Convergence and Stability of a New Quadrature Rule for Evaluating Hilbert Transform

Maria Rosaria Capobianco

CNR - National Research Council of Italy
Institute for Computational Applications "Mauro Picone", Naples, Italy.

G. Criscuolo

Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli "Federico II", Naples, Italy.

w nonnegative weight function on the interval $[a, b]$,

$$-\infty \leq a < b \leq \infty \;, \qquad 0 < \int_a^b w(x)dx < \infty.$$

Weighted Hilbert Transform

$$H(wf; x) := \int_a^b \frac{f(t)}{t-x} w(t)dt = \lim_{\varepsilon \to 0^+} \int_{|t-x| \geq \varepsilon} \frac{f(t)}{t-x} w(t)dt, \qquad t \in (a, b).$$

▶ Causality and generalization of the phaser idea beyond pure alternating current

1 R. Bracewell, *The Fourier Transform and its Applications*, Electrical and Electronic Engineering Series, McGraw–Hill, New York, 1965.

▶ Boundary Value Problems as Singular Integral Eequations Involving Cauchy Principal Value Integrals

2 S.G. Mikhlin, S. Prössdorf, *Singular Integral Operators*, Springer–Verlag, Berlin, 1986.

3 N.I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, 1977.

▶ Numerical Evaluation of $H(wf)$ when $a - \infty < a < b < \infty$

4 G. Criscuolo, *A new algorithm for Cauchy principal value and Hadamard finite–part integrals*, J. Comput. Appl. Math. **78** (1997), 255–275.

5 G. Criscuolo, L. Scuderi *The numerical evaluation of Cauchy principal value integrals with non–standard weight functions*, BIT **38** (1998), 256–274.

▶ Numerical Evaluation of the Weighted Hilbert Transform on the Real Line

6 M.R. Capobianco, G. Criscuolo, R. Giova, *Approximation of the Hilbert transform on the real line by an interpolatory process*, BIT **41** (2001), 666–682.

7 M.R. Capobianco, G. Criscuolo, R. Giova, *A stable and convergent algorithm to evaluate Hilbert transform*, Numerical Algorithms **28** (2001), 11–26.

8 S.B. Damelin, K. Diethelm, *Interpolatory product quadrature for Cauchy principal value integrals with Freud weights*, Numer. Math. **83** (1999), 87–105.

9 S.B. Damelin, K. Diethelm, *Boundedness and uniform numerical approximation of the weighted Hilbert transform on the real line*, J. Funct. Anal. Optim. **22** n.1–2 (2001), 13–54.

10 K. Diethelm, *A method for the practical evaluation of the Hilbert transform on the real line*, J. Comp. Appl. Math. **112** (1999), 45–53.

Essentially two kinds of quadrature rules of interpolatory type have been proposed

Gaussian Rules

Product Rules

Hypothesis on the function f

$$W_0^\infty := \left\{ f \in C^0_{LOC}(\mathbb{R}) \ : \ \lim_{|t|\to\infty} f(t)e^{-t^2/2} = 0 \right\},$$

where $C^0_{LOC}(\mathbb{R})$ is the set of all locally continuous functions on $\mathbb{R}$ and f satisfies a Dini type condition by the Ditzian–Totik modulus of continuity, then $H(wf)$ is bounded on $\mathbb{R}$

(see Theorem 1.2 in S.B. Damelin, K. Diethelm, *Boundedness and uniform numerical approximation of the weighted Hilbert transform on the real line*, J. Funct. Anal. Optim. **22** n.1–2 (2001), 13–54.)

$$\{p_m(w)\}_{m=0}^{\infty}$$

sequence of the orthonormal Hermite polynomials associated with the weight $w(t) = e^{-t^2}$,

$$p_m(w;t) = \gamma_m t^m + \cdots \quad , \qquad \gamma_m > 0$$
$$-\infty < t_{m,m} < t_{m,m-1} < \cdots < t_{m,2} < t_{m,1} < +\infty$$
$$t_{m,1} = -t_{m,m} < \sqrt{2m+1}$$

For any $x \in \mathbb{R},\ m \in \mathbb{N}$ we denote by $t_{m,c}$ the zero of $p_m(w)$ closest to x, defined by

$$|t_{m,c} - x| = \min_{1 \le k \le m} |t_{m,k} - x|.$$

When x is equidistant between two zeros, i.e. $x = (t_{m,k} + t_{m,k+1})/2$ for some $k \in \{1, 2, \cdots, m-1\}$, it makes no difference for the subsequent analysis to define $t_{m,c} = t_{m,k}$ or $t_{m,c} = t_{m,k+1}$.

$$H(wf;x) = \int_{-\infty}^{+\infty} \frac{f(t) - f(x)}{t - x}\, e^{-t^2} dt + f(x) \int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t - x} dx$$

and approximate the first integral by interpolating the function

$$\frac{f - f(x)}{e_1 - x}, \qquad e_1(t) = t,$$

on the set of nodes $\{t_{m,k},\ k = 1, 2, \cdots, m,\ k \neq c\}$, all of which are far from the singularity x.

The Lagrange polynomial $\mathcal{L}_{m-1}(g)$ which interpolates the function g at these points may written as

$$\mathcal{L}_{m-1}(g;t) = \sum_{k=1,k\neq c}^{m} \ell_{m,k}(w;t)\frac{t_{m,k}-t_{m,c}}{t-t_{m,c}} g(t_{m,k}),$$

where

$$\ell_{m,k}(w;t) = \frac{p_m(w;t)}{p'_m(w;t_{m,k})(t-t_{m,k})}, \qquad k = 1, 2, \cdots, m.$$

$$\int_{-\infty}^{+\infty} \frac{\ell_{m,k}(w;t)}{t-t_{m,c}}\, e^{-t^2} dt,$$

can be computed by the Gaussian rule with respect to the Hermite weight

$$\int_{-\infty}^{+\infty} \frac{\ell_{m,k}(w;t)}{t-t_{m,c}}\, e^{-t^2} dt = \frac{1}{p'_m(w;t_{m,k})} \left\{ \lambda_{m,k} \frac{p'_m(w;t_{m,k})}{t_{m,k}-t_{m,c}} + \lambda_{m,c} \frac{p'_m(w;t_{m,c})}{t_{m,c}-t_{m,k}} \right\},$$

where $\lambda_{m,k} = \lambda_{m,k}(w)$, $k = 1, 2, \cdots, m$, are the Cotes numbers with respect to the Hermite weight. Since

$$p'_m(w;t_{m,j}) = \frac{\gamma_m}{\gamma_{m-1}} \frac{1}{\lambda_{m,j} p_{m-1}(w;t_{m,j})}, \qquad j = 1, 2, \cdots, m,$$

we get

$$\int_{-\infty}^{+\infty} \frac{\ell_{m,k}(w;t)}{t-t_{m,c}}\, e^{-t^2} dt = \frac{\lambda_{m,k}}{t_{m,k}-t_{m,c}} \left\{ 1 - \frac{p_{m-1}(w;t_{m,k})}{p_{m-1}(w;t_{m,c})} \right\}, \qquad k = 1, 2, \cdots, m,$$

and

$$\int_{-\infty}^{+\infty} g(t)e^{-t^2}dt = \sum_{k=1,k\neq c}^{m} \frac{\lambda_{m,k}}{t_{m,k}-t_{m,c}} \left\{ 1 - \frac{p_{m-1}(w;t_{m,k})}{p_{m-1}(w;t_{m,c})} \right\} g(t_{m,k}) + R_m(w;g).$$

Finally, we arrive at the formula

$$H(wf;x) = H_m(w;f;x) + R_m^H(w;f;x),$$

where

$$H_m(w;f;x) = \sum_{k=1,k\neq c}^{m} \frac{\lambda_{m,k}}{t_{m,k}-t_{m,c}} \left\{1 - \frac{p_{m-1}(w;t_{m,k})}{p_{m-1}(w;t_{m,c})}\right\} \frac{f(t_{m,k})-f(x)}{t_{m,k}-x} +$$

$$f(x)\int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t-x}dt,$$

and

$$R_m^H(w;f;x) = R_m\left(w; \frac{f-f(x)}{e_1-x}\right),$$

is the error.

The quadrature rule has degree of exactness at least $m-1$, i.e. $R_m^H(w;f) \equiv 0$ whenever f is a polynomial of degree $m-1$.

If, as required sometimes in applications, we want to use only the values of the function f at the interpolation points, then it is convenient to apply () rewritten as

$$H_m(w;f;x) = A_m(x)f(x) + \sum_{k=1,k\neq c}^{m} A_{m,k}(x)f(t_{m,k}),$$

where

$$A_m(x) = H(w;x) - \sum_{k=1,k\neq c}^{m} \frac{\lambda_{m,k}}{t_{m,k}-x}\left\{1 - \frac{p_{m-1}(w;t_{m,k})}{p_{m-1}(w;t_{m,c})}\right\},$$

and

$$A_{m,k}(x) = \frac{\lambda_{m,k}}{t_{m,k}-x}\left\{1 - \frac{p_{m-1}(w;t_{m,k})}{p_{m-1}(w;t_{m,c})}\right\}, \qquad k = 1,2,\cdots,m, \quad k \neq c.$$

We define the **Amplification Factor**

$$K_m(w;x) := |A_m(x)|\, w^{-1/2}(x) + \sum_{k=1, k\neq c}^{m} |A_{m,k}(x)|\, w^{-1/2}(t_{m,k}), \qquad x \in \mathbb{R},$$

THEOREM

Let $w(t) = e^{-t^2}$. Then

$$K_m(w;x) \leq \; C \begin{cases} \log m, & \text{if } |x| \leq \varrho\sqrt{2m}, \quad 0 < \varrho < 1, \\ m^{1/6} \log m, & \text{if } |x| \leq 2t_{m,1} - t_{m,2}, \end{cases}$$

with some constant C independent of m and x.

For any function

$$f \in C^0_{\sqrt{w}} := \left\{ f \text{ continuous on } \mathbb{R} \text{ and } \lim_{|t| \to \infty} f(t)\sqrt{w(t)} = 0 \right\},$$

we define the norm

$$\|f\|_{C^0_{\sqrt{w}}} := \|f\sqrt{w}\|_\infty = \max_{t \in \mathbb{R}} |f(t)\sqrt{w(t)}|.$$

We set

$$E_m(f)_{\sqrt{w},\infty} := \inf_{P \in \mathbb{P}_n} \|(f - p)\sqrt{w}\|_\infty,$$

for any $f \in C^0_{\sqrt{w}}$, and where $\mathbb{P}_n$ denotes the set of the polynomials of degree at most m.
Denoting by $\omega(f,t)_{\sqrt{w},\infty}$ the Ditzian–Lubinsky weighted modulus of smoothness, we can state the following result.

THEOREM

Assume that $f \in C^0_{\sqrt{w}}$ satisfies the condition

$$\int_0^1 t^{-1}\omega(f,t)_{\sqrt{w},\infty} dt < \infty.$$

$$\left|R_m^H(w;f;x)\right| \le C \begin{cases} \log m \; E_{m-1}(f)_{\sqrt{w},\infty} + \int_0^{1/\sqrt{m}} \frac{\omega(f,t)_{\sqrt{w},\infty}}{t} dt, \\ \qquad \text{if } |x| \le \varrho\sqrt{2m}, \quad 0 < \varrho < 1, \\ \\ m^{1/6} \log m \; E_{m-1}(f)_{\sqrt{w},\infty} + \int_0^{1/\sqrt{m}} \frac{\omega(f,t)_{\sqrt{w},\infty}}{t} dt, \\ \qquad \text{if } |x| \le 2t_{m,1} - t_{m,2}, \end{cases}$$

for some constant C independent of f, m and x.

$$E_m(f)_{\sqrt{w},\infty} \le \text{const}\ \omega(f, m^{-1/2})_{\sqrt{w},\infty}, \qquad m \ge 0, \quad w(t) = e^{-t^2},$$

Corollary

Assume that $w(t) = e^{-t^2}$ and $f \in C^0_{\sqrt{w}}$ satisfies the condition $\omega(f,t)_{\sqrt{w},\infty} = O(t^\alpha)$, $\alpha > 0$. Then

$$\left| R_m^H(w; f; x) \right| \le C\ m^{-\alpha/2} \log m, \qquad |x| \le \varrho\sqrt{2m}, \quad 0 < \varrho < 1,$$

for some constant C independent of f, m and x.
Further, if $\alpha > 1/3$, then

$$\left| R_m^H(w; f; x) \right| \le C\ m^{-\alpha/2+1/6}, \qquad |x| \le 2t_{m,1} - t_{m,2},$$

with some constant C independent of f, m and x.

REMARK

We have assumed $|x| \leq 2t_{m,1} - t_{m,2}$. If $|x| > 2t_{m,1} - t_{m,2}$ then x is "sufficiently" far from all the nodes $t_{m,k},\ k = 1, 2, \cdots, m$. So that the Gaussian rule converges and does not present the numerical cancelation problem . Thus, the quadrature rule we have introduced in this paper is meaningful exactly when $|x| \leq 2t_{m,1} - t_{m,2}$.

Amplification Factor $K_m(w;x)$

Table 1: x=0

m	K_m	K_m gaussian
4	7.5	3.9
5	4.2	7.d+15
8	9.2	4.6
9	5.5	1.d+16
16	10.9	5.2
17	6.8	4.d+15
32	12.5	5.9
33	8.2	4.d+15
64	14.0	6.6
65	9.6	2.d+15

Table 2: x=0.00001

m	K_m	K_m gaussian
4	7.5	3.9
5	6.4	1.9d+5
8	9.2	4.6
9	8.5	1.4d+5
16	10.9	5.2
17	10.6	1.1d+5
32	12.5	5.9
33	12.7	7.7d+4
64	14.0	6.6
65	14.8	5.5d+4

Table 3: x=0.001

m	K_m	K_m gaussian
4	7.5	3.9
5	4.2	1892.1
8	9.2	4.6
9	5.5	1442.4
16	10.9	5.3
17	6.9	1064.4
32	12.4	5.9
33	8.2	770.8
64	13.9	6.6
65	9.6	552.8

Table 4: x=0.01

m	K_m	K_m gaussian
4	7.3	4.0
5	4.3	190.5
8	9.0	4.7
9	5.6	146.0
16	10.6	5.4
17	6.9	108.7
32	12.1	6.2
33	8.4	79.9
64	13.4	7.0
65	9.8	58.6

Table 5: x=0.1

m	K_m	K_m gaussian
4	6.1	5.0
5	4.8	20.0
8	7.4	6.2
9	6.3	15.9
16	8.6	7.9
17	8.0	12.6
32	9.5	10.6
33	9.9	10.1
64	10.2	17.8
65	12.3	8.4

Table 6: x=1.

m	K_m	K_m gaussian
4	5.9	3.5
5	3.3	30.9
8	3.5	7.2
9	6.3	4.6
16	6.6	5.6
17	4.5	11.4
32	5.9	23.0
33	6.6	5.0
64	7.0	13.9
65	7.4	5.8

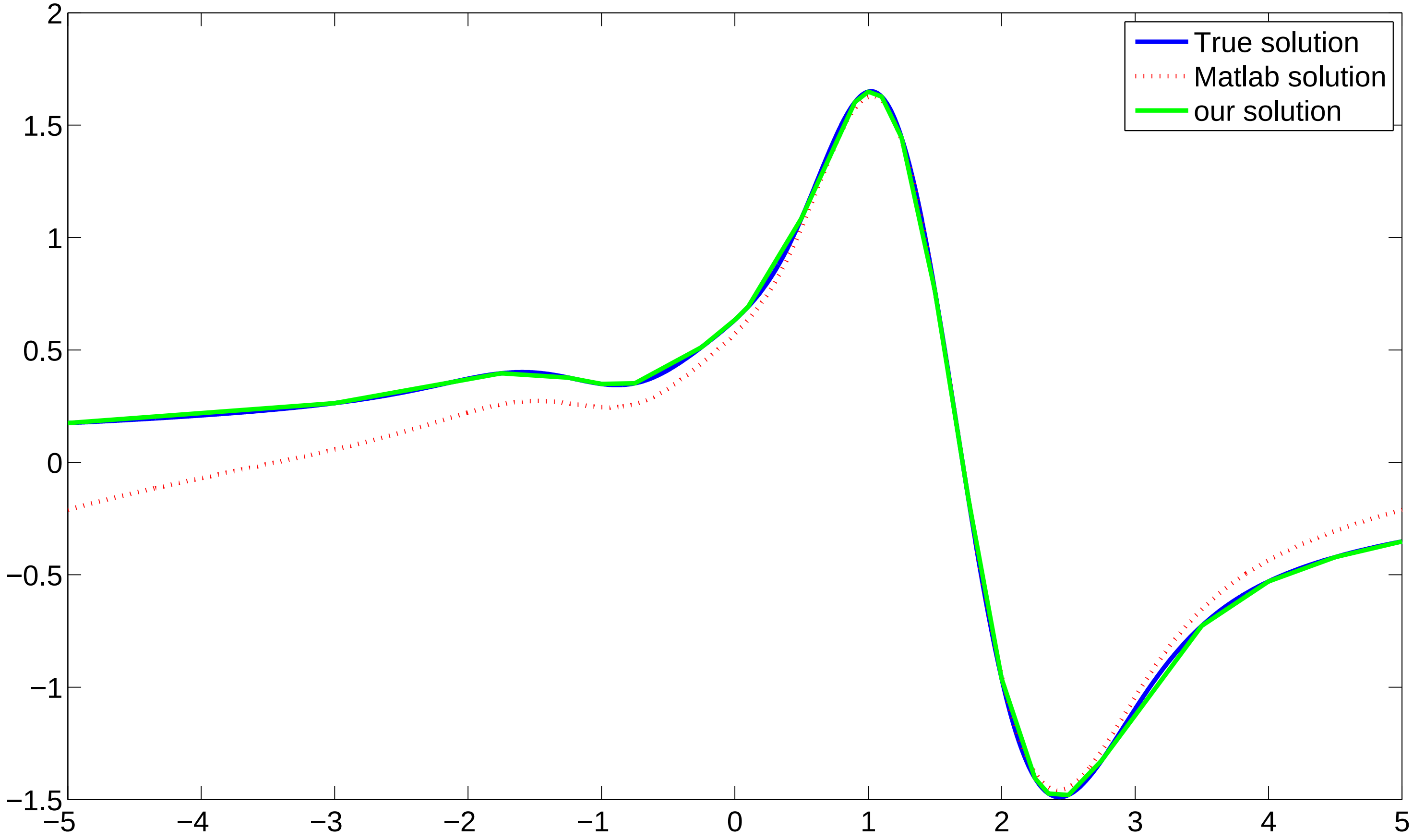
In the first example the exact solution $H(wf;x)$ is given and we have computed it by using the computer algebra package Mathematica . Figure shows the plots of the exact solution, our solution and the solution evaluated with the help of the Matlab routine Hilbert.

$$H(wf;x) = \int_{-\infty}^{+\infty} \frac{\exp(t)t^4 \exp(-t^2)}{t-x} dt =$$

$$= \frac{1}{8}\exp(\frac{1}{4})\sqrt{\pi}(7+6x+4x^2+8x^3) + x^4 \exp(\frac{1}{4}) H(w; x-\frac{1}{2})$$

where $f(t) = \exp(t)t^4$.

Comparison Between Exact, Matlab and Our Solution



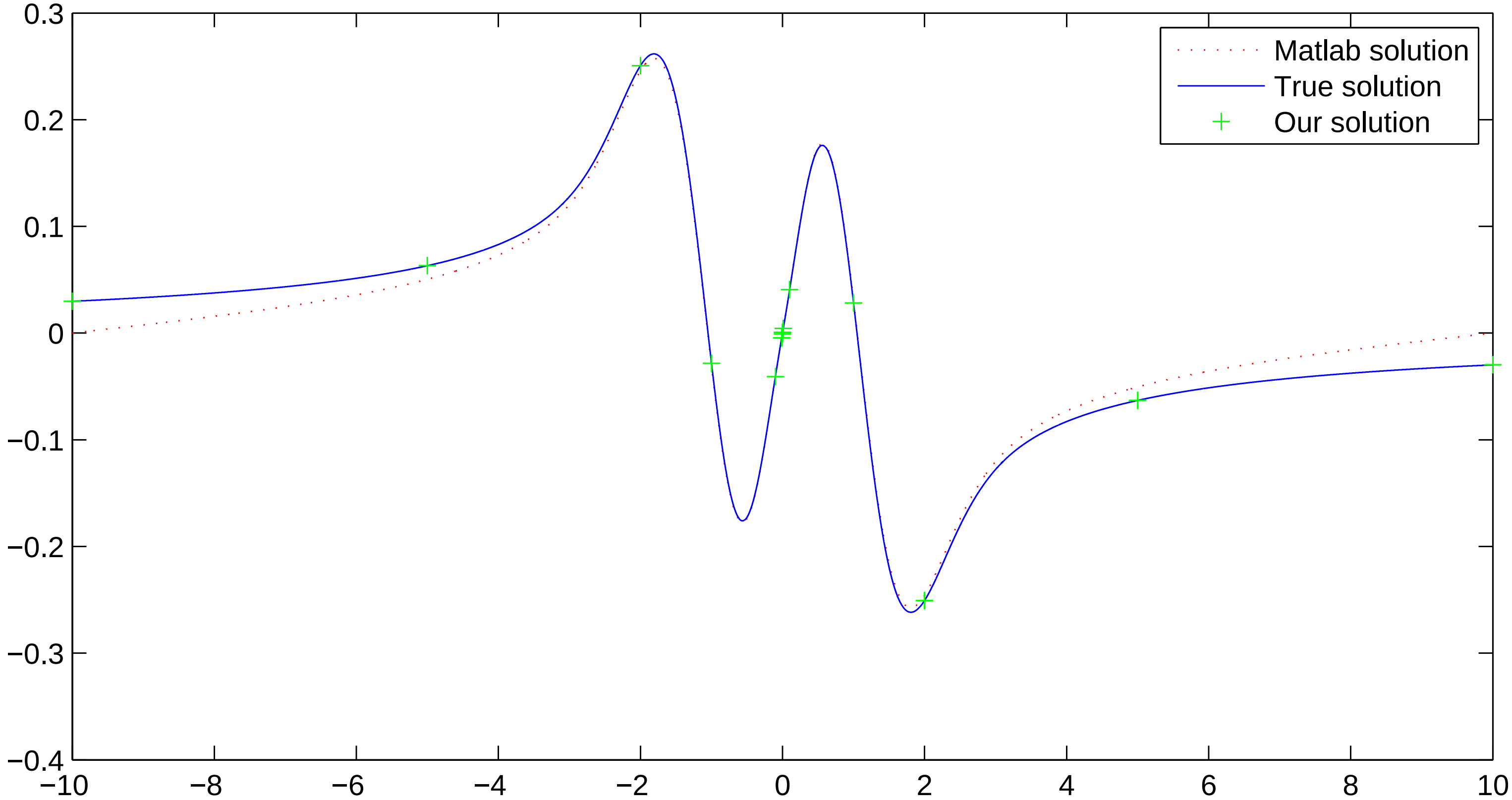
Also in the second example the exact solution $H(wf;x)$ is given and we have computed it by using the computer algebra package Mathematica . Figure shows the plots of the exact solution, our solution and the solution evaluated with the help of the Matlab routine Hilbert.

$$H(wf;x) = \int_{-\infty}^{+\infty} \frac{|t|^{\frac{5}{2}} \exp(-t^2)}{t-x} dt =$$

$$= x\Gamma(\frac{3}{4}) + i\exp(-x^2)(\frac{1}{x^2})^{\frac{1}{4}} x^3 (\pi - (-1)^{\frac{1}{4}} \Gamma(\frac{3}{4}) \Gamma(\frac{1}{4}, -x^2))$$

where $f(t) = |t|^{\frac{5}{2}}$.

Comparison Between Exact, Matlab and Our Solution



In the last example again the exact solution $H(wf;x)$ is given and we have computed it by using the computer algebra package Mathematica . Figure shows the plots of the exact solution, our solution and the solution evaluated with the help of the Matlab routine Hilbert.

$$H(wf;x) = \int_{-\infty}^{+\infty} \frac{\exp(-t^2)}{(1+t^2)(t-x)} dt =$$

$$= \frac{H(w;x) + e\pi x(erf(1) - 1)}{1 + x^2}$$

where $f(t) = \frac{1}{1+t^2}$.

Comparison Between Exact, Matlab and Our Solution