Transform (FT) based noise filtering

(2) FT is linear, treats signal and noise on the same footing

(3) Discrete FT is Z-transform evaluated at roots of unity

(4) But roots of unity are "attractors" of noise

(5) We had success in detecting faint transients buried in heavy noise

Open problems

(1) What is the norm of noise related eigenfunctions of J-operator?

(2) Separation in Froissart doublets decreases exponentially with the length of the series?

(3) Prove universality for statistical distribution of poles and zeros

(4) Extend P. Barone's results for Gaussian noise.
[P. Barone, *Journal of Approximation Theory* **132**, 224 (2005)]