Quadrature on the positive real line with quasi and pseudo orthogonal rational functions

Adhemar Bultheel
(joint work with P. González-Vera, E. Hendriksen, O. Njåstad)

Department of Computer Science
K.U.Leuven

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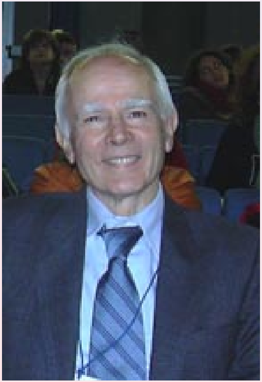
Happy birthday Claude and Sebastiano



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Survey

- Quadrature on the positive real line and OP
- ORF (incl. OP /OLP)
- Quasi and Pseudo versions
- Quadrature with QORF and PORF
- Differences and similarities
- Numerical examples

Gauss-type quadrature

- Consider integrals $I_\mu(f) = \int_0^\infty f(x)d\mu(x)$, $\mu > 0$.
- Gauss-type QF: inner product $\langle f, g\rangle_\mu = \int_0^\infty f(x)g(x)d\mu(x)$
- Construct OP $\{\varphi_n\}$: $\varphi_n \perp \mathcal{L}_{n-1} = \Pi_{n-1}$
- Zeros φ_n: $\{x_{k,n}\}_{k=1}^n \subset \mathbb{R}_+ \Rightarrow$ nodes
- Interpolating polynomial for f in these nodes $\Rightarrow$ weights $\{\lambda_{k,n} > 0\}_{k=1}^n$
- $$I_\mu(f) \approx I_{\mu_n}(f) = \sum_{k=1}^n \lambda_{k,n} f(x_{k,n}).$$
- Equality for $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-1} = \mathcal{L}_{2n-1} = \Pi_{2n-1}$

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Orthogonal Rational Functions

- Introduce poles: define $\mathcal{L}_n = \{f_n/d_n : f_n \in \Pi_n\}$
- $d_n = r_1 r_2 \cdots r_n, \quad r_k(x) = \begin{cases} \zeta_k - x, & \text{if } -\infty < \zeta_k \le 0 \\ 1, & \text{if } \zeta_k = -\infty \end{cases}$
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- Define

$$Q_n(x,\tau) = \varphi_n(x) - \tau \frac{r_{n-1}(x)}{r_n(x)} \varphi_{n-1}(x), \quad \tau \in \mathbb{R} \cup \{\infty\}.$$

$\left(Q_n(x,\infty) = \frac{r_{n-1}(x)}{r_n(x)} \varphi_{n-1}(x)\right)$

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Zeros of QORF

Theorem

The zeros of $Q_n(x,\tau)$ and $Q_{n-1}(x,\tau)$ interlace.

- $Q_n(x,\tau) = \varphi_n(x) - \tau \frac{r_{n-1}(x)}{r_n(x)} \varphi_{n-1}(x) = \frac{q_n(x,\tau)}{d_n(x)}$
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  (if $\zeta_{n-1} \neq \infty$)
- $\mathcal{Z}\{Q_n(x,\tau)\} = \{x_{1,n}(\tau), \ldots, x_{n,n}(\tau)\}$
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positive weights

Theorem

If the zero $x_{1,n}(\tau) \geq 0$ or $x_{1,n}(\tau) \neq \zeta_k$, $\forall k$
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then the weights of the QF $\lambda_{k,n}(\tau) > 0$, if $x_{k,n}(\tau) > 0$ PORF.

Skip arguments

- ORF of 2nd kind: $\sigma_n(x) = \int_0^\infty \frac{\varphi_n(t)-\varphi_n(x)}{t-x} dt$.
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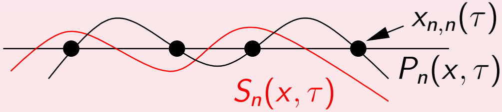
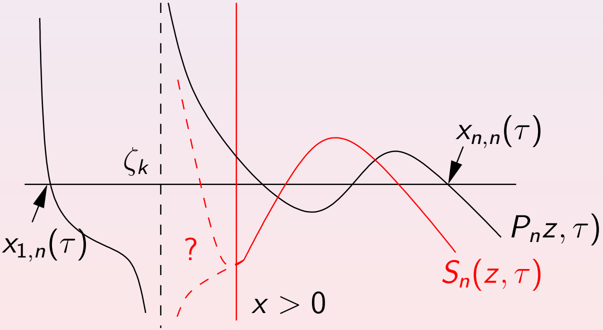
- $\mathcal{Z}(P_n(x,\tau))$ and $\mathcal{Z}(S_n(x,\tau))$ interlace
- $\mathrm{sgn}(P_n'(x,\tau)) = \mathrm{sgn}(S_n(x,\tau))$ for $x > x_{n,n}(\tau)$
  $\Rightarrow \mathrm{sgn}(P_n'(x_{k,n}(\tau),\tau)) = \mathrm{sgn}(S_n(x_{k,n}(\tau),\tau))$, $\forall k$



positive weights

- This may not work if $\exists k$: $x_{1,n}(\tau) < \zeta_k < 0$



Computation

- Recurrence

$$\varphi_n(x) \sim \frac{x^{\gamma_{n-1}} - c_n r_{n-1}(x)}{r_n(x)} \varphi_{n-1}(x) - d_{k-1} \frac{r_{n-2}(x)}{r_n(x)} \varphi_{n-2}(x)$$

$\gamma_n = 1$ if $\zeta_n = \infty$, $\gamma_n = 0$ otherwise

$$J_n = \mathrm{tridiag}\begin{pmatrix} d_1, d_2, \ldots, d_{n-1} \\ c_1, c_2, c_3, \ldots, c_{n-1}, c_n \\ d_1, d_2, \ldots, d_{n-1} \end{pmatrix}$$

$$I_n^\gamma = \mathrm{diag}(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}), \quad I_n^{1-\gamma} = I_n - I_n^\gamma$$

$$Z_n = \mathrm{diag}(\rho_0, \rho_1, \ldots, \rho_{n-1})$$

$\rho_n = \zeta_n$ if $\zeta_n \neq \infty$, $\rho_n = 1$ otherwise

$$A_n = J_n Z_n - I_n^{1-\gamma}, \quad B_n = J_n I_n^{1-\gamma} + I_n^\gamma.$$

Computation

Theorem

Consider the pencil (A_n, B_n) as above.
Then the nodes $x_{k,n}$ of the rational Gauss QF are its eigenvalues.
If $\mathbf{e}_k = [e_{1,n}(k), \ldots, e_{n,n}(k)]^T$ is the eigenvector corresponding to $x_{k,n}(\tau = 0)$ with $\mathbf{e}_k^T \mathbf{e}_k = 1$, then the corresponding weight in the rational Gauss QF is $\lambda_{k,n}(\tau = 0) = e_{1,n}(k)^2$.

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- Note that in the polynomial case all $\gamma_k = 1$, $I_n^\gamma = I_n$, $Z_n = I_n$, and thus $(A_n, B_n) = (J_n Z_n - I_n^{1-\gamma}, J_n I_n^{1-\gamma} + I_n^\gamma) = (J_n, I_n)$

Computation QORF

- Use $Q_n(x,\tau) = \varphi_n(x) - \tau\frac{r_{n-1}(x)}{r_n(x)}\varphi_{n-1}(x)$ to get
  $Q_n(x,\tau) \sim \frac{x^{\gamma_{n-1}} - c_n(\tau) r_{n-1}(x)}{r_n(x)}\varphi_{n-1}(x) - d_{n-1}\frac{r_{n-2}(x)}{r_n(x)}\varphi_{n-2}(x)$
  $c_n(\tau)$ is a simple modification of c_n involving τ.
  The same procedure as in the usual rational Gauss QF applies for nodes and weights.

Computation PORF

- Use $P_n(x,\tau) = \varphi_n(x) - \tau\varphi_{n-1}(x)$ to get
  $Q_n(x,\tau) \sim \frac{x^{\gamma_{n-1}} - c_n'(\tau) r_{n-1}(x)}{r_n(x)} \varphi_{n-1}(x) - d_{n-1}' \frac{r_{n-2}(x)}{r_n(x)} \varphi_{n-2}(x)$
  Now c_n' and d_{n-1}' are more complex modifications of c_n and d_{n-1} involving τ but also ζ_n and ζ_{n-1}.
  Moreover the symmetry of the pencil is lost.
  The computation of the weights from the eigenvectors becomes a nonlinear procedure:
  (we drop n and τ from the notation):
  $1/\lambda_k(\omega) = \sum_{i=0}^{n-2} \varphi_i^2(x_k) + \omega^{-1}\varphi_{n-1}^2(x_k)$, where
  $\sum_{k=1}^{n} \lambda_k(\omega) = 1$ (normalized measure: $\int d\mu = 1$).

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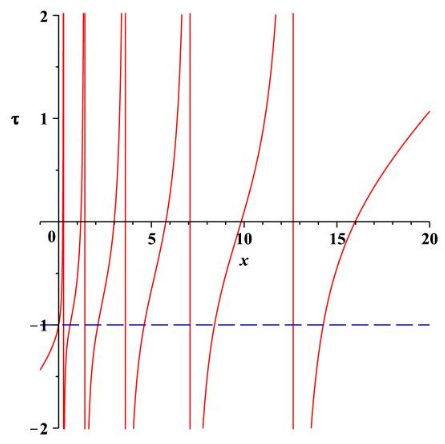
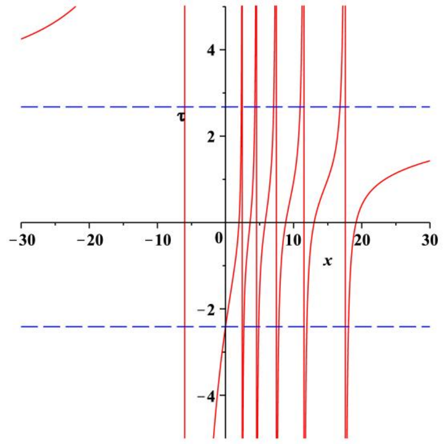
- It's a mess

Example 1: Laguerre polynomials, PORF



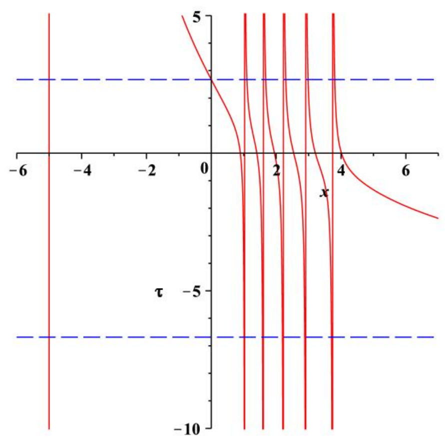
$d\mu(x) = \exp(-x)dx,$
$n = 6,$
all $\zeta_k = \infty$

Example 2, PORF



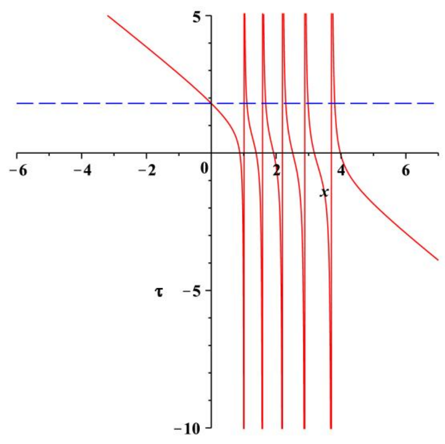
$$d\mu(x) = x^{10}\exp(-x)dx,$$
$$n = 6,$$
$$\zeta_k = -k,\ k = 1, 2, 3, 4, 5, 6$$

Example 3, PORF



$d\mu(x) = x^{10}\exp(-x^2)dx,$
$n = 6,$
$\zeta_k = \infty, -1, \infty, -3, \infty, -5$

Example 4, PORF



$$d\mu(x) = x^{10}\exp(-x^2)dx,$$
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$$\zeta_k = -1, \infty, -3, \infty, -5, \infty$$

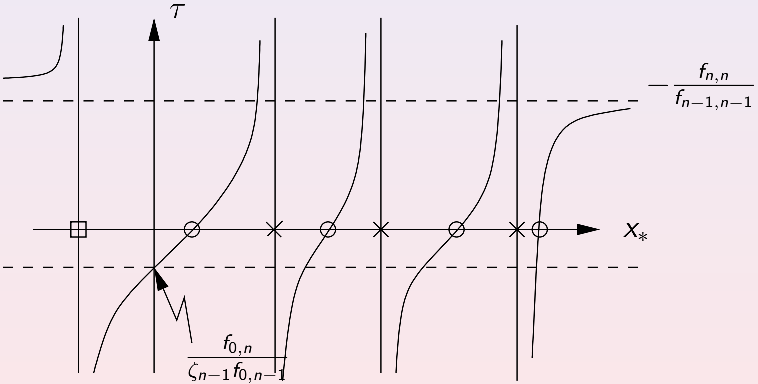
The end

Thank you

Reference

A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad.
Quadratures associated with pseudo-orthogonal rational functions on the real half line with poles in $[-\infty, 0]$.
Technical Report TW587, Department of Computer Science, K.U. Leuven, March 2011.

QORF $\tau(x_*) = \dfrac{f_n(x_*)}{r_{n-1}(x_*)f_{n-1}(x_*)}$ $(n = 4,\ \zeta_{n-1} \neq \infty)$



square = ζ_{n-1}; circles = $\mathcal{Z}(\varphi_n)$; crosses = $\mathcal{Z}(\varphi_{n-1})$
note interlacing

Back

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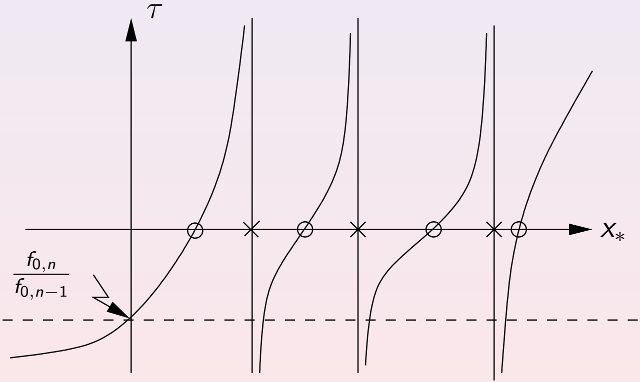

circles $= \mathcal{Z}(\varphi_n)$; crosses $= \mathcal{Z}(\varphi_{n-1})$
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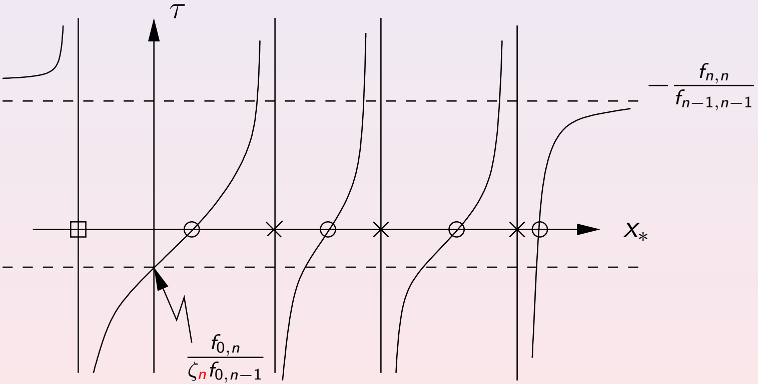
PORF $\tau(x_*) = \frac{f_n(x_*)}{r_n(x_*)f_{n-1}(x_*)} \quad (n = 4,\ \zeta_n \neq \infty)$



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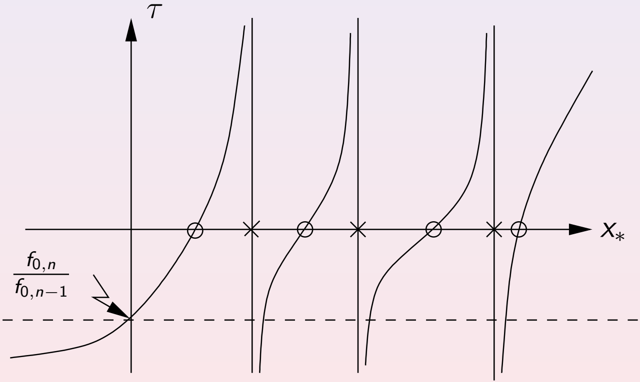

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