# Convergence features of the $\delta$ transformation for the Euler series

R. Borghi[1] and E. J. Weniger[2]

[1] Università "Roma Tre", Rome, Italy
[2] University of Regensburg, Regensburg, Germany

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- Factorially divergent series often arise in special function theory and from perturbative treatments of several problems in different branches of physics and engineering, like quantum mechanics, optics, elasticity theory, fluid mechanics, thermodynamics.

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- Divergent series can be invaluable numerical tools, provided we find suitable decoding techniques
- Among them, sequence transformations turn out to be particularly efficient
- Many numerical examples are known which show that sequence transformations can be extremely useful. However, the convergence theory is still in an underdeveloped stage

# The Euler series

- We consider the Euler series (ES for short), namely

$$\mathcal{E}(z) \sim \sum_{m=0}^{\infty} (-1)^m z^m \, \Gamma(m+1) \,, \qquad z \to 0$$

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- If $|\arg(z)| < \pi$, the ES is an asymptotic series for the Euler integral (EI for short)

$$\mathcal{E}(z) = \int_0^\infty \frac{\exp(-t)}{1+zt} \, \mathrm{d}t = \frac{\exp(1/z)}{z} E_1(1/z),$$

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  where $E_1(\cdot)$ denotes the exponential integral function.
- The task is to construct an approximation to the EI from the terms of the ES

# Partial sum decomposition

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$$s_n \;=\; s \,+\, r_n \,,$$

- where $s$ is the antilimit and $r_n$ denotes the $n$th-order remainder

- In this way, the divergence of the partial sum sequence $\{s_n\}$ can be ascribed to the (divergent) behavior of the remainder sequence $\{r_n\}$

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- Sequence transformations provide approximations to $s$ by eliminating an approximation of $r_n$ from $s_n$
- Such elimination is accomplished by using suitable series representations for $r_n$, based on *a priori* information about the asymptotic behavior of the single terms of the series

# Decoding ES via $\delta$ transformation

- For factorially divergent series, in particular, the following factorial series representation of the remainder is a good model:

$$\frac{s - s_n}{a_{n+1}} = \sum_{m=0}^{\infty} \frac{c_m}{(n+\beta)_m},$$

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- $(\cdot)_m$: Pochhammer symbol, $\beta > 0$. The coefficients $c_m$ are *independent of* $n$.
- Due to the linearity with respect to both $s$ and the coefficients $c_m$, the elimination of a finite number of them can be achieved, for instance, by Cramer's rule

- We consider the sequence transformation defined by[1]

$$\delta_k^{(n)}(\beta) = \frac{\Delta^k \left\{ (n+\beta)_{k-1} \, \dfrac{s_n}{a_{n+1}} \right\}}{\Delta^k \left\{ (n+\beta)_{k-1} \, \dfrac{1}{a_{n+1}} \right\}},$$

acting on the sequence $s_n = \sum_{k=0}^{n} a_k$

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- here, $\Delta$ denotes the forward difference operator with respect to $n$,

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- We want to prove that the $\delta$ transformation of the divergent ES converges

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- The $\delta$ transformation to the ES produces ratio of two polynomials
- The zeros of the denominator polynomial play a key role. They must lie on the negative real axis (cut). This is the first thing we have to prove

# Simulating the cut

- The following closed-form expression for the denominator, in terms of hypergeometric polynomials ${}_2F_2$, can be established:

$$\Delta^k \left\{ (n+\beta)_{k-1} \frac{1}{a_{n+1}} \right\} = \frac{(-1)^k}{(-z)^{n+1}} \frac{(n+\beta)_{k-1}}{\Gamma(n+2)} \, {}_2F_2 \left( \begin{matrix} -k, k+n+\beta-1 \\ n+\beta, n+2 \end{matrix} ; -\frac{1}{z} \right)$$

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- For $z < 0$ the terms of the polynomials ${}_2F_2$ have strictly alternating signs
- All zeros of this polynomials in $z$ are *real and negative*
- The $\delta$ transformation simulates the cut of EI

# Error analysis

- We now attempt to evaluate directly the truncation error, $\delta_k^{(n)}(\beta) - \mathcal{E}(z)$, which is given by the following expression:[2]

$$\delta_k^{(n)}(\beta) - s = \frac{\Delta^k \left\{ (n+\beta)_{k-1} \, \dfrac{r_n}{a_{n+1}} \right\}}{\Delta^k \left\{ (n+\beta)_{k-1} \, \dfrac{1}{a_{n+1}} \right\}} .$$

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- Our problem is to find manageable estimates of the numerator

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- Our analysis is based on the following factorial series representation of the ES remainder:[3]

$$\boxed{\frac{r_n}{a_{n+1}} = -\sum_{k=0}^{\infty} \frac{L_k^{(-1)}(1/z)}{z} \frac{k!}{(n+1)_{k+1}}}$$

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- With the help of such factorial series we could derived an integral representation for the truncation error

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- We recall that, given the factorial series

$$\Omega(z) = \frac{b_0}{z} + \frac{b_1 1!}{z(z+1)} + \frac{b_2 2!}{z(z+1)(z+2)} + \cdots = \sum_{k=0}^{\infty} \frac{b_k\, k!}{(z)_{k+1}}$$

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- $\Omega(z)$ also possesses the following integral representation:[4]

$$\Omega(z) \;=\; \int_0^1 t^{z-1}\, \varphi_\Omega(t)\, \mathrm{d}t\,, \qquad \Re(z) > 0$$

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- This leads to the following integral representation of the remainder:

$$\frac{r_n}{a_{n+1}} \;=\; -\frac{1}{z}\int_0^1 \mathrm{d}t\, t^n \exp\left(-\frac{1-t}{zt}\right)$$

# Integral representation of the numerator

- After some algebraic "gymnastics" we obtain the following integral representation:

$$\Delta^k \left\{ (n+\beta)_{k-1} \frac{r_n}{a_{n+1}} \right\} = -\frac{(-1)^k}{z} (n+\beta)_{k-1} \times \int_0^1 \mathrm{d}t \, t^n \exp\left( -\frac{1-t}{zt} \right) {}_2F_1 \left( \begin{matrix} -k, k+n+\beta-1 \\ n+\beta \end{matrix} ; t \right)$$

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- The hypergeometric polynomial ${}_2F_1$ can be expressed as a Jacobi polynomial $P_k^{(\alpha,\beta)}$

$${}_2F_1\left( {-k, k+n+\beta-1 \atop n+\beta} ; t \right) = (-1)^k \binom{k+n+\beta-1}{n+\beta-1}^{-1} P_k^{(-1,n+\beta-1)}(2t-1)$$

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- This yields the following integral representation:

$$\delta_k^{(0)}(1) - \mathcal{E}(z) = \frac{(-1)^k}{2}\,\exp(1/z)$$

$$\times \boxed{\frac{\displaystyle\int_{-1}^{1} \mathrm{d}x \left(\frac{x+1}{2}\right)^n \exp\left(-\frac{2/z}{x+1}\right) P_k^{(-1,0)}(x)}{{}_2F_2\left(\begin{matrix}-k, k\\ 1, 2\end{matrix}; -\frac{1}{z}\right)}}$$

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- We now have to find an approximation to the numerator which demonstrates the convergence of $\delta$ transformation
- We separately analyze the numerator and the denominator

# Analytical approximant of the denominator

- The hypergeometric function in the denominator can be approximated, for large $k$, as follows:[5]

$$
{}_2F_2\left(\begin{matrix}-k,k\\1,2\end{matrix};-\frac{1}{z}\right)\simeq\frac{1}{2\pi\sqrt{3}}\left(\frac{k^2}{z}\right)^{-\frac{2}{3}}\exp\left(3z^{-\frac{1}{3}}\,k^{\frac{2}{3}}-\frac{1}{3z}\right)
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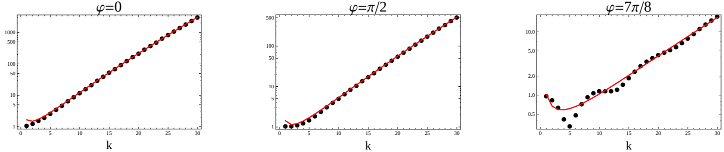
- Some examples of the above approximation are given for $z=10\exp(\mathrm{i}\varphi)$, with several values of $\varphi$

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# Analytical approximant of the numerator (1/ 3)

- For the numerator, consider first the following approximation of Jacobi polynomials:

$$P_k^{(-1,0)}(x) \simeq \frac{1}{\sqrt{\pi k}} \mathrm{Re}\left[\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}} \exp\left(\mathrm{i} k \arccos x + \frac{\mathrm{i}\pi}{4}\right)\right]$$

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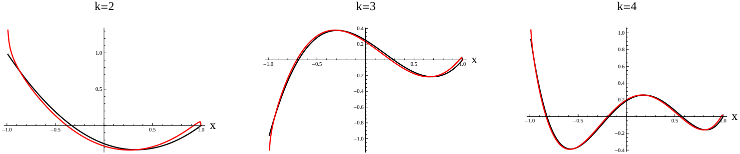
- 

# Analytical approximant of the numerator (2/3)

- The integral simplifies as

$$\int_{-1}^{1} \mathrm{d}x \left(\frac{x+1}{2}\right)^n \exp\left(-\frac{2/z}{x+1}\right) P_k^{(-1,0)}(x) \simeq \frac{2}{\sqrt{\pi k}} \mathrm{Re}\left[\exp\left(\frac{\mathrm{i}\pi}{4}\right) \mathcal{I}_k\left(\frac{1}{z}\right)\right]$$

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- where

$$\mathcal{I}_k(\alpha) = \frac{1}{2} \int_{-1}^{1} \mathrm{d}x \left(\frac{x+1}{2}\right)^n \exp\left(-\frac{2\alpha}{x+1}\right) \left(\frac{1-x}{1+x}\right)^{\frac{1}{4}} \exp(\mathrm{i}k \arccos x)$$

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- The evaluation of the last integral can be carried out, for large $k$, by standard asymptotic techniques

## Asymptotic evaluation of $\mathcal{I}_k(\alpha)$ (1/3)

- To this end, we first recast the integral as follows:

$$\mathcal{I}_k(\alpha) = 2\,\exp(-\alpha) \int_0^\infty \mathrm{d}\tau\, g(\tau)\, \exp[f(\tau)],$$

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- The saddles of the function $f(\tau)$ are solutions of the equation $f'(\tau) = 0$

$$4\alpha\tau^4 + (4\alpha - 3)\tau^2 - 4\mathrm{i}k\tau + 3 = 0$$

# Asymptotic evaluation of $\mathcal{I}_k(\alpha)$ (2/3)

- Last equation is solved, for large $k$, by the ansatz $\tau \simeq Ak^{\gamma}$. The unknowns can be found by substituting this ansatz into the saddle equation and by solving it in the limit $k \gg 1$

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- After some algebra, the following estimates of the four saddles $\tau_0, \ldots, \tau_3$ are found:

$$\tau_m \simeq (k/\alpha)^{1/3} \exp\left(\frac{\mathrm{i}\pi}{6} + \frac{\mathrm{i}2\pi m}{3}\right), \qquad m = 0, ..., 2,$$

$$\tau_3 \simeq \frac{3\mathrm{i}}{4k}$$

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- Numerical trials shown that it is sufficient to consider only $\tau_0$ to obtain a meaningful estimate of the above integral

- Accordingly, $\mathcal{I}_k(1/z)$ can be approximated as follows:

$$\frac{1}{\sqrt{\pi k}}\,\mathrm{Re}\left\{\exp\left(\frac{\mathrm{i}\pi}{4}\right)\mathcal{I}_k\left(\frac{1}{z}\right)\right\} \simeq \frac{2(-1)^k}{\sqrt{3}}\,\exp\left(-\frac{1}{3z}\right)\,z^{-\frac{1}{3}}\,k^{-\frac{4}{3}}$$

$$\times\exp\left(-\frac{3}{2}\,z^{-1/3}\,k^{2/3}\right)\,\cos\left(\frac{3\sqrt{3}}{2}\,z^{-1/3}\,k^{2/3}+\frac{\pi}{6}\right)$$

# Analytical approximant of the error

- We finally obtain the following analytical estimate of the truncation error:

$$\delta_k^{(0)}(1) - \mathcal{E}(z) \simeq \frac{4\pi}{z}\,\exp\left(\frac{1}{z}\right)\,\exp\left(-\frac{9}{2}z^{-\frac{1}{3}}k^{\frac{2}{3}}\right)\,\cos\left(\frac{3\sqrt{3}}{2}z^{-\frac{1}{3}}k^{\frac{2}{3}}+\frac{\pi}{6}\right)$$

# Numerical results

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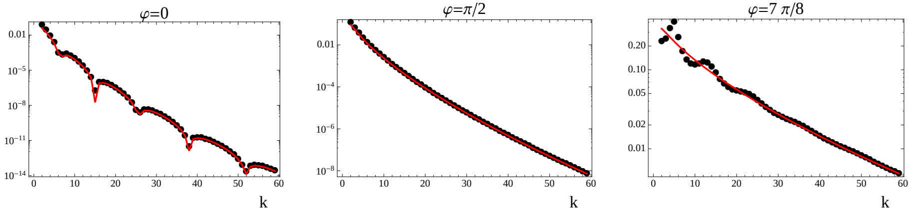
- We plot the modulus of the truncation error $|\delta_k^{(0)}(1) - \mathcal{E}(z)|$ as a function of $k$.



# Comparison to Padé approximants

- For the special case of ES, the Padé approximants can be computed by Drummond sequence transformation, yielding the following truncation error:

$$[n+k/k] - \mathcal{E}(z) = \frac{\Delta^k \left\{ \dfrac{r_n}{a_{n+1}} \right\}}{\Delta^k \left\{ \dfrac{1}{a_{n+1}} \right\}},$$

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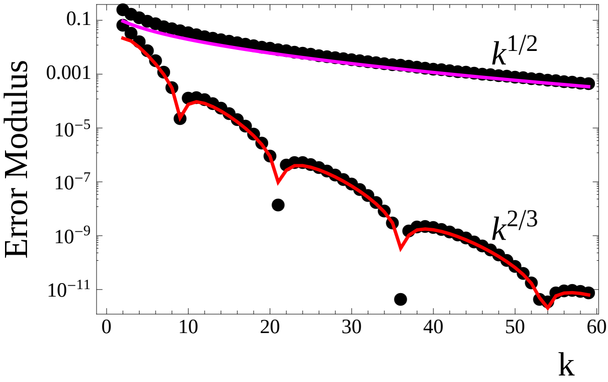
- Here the same approach is possible and, for $n=0$, we obtain the following analytical estimate to the truncation error:

$$[k/k]-\mathcal{E}(z)\simeq\frac{2\pi}{z}\,\exp\left(\frac{1}{z}\right)\,\exp\left(-4z^{-\frac{1}{2}}k^{\frac{1}{2}}\right)$$

- Truncation errors and analytical estimates for $\delta$ and Padé for $z = 10$. Dots: experimental. Curves: analytical estimates



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- The observed superiority of the $\delta$ transformation over Padé has been confirmed by our estimates
- Our approach did not lead to manageable results in the case of the Levin transformation. **OPEN PROBLEM**