SHELL MODELS: OLD AND NEW

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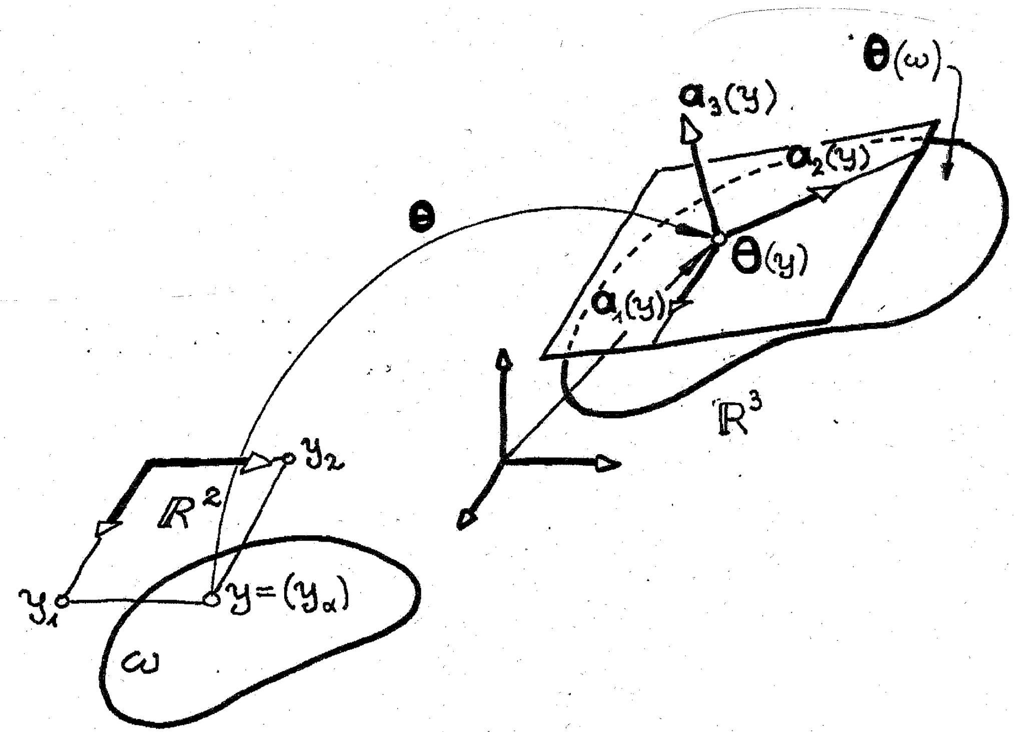
Outline

1. The two fundamental forms of a surface
2. Nonlinear shell theory – The classical and intrinsic approaches
3. A nonlinear Korn inequality on a surface
4. Classical linear shell theory – Korn's inequality on a surface
5. Intrinsic linear shell theory: Compatibility conditions of Saint–Venant type

1. THE TWO FUNDAMENTAL FORMS OF A SURFACE

$\alpha, \beta, \ldots \in \{1, 2\}$
$i, j, \ldots \in \{1, 2, 3\}$

Summation convention
ω: open in $\mathbb{R}^2$
$\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to \boldsymbol{\theta}(\omega) \subset \mathbb{R}^3$
$\boldsymbol{\theta}$ is "smooth enough"



$\boldsymbol{\theta}(\omega)$: **surface**

y_1, y_2: **curvilinear coordinates**

Assume $\boldsymbol{\theta}$ is an immersion: $\partial_\alpha \boldsymbol{\theta}$ linearly independent in ω

covariant basis: $\boldsymbol{a}_\alpha \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{\theta}, \quad \boldsymbol{a}_3 \stackrel{\text{def}}{=} \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$

First fundamental form: $a_{\alpha\beta} \stackrel{\text{def}}{=} \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta}$

Second fundamental form: $b_{\alpha\beta} \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \dfrac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}$

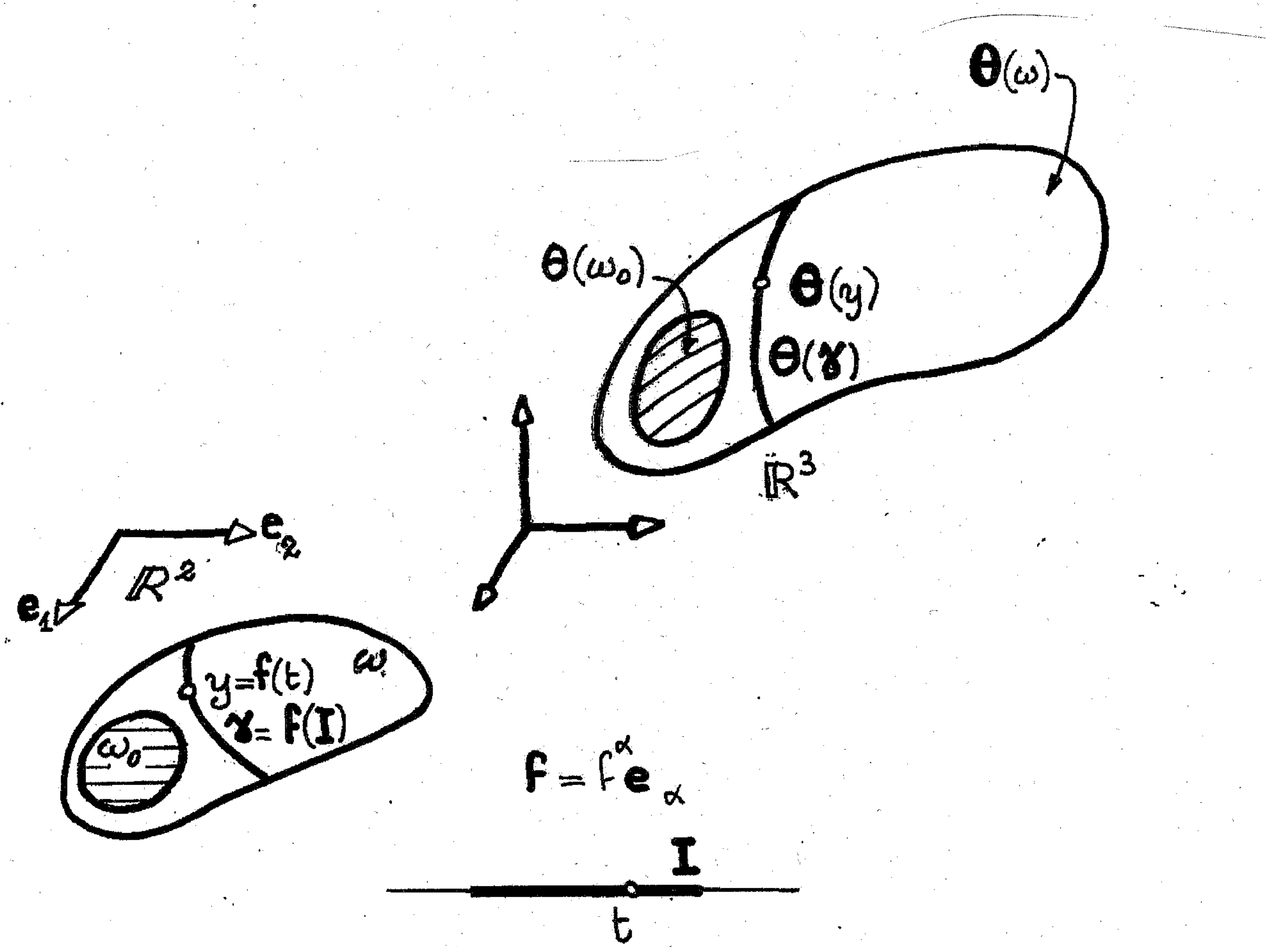
First fundamental form: *metric notions*, such as lengths, areas, angles $\therefore$ a.k.a. **metric tensor**

$(a_{\alpha\beta})$: symmetric positive-definite matrix field

Second fundamental form: *curvature notions*

$(b_{\alpha\beta})$: symmetric matrix field

$$\text{area}\,\boldsymbol{\theta}(\omega_0) = \int_{\omega_0} \sqrt{\det(a_{\alpha\beta}(y))}\,\mathrm{d}y$$

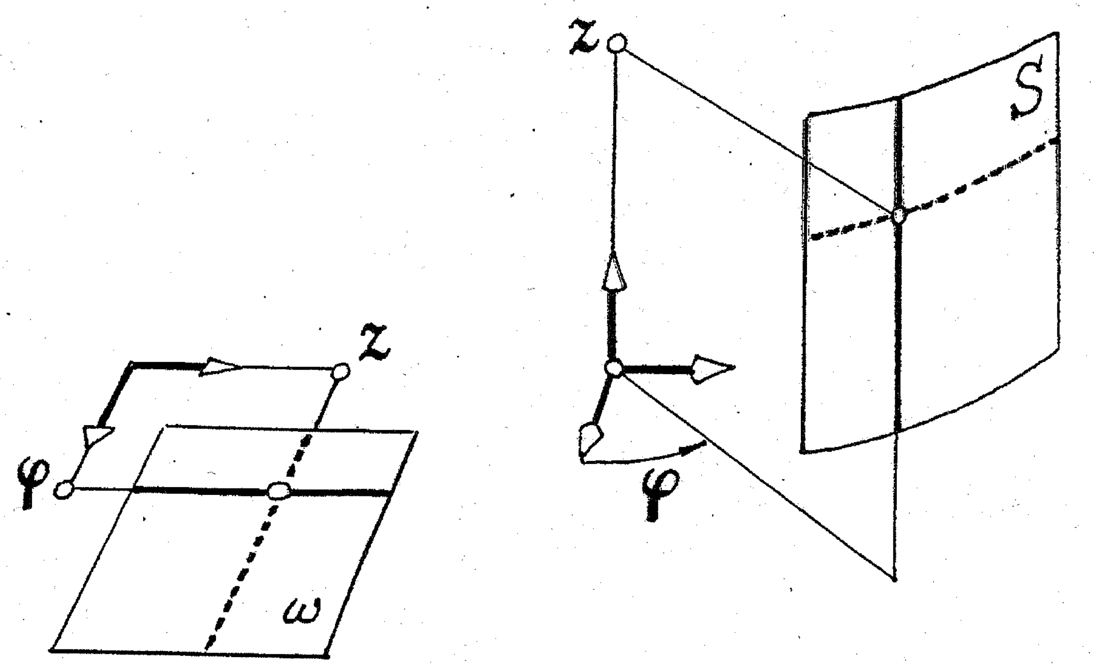


$$\boxed{\textbf{length of } \boldsymbol{\theta}(\boldsymbol{\gamma}) = \int_I \sqrt{a_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}\,\mathrm{d}t}$$

Curvature of $\boldsymbol{\theta}(\boldsymbol{\gamma})$ at $\boldsymbol{\theta}(y)$, $y = \boldsymbol{f}(t)$, when $\boldsymbol{\theta}(\boldsymbol{\gamma})$ lies in a plane normal to the surface $\boldsymbol{\theta}(\omega)$ at $\boldsymbol{\theta}(y)$:

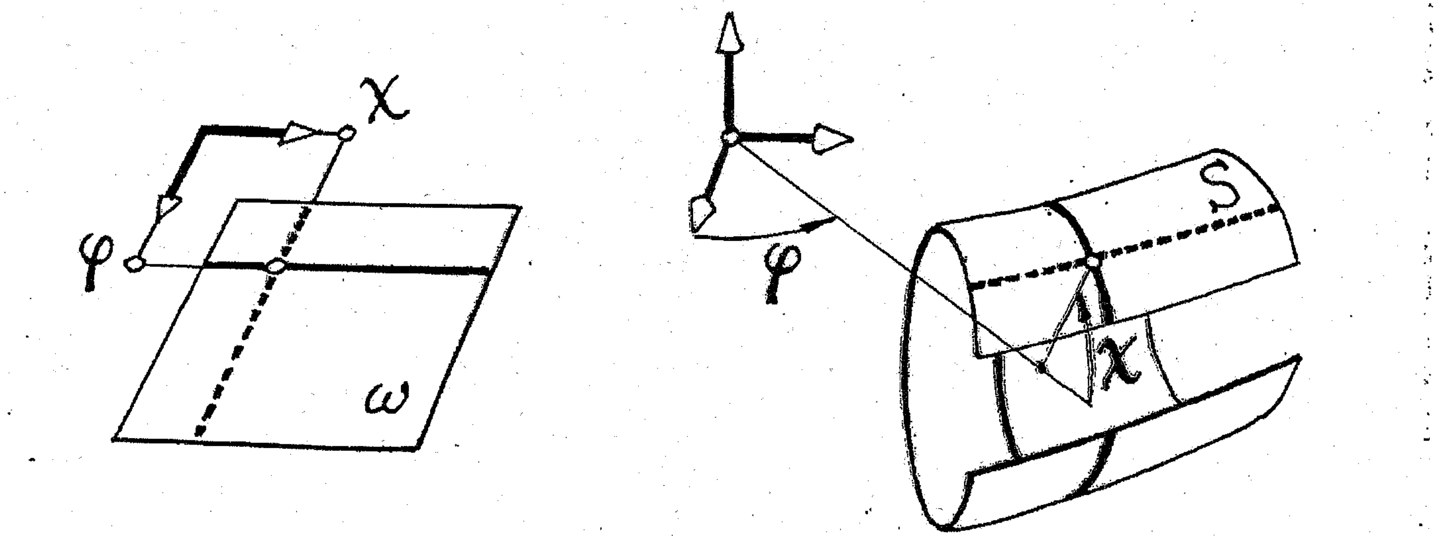
$$\frac{1}{R} = \frac{b_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}{a_{\alpha\beta}(\boldsymbol{f}(t))\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\frac{\mathrm{d}f^\beta}{\mathrm{d}t}(t)}$$

Portion of a **cylinder**



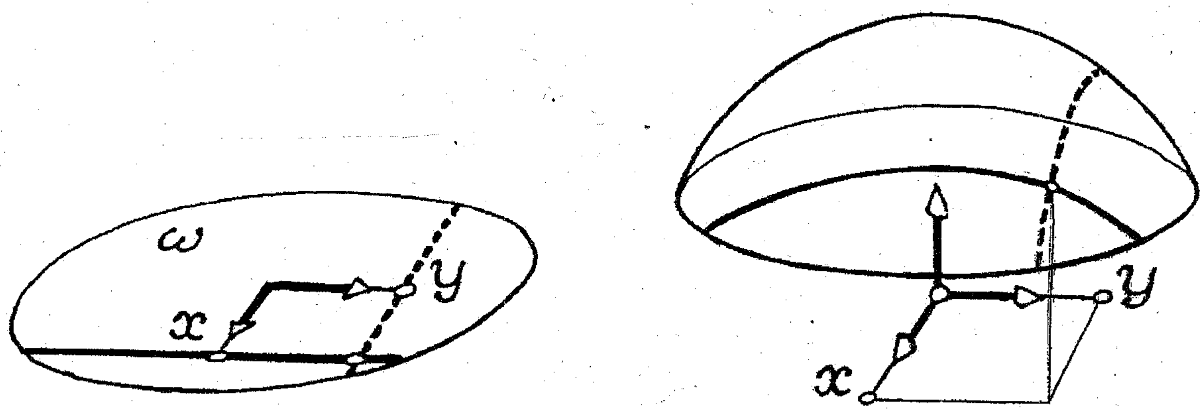
$$\boldsymbol{\theta} : (\varphi, z) \to \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

Portion of a **torus**



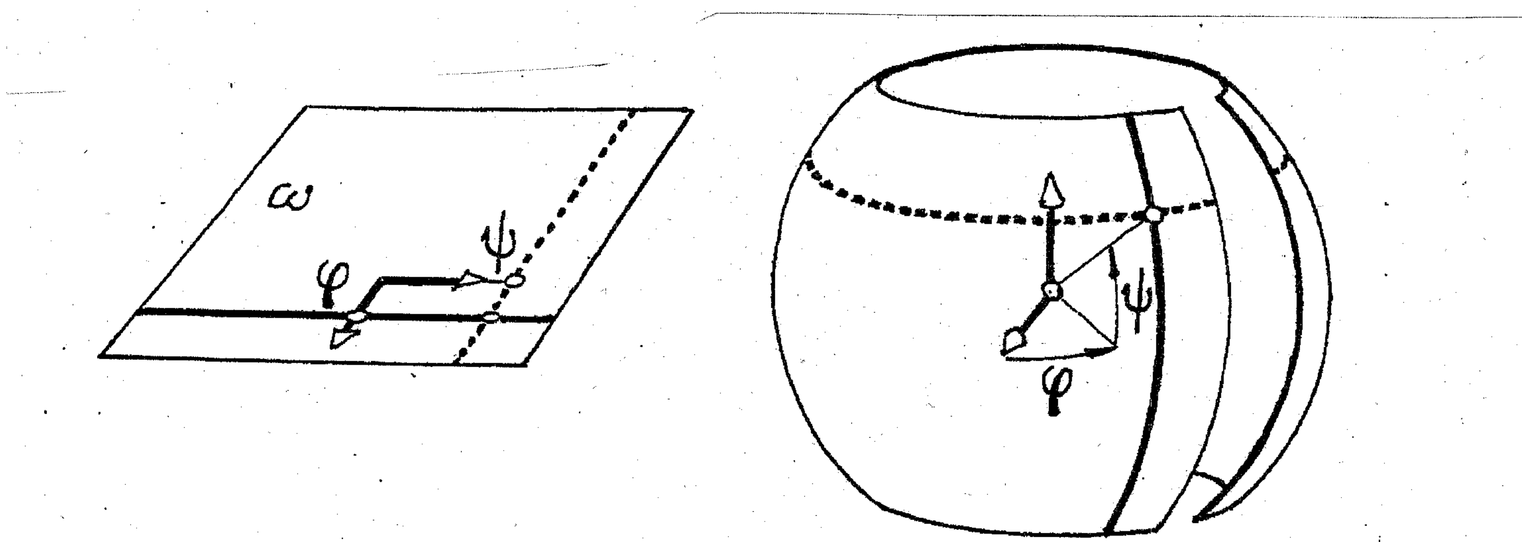
$$\boldsymbol{\theta} : (\varphi, \chi) \to \begin{pmatrix} (R + r\cos\chi)\cos\varphi \\ (R + r\cos\chi)\sin\varphi \\ r\sin\chi \end{pmatrix}$$

Cartesian coordinates



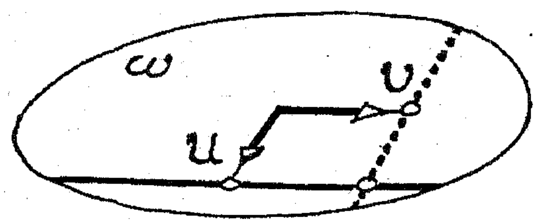
$$\boldsymbol{\theta} : (x, y) \to \begin{pmatrix} x \\ y \\ \sqrt{R^2 - (x^2 + y^2)} \end{pmatrix}$$

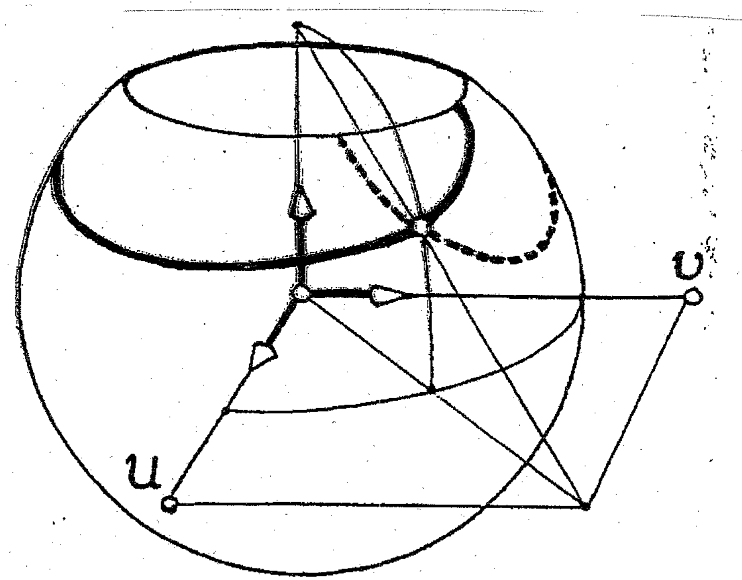
Spherical coordinates



$$\boldsymbol{\theta} : (\varphi, \psi) \rightarrow \begin{pmatrix} R\cos\psi\cos\varphi \\ R\cos\psi\sin\varphi \\ R\sin\psi \end{pmatrix}$$

Stereographic coordinates





$$\boldsymbol{\theta} : (u, v) \to \frac{1}{(u^2 + v^2 + R^2)} \begin{pmatrix} 2R^2 u \\ 2R^2 v \\ R(u^2 + v^2 - R^2) \end{pmatrix}$$

The components $a_{\alpha\beta} : \omega \to \mathbb{R}$ and $b_{\alpha\beta} : \omega \to \mathbb{R}$ of the two fundamental forms *cannot be arbitrary functions*: Let

$$\boxed{(a^{\sigma\tau}) \stackrel{\text{def}}{=} (a_{\alpha\beta})^{-1}, \quad \Gamma_{\alpha\beta\tau} \stackrel{\text{def}}{=} \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau \quad \text{and} \quad \Gamma^\sigma_{\alpha\beta} \stackrel{\text{def}}{=} a^{\sigma\tau}\Gamma_{\alpha\beta\tau}}$$

The functions $\Gamma_{\alpha\beta\tau}$ and $\Gamma^\sigma_{\alpha\beta}$ are the **Christoffel symbols**

Then it is easy to see that:

$$\begin{aligned} \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau &= \partial_\sigma \Gamma_{\alpha\beta\tau} - \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - b_{\alpha\beta}b_{\sigma\tau}, \\ \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 &= \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\beta} b_{\sigma\mu}. \end{aligned}$$

Besides,

$$\partial_{\alpha\sigma\beta}\boldsymbol{\theta} = \partial_{\alpha\beta\sigma}\boldsymbol{\theta} \iff \partial_{\alpha\sigma}\boldsymbol{a}_\beta = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \iff \begin{cases} \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_\tau = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_\tau \\ \partial_{\alpha\sigma}\boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta}\boldsymbol{a}_\sigma \cdot \boldsymbol{a}_3 \end{cases}$$

Necessary conditions:

$$\partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \Gamma^{\mu}_{\alpha\beta}\Gamma_{\sigma\tau\mu} - \Gamma^{\mu}_{\alpha\sigma}\Gamma_{\beta\tau\mu} = b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau} \quad \text{in } \omega$$

Gauß equations

$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^{\mu}_{\alpha\sigma} b_{\beta\mu} - \Gamma^{\mu}_{\alpha\beta} b_{\sigma\mu} = 0 \quad \text{in } \omega$$

Codazzi-Mainardi equations

Remarkably, these conditions are also *sufficient* if ω is *simply-connected* (see next theorem). Observe that the Christoffel symbols $\Gamma_{\alpha\beta\tau}$ and $\Gamma^{\sigma}_{\alpha\beta}$ can be expressed solely in terms of the components of the first fundamental form:

$$\Gamma_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \quad \text{and} \quad \Gamma^{\sigma}_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta\tau} \ \text{ with } (a^{\sigma\tau}) = (a_{\alpha\beta})^{-1}$$

Consequently, *the Gauß and Codazzi-Mainardi equations are* (nonlinear) *relations between the first and second fundamental forms*.

$$\mathbb{S}^2 \stackrel{\text{def}}{=} \{\text{ symmetric } 2\times 2 \text{ matrices }\}$$
$$\mathbb{S}^2_> \stackrel{\text{def}}{=} \{\text{ symmetric positive-definite } 2\times 2 \text{ matrices }\}$$
$$\mathbb{O}^3_+ \stackrel{\text{def}}{=} \{\text{ proper orthogonal } 3\times 3 \text{ matrices }\}$$

FUNDAMENTAL THEOREM OF SURFACE THEORY:

$\omega \subset \mathbb{R}^2$: *open, connected, simply connected. Let there be given* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ *satisfying the* **Gauß** *and* **Codazzi-Mainardi** *equations in* ω.
Then there exists $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ *such that:*

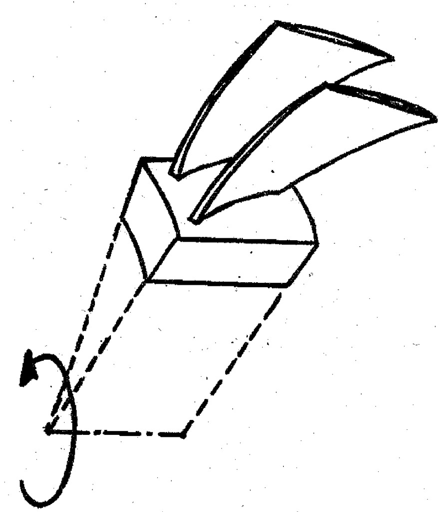
$$\boxed{a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \quad \text{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \quad \text{in } \omega}$$

Uniqueness holds modulo isometries of $\mathbb{R}^3$*: All other solutions are:*

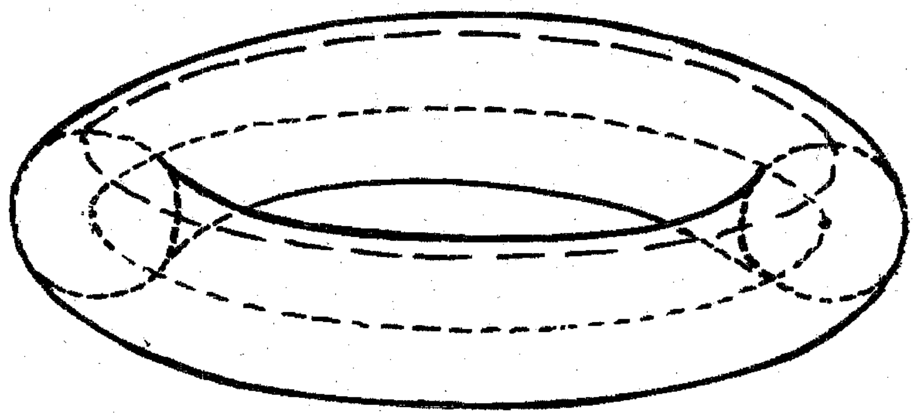
$$\boxed{y \in \omega \to \boldsymbol{\chi}(y) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\theta}(y) \quad \text{with } \boldsymbol{a} \in \mathbb{R}^3,\ \boldsymbol{Q} \in \mathbb{O}^3_+ \iff (\boldsymbol{\chi}, \boldsymbol{\theta}) \in \mathcal{R}}$$

S. Mardare (2003): $(a_{\alpha\beta}) \in W^{1,p}(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^p(\omega; \mathbb{S}^2), p > 2$. Then $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{R}^3)$

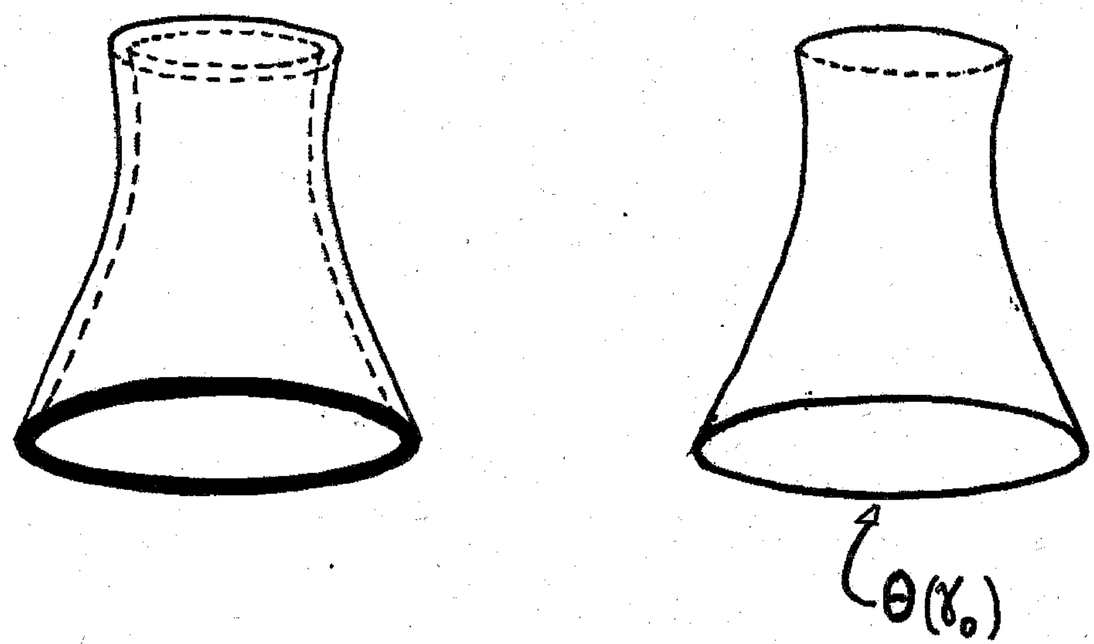
EXAMPLES OF SHELLS:



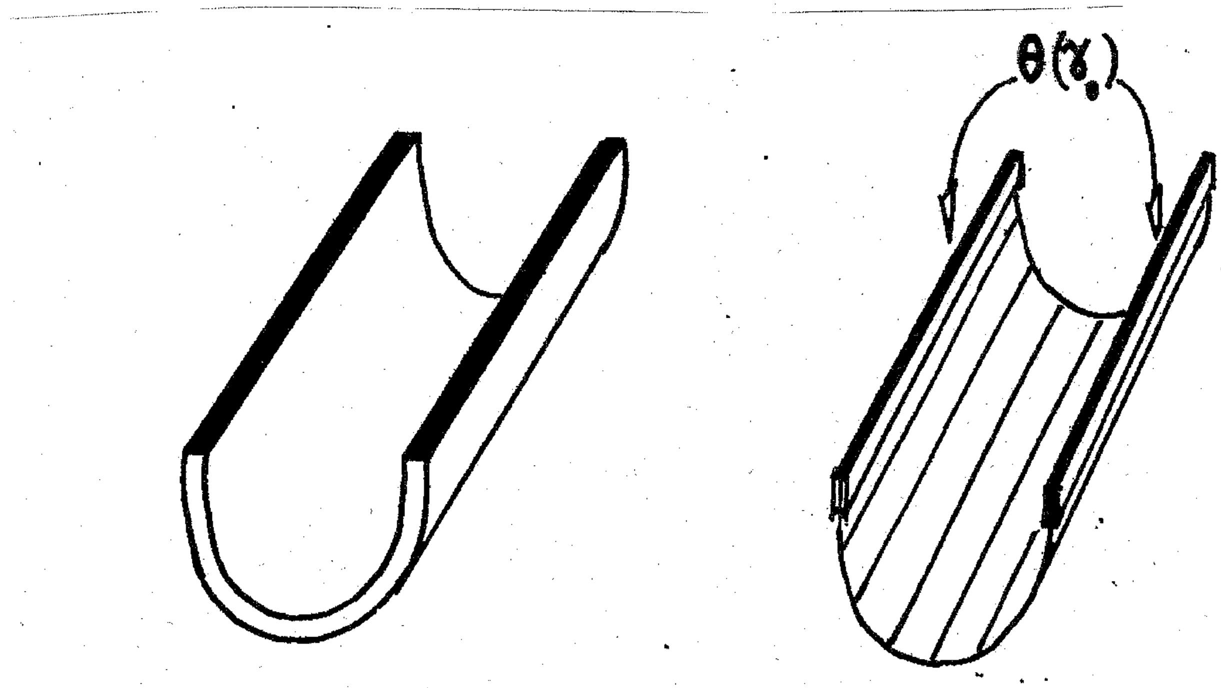
Blades of a rotor



Inner tube

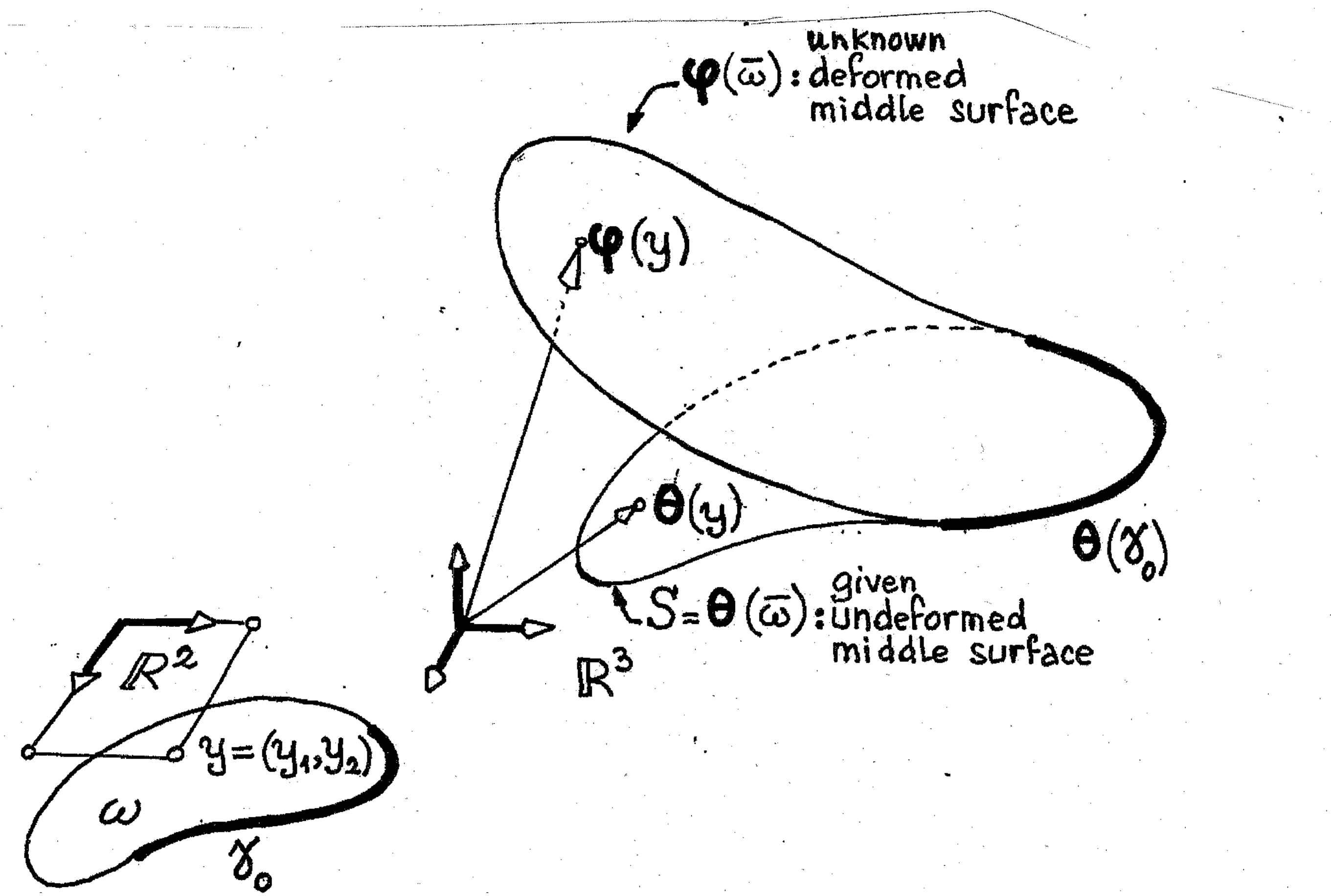


Cooling tower



Hangar for Zeppelins (upside down)

HOW IS A SHELL PROBLEM POSED?



CLASSICAL APPROACH

Unknown: $\boldsymbol{\varphi} = (\varphi_i) : \overline{\omega} \to \mathbb{R}^3$: **deformation** of middle surface S

Boundary conditions: $\boldsymbol{\varphi} = \boldsymbol{\theta}$ on γ_0 (simple support), or
$\boldsymbol{\varphi} = \boldsymbol{\theta}$ and $\partial_\nu \boldsymbol{\varphi} = \partial_\nu \boldsymbol{\theta}$ on γ_0 (clamping) $(\text{length}\, \gamma_0 > 0)$

Applied forces: $(f^i) : \omega \to \mathbb{R}^3$

Lamé constants of the elastic material: $\lambda > 0$, $\mu > 0$

$$A^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}), \quad \text{where } (a^{\sigma\tau}) = (a_{\alpha\beta})^{-1}$$

There exists $c_0 > 0$ such that $A^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} \geq c_0 \sum_{\alpha,\beta} |t_{\alpha\beta}|^2$ for all $y \in \overline{\omega}$, $(t_{\alpha\beta}) \in \mathbb{S}^2$

Thickness of the shell: $2\varepsilon > 0$

Area element along $S : \sqrt{a}\, \mathrm{d}y$ where $a = \det(a_{\alpha\beta})$

P.G. Ciarlet: *An Introduction to Differential Geometry with Applications to Elasticity*, Springer, 2005

Problem: To find $\boldsymbol{\varphi} : \overline{\omega} \to \mathbb{R}^3$ such that:

$$\boxed{J(\boldsymbol{\varphi}) = \inf\{\ J(\widetilde{\boldsymbol{\varphi}});\ \widetilde{\boldsymbol{\varphi}} : \overline{\omega} \to \mathbb{R}^3 \text{ smooth enough};\ \widetilde{\boldsymbol{\varphi}} = \boldsymbol{\theta} \text{ on } \gamma_0\ \}}$$

Total energy of the shell – W.T. Koiter (1966):

$$\begin{aligned}
J(\widetilde{\boldsymbol{\varphi}}) &= \frac{\varepsilon}{2}\int_\omega A^{\alpha\beta\sigma\tau}(\widetilde{a}_{\sigma\tau} - a_{\sigma\tau})(\widetilde{a}_{\alpha\beta} - a_{\alpha\beta})\sqrt{a}\,\mathrm{d}y \\
&+ \frac{\varepsilon^3}{6}\int_\omega A^{\alpha\beta\sigma\tau}(\widetilde{b}_{\sigma\tau} - b_{\sigma\tau})(\widetilde{b}_{\alpha\beta} - b_{\alpha\beta})\sqrt{a}\,\mathrm{d}y \\
&- \int_\omega f^i \widetilde{\varphi}_i \sqrt{a}\,\mathrm{d}y, \\
\widetilde{a}_{\alpha\beta} - a_{\alpha\beta} &\overset{\text{def}}{=} \partial_\alpha \widetilde{\boldsymbol{\varphi}} \cdot \partial_\beta \widetilde{\boldsymbol{\varphi}} - a_{\alpha\beta} \\
\widetilde{b}_{\alpha\beta} - b_{\alpha\beta} &\overset{\text{def}}{=} \partial_{\alpha\beta}\widetilde{\boldsymbol{\varphi}} \cdot \frac{\partial_1\widetilde{\boldsymbol{\varphi}} \wedge \partial_2\widetilde{\boldsymbol{\varphi}}}{|\partial_1\widetilde{\boldsymbol{\varphi}} \wedge \partial_2\widetilde{\boldsymbol{\varphi}}|} - b_{\alpha\beta}
\end{aligned}$$

◀ **membrane energy**

◀ **flexural energy**

◀ **forces**

◀ **change of metric tensor**

◀ **change of curvature tensor**

INTRINSIC APPROACH:

Another look at the energy of the shell:

$$
\boxed{
\begin{aligned}
J(\widetilde{\boldsymbol{\varphi}}) &= \frac{\varepsilon}{2}\int_\omega A^{\alpha\beta\sigma\tau}(\widetilde{a}_{\sigma\tau} - a_{\sigma\tau})(\widetilde{a}_{\alpha\beta} - a_{\alpha\beta})\sqrt{a}\,\mathrm{d}y \\
&+ \frac{\varepsilon^3}{6}\int_\omega A^{\alpha\beta\sigma\tau}(\widetilde{b}_{\sigma\tau} - b_{\sigma\tau})(\widetilde{b}_{\alpha\beta} - b_{\alpha\beta})\sqrt{a}\,\mathrm{d}y \\
&- \int_\omega f^i \widetilde{\varphi}_i \sqrt{a}\,\mathrm{d}y
\end{aligned}
}
$$

◀ **membrane energy**

◀ **flexural energy**

◀ **forces**

Hence the **fundamental forms** $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ **of the unknown surface** $\widetilde{\boldsymbol{\varphi}}(\omega)$ appear as **natural unknowns**

This is the basis of the **intrinsic approach**:

J.L. Synge & W.Z. Chien (1941); W.Z. Chien (1944)

S.S. Antman (1976)

W. Pietraszkiewicz (2001); S. Opoka & W. Pietraszkiewicz (2004)

But, if $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ are chosen as the primary unknowns:

– How to express *in terms of* $(\widetilde{a}_{\alpha\beta})$ *and* $(\widetilde{b}_{\alpha\beta})$ the integral $\int_\omega \boldsymbol{f} \cdot \widetilde{\boldsymbol{\varphi}} \sqrt{a}\,\mathrm{d}y$ taking into account the **forces** in the energy?

– How to express *in terms of* $(\widetilde{a}_{\alpha\beta})$ *and* $(\widetilde{b}_{\alpha\beta})$ the **boundary condition**, e.g., $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\theta}$ on Γ_0, that the admissible deformations must satisfy?

– How to handle such expressions if **minimizing sequences** are considered:

$$\widetilde{a}^k_{\alpha\beta} \underset{k\to\infty}{\longrightarrow} \widetilde{a}_{\alpha\beta} \quad \text{and} \quad \widetilde{b}^k_{\alpha\beta} \underset{k\to\infty}{\longrightarrow} \widetilde{b}_{\alpha\beta} \quad \Longrightarrow \quad \widetilde{\boldsymbol{\varphi}}^k \to \widetilde{\boldsymbol{\varphi}}\ ?$$

– **Constrained minimization problem**: The new unknowns $\widetilde{a}_{\alpha\beta}$ and $\widetilde{b}_{\alpha\beta}$ must satisfy the (*highly nonlinear*) **Gauß** and **Codazzi-Mainardi equations**

3. A NONLINEAR KORN INEQUALITY ON A SURFACE

Like in linear shell theory (Part 4), a *nonlinear Korn inequality on a surface* could perhaps provide an existence theorem in *nonlinear shell theory*.
The inequality found in this section constitutes a first step in this direction.

In what follows: $\boxed{p \geq 2}$

$$\left.\begin{aligned}&\boldsymbol{\theta} \in W^{1,p}(\omega;\mathbb{R}^3), \quad \boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}\\ &\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \neq \mathbf{0} \text{ a.e. in } \omega\\ &\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in W^{1,p}(\omega;\mathbb{R}^3)\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned}&a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \in L^{p/2}(\omega)\\ &b_{\alpha\beta} = -\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\beta \in L^{p/2}(\omega)\\ &c_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3 \in L^{p/2}(\omega)\end{aligned}\right.$$

$\widetilde{R}_1$ and $\widetilde{R}_2$: principal radii of curvature of the surface $\widetilde{\boldsymbol{\theta}}(\omega)$

THEOREM: $\omega \subset \mathbb{R}^2$ *bounded, open, connected, Lipschitz boundary*
Let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$*: immersion such that* $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$*.*
Given $\varepsilon > 0$*, there exists a constant* $c(\varepsilon)$ *with the following property:*
Given any $\widetilde{\boldsymbol{\theta}} \in W^{1,p}(\omega; \mathbb{R}^3)$ *such that* $\widetilde{\boldsymbol{a}}_1 \wedge \widetilde{\boldsymbol{a}}_2 \neq 0$ *a.e. in* ω*,* $\widetilde{\boldsymbol{a}}_3 \in W^{1,p}(\omega; \mathbb{R}^3)$*,*
$|\widetilde{R}_1| \geq \varepsilon$ *and* $|\widetilde{R}_2| \geq \varepsilon$ *a.e. in* ω*,*
there exist $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{\theta}, \widetilde{\boldsymbol{\theta}}, \varepsilon) \in \mathbb{R}^3$ *and* $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{\theta}, \widetilde{\boldsymbol{\theta}}, \varepsilon) \in \mathbb{O}^3_+$ *such that*

$$\overbrace{\|(\boldsymbol{a} + \boldsymbol{Q}\widetilde{\boldsymbol{\theta}}) - \boldsymbol{\theta}\|_{W^{1,p}(\omega;\mathbb{R}^3)} + \|\boldsymbol{Q}\widetilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{W^{1,p}(\omega;\mathbb{R}^3)}}^{\text{“distance” between surfaces } \boldsymbol{\theta}(\omega) \text{ and } \boldsymbol{a}+\boldsymbol{Q}\widetilde{\boldsymbol{\theta}}(\omega)}$$

$$\leq c(\varepsilon)\Big\{ \underbrace{\|(\widetilde{a}_{\alpha\beta} - a_{\alpha\beta})\|^{1/2}_{L^{p/2}(\omega;\mathbb{S}^2)} + \|(\widetilde{b}_{\alpha\beta} - b_{\alpha\beta})\|^{1/2}_{L^{p/2}(\omega;\mathbb{S}^2)} + \|(\widetilde{c}_{\alpha\beta} - c_{\alpha\beta})\|^{1/2}_{L^{p/2}(\omega;\mathbb{S}^2)} \Big\}}_{\text{“change of metric” and “change of curvature”}}$$

As a corollary: *Sequential continuity of a surface as a function of its fundamental forms with respect to Sobolev norms:*

THEOREM: $\omega \subset \mathbb{R}^2$ *bounded, open, connected, Lipschitz boundary*
Let $\boldsymbol{\theta}^k \in W^{1,p}(\omega; \mathbb{R}^3)$ *such that* $\boldsymbol{a}_3^k \in W^{1,p}(\omega; \mathbb{R}^3), k \geq 1$, *and there exists* $\varepsilon > 0$, *such that the principal radii of curvature* R_1^k *and* R_2^k *of each surface* $\boldsymbol{\theta}^k(\omega), k \geq 1$, *satisfy*

$$|R_1^k| \geq \varepsilon \quad \text{and} \quad |R_2^k| \geq \varepsilon \quad \text{for all } k \geq 1.$$

Let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ *be an immersion such that* $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$. *Assume that:*

$$\boxed{a_{\alpha\beta}^k \underset{k\to\infty}{\longrightarrow} a_{\alpha\beta}, \quad b_{\alpha\beta}^k \underset{k\to\infty}{\longrightarrow} b_{\alpha\beta}, \quad c_{\alpha\beta}^k \underset{k\to\infty}{\longrightarrow} c_{\alpha\beta} \quad \text{in } L^{p/2}(\omega)}$$

Then there exist $\boldsymbol{a}^k \in \mathbb{R}^3, \boldsymbol{Q}^k \in \mathbb{O}_+^3, k \geq 1$, *such that*

$$\boxed{\boldsymbol{a}^k + \boldsymbol{Q}^k \boldsymbol{\theta}^k \underset{k\to\infty}{\longrightarrow} \boldsymbol{\theta} \quad \text{in } W^{1,p}(\omega; \mathbb{R}^3)}$$

Proofs rely on

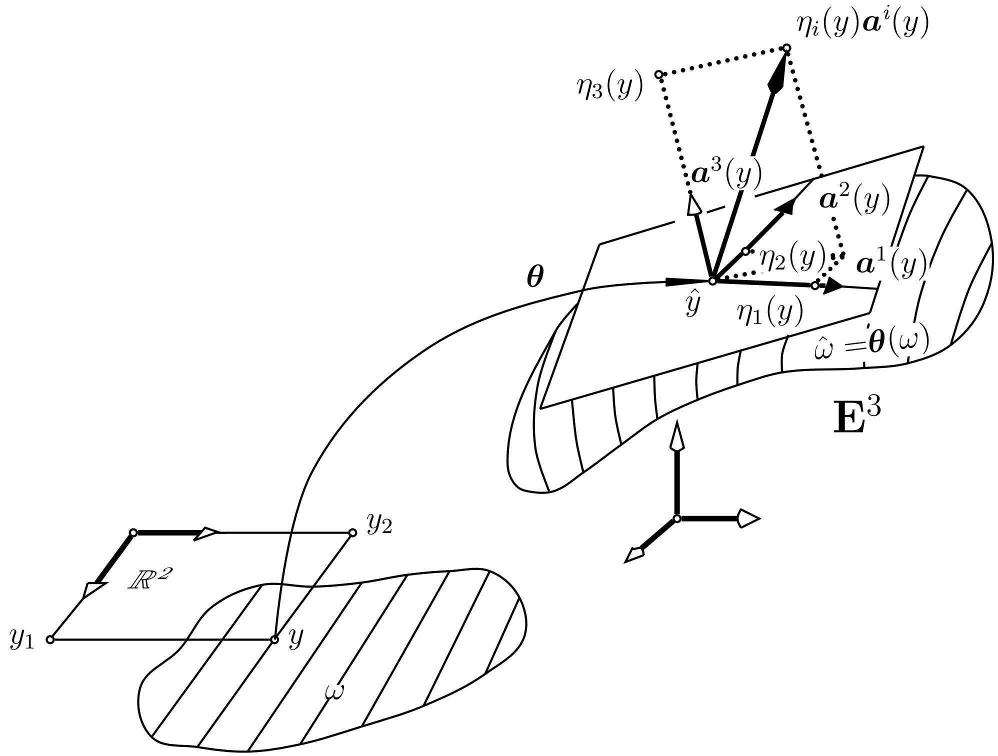
(a) the "**geometric rigidity lemma**":
There exists a constant $\Lambda(\Omega)$ such that, for each $\boldsymbol{\theta} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \boldsymbol{\nabla\theta} > 0$ a.e. in Ω, there exists $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\theta}) \in \mathbb{O}^n_+$ such that

$$\|\boldsymbol{\nabla\theta} - \boldsymbol{R}\|_{L^2(\Omega;\mathbb{M}^n)} \leq \Lambda(\Omega) \left\| \mathrm{dist}(\boldsymbol{\nabla\theta}, \mathbb{O}^n_+) \right\|_{L^2(\Omega)}$$

G. Friesecke, R.D. James, S. Müller (2002).
This lemma was extended to the "L^p-case" by Conti (2004).

(b) a "**nonlinear 3d-Korn inequality**": P.G. Ciarlet, C. Mardare (2004).
See also: Y.G. Reshetnyak (2003)

Contravariant basis $(\boldsymbol{a}^i)$: $\boldsymbol{a}^\alpha = \boldsymbol{a}^{\alpha\beta}\boldsymbol{a}_\beta,\ (a^{\alpha\beta}) = (a_{\sigma\tau})^{-1},\ \boldsymbol{a}^3 = \boldsymbol{a}_3$. Then $\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j$.
$\Gamma^\sigma_{\alpha\beta} = \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta$



$\widetilde{\boldsymbol{\eta}} = \eta_i \boldsymbol{a}^i : \omega \to \mathbb{R}$: displacement field (note that $\boldsymbol{\varphi} = \boldsymbol{\theta} + \widetilde{\boldsymbol{\eta}}$)
$\boldsymbol{\eta} = (\eta_i) : \omega \to \mathbb{R}^3$
Undeformed surface: $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$; deformed surface: $(a_{\alpha\beta}(\boldsymbol{\eta}))$ and $(b_{\alpha\beta}(\boldsymbol{\eta}))$.

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) \stackrel{\text{def}}{=} \frac{1}{2}\left[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}\right]^{\text{lin}} = \frac{1}{2}(\partial_\alpha \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\beta \widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha)$$
$$= \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$$

Linearized change of metric tensor

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) \stackrel{\text{def}}{=} \left[b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}\right]^{\text{lin}} = (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3$$
$$= \eta_{3|\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha \eta_{\sigma|\beta} + b^\tau_\beta \eta_{\tau|\alpha} + b^\tau_\beta|_\alpha \eta_\tau$$
$$= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3$$
$$+ b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma)$$
$$+ (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau$$

Linearized change of curvature tensor

$\eta_\alpha \in H^1(\omega)$ and $\eta_3 \in L^2(\omega) \Longrightarrow \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$

$\eta_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega) \Longrightarrow \rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$

Koiter's linear shell equations (Koiter [1970])

ω: open, bounded, connected in $\mathbb{R}^2$, Lipschitz boundary

$\gamma_0 \subset \partial\omega$ with length $\gamma_0 > 0$

$$\begin{aligned}
\boldsymbol{\zeta} &= (\zeta_i) \in \boldsymbol{V}(\omega) \stackrel{\text{def}}{=} \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\} \\
j(\boldsymbol{\zeta}) &= \inf\{j(\boldsymbol{\eta});\ \boldsymbol{\eta} \in \boldsymbol{V}(\omega)\}, \text{ where} \\
j(\boldsymbol{\eta}) &= \frac{\varepsilon}{2} \int_\omega A^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y \\
&\quad + \frac{\varepsilon^3}{6} \int_\omega A^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y \\
&\quad - \int_\omega f^i \eta_i \sqrt{a}\, \mathrm{d}y
\end{aligned}$$

THEOREM: KORN'S INEQUALITY ON A SURFACE

There exists $c > 0$ *such that*

$$\overbrace{\left\{ \sum_\alpha \|\eta_\alpha\|^2_{H^1(\omega)} + \|\eta_3\|^2_{H^2(\omega)} \right\}^{1/2}}^{\text{norm on } H^1(\omega)\times H^1(\omega)\times H^2(\omega)}$$

$$\leq c \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\eta)\|^2_{L^2(\omega)} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\eta)\|^2_{L^2(\omega)} \right\}^{1/2} \quad \text{for all } \eta \in \boldsymbol{V}(\omega)$$

Existence then follows by the Lax-Milgram lemma

M. Bernadou & Ciarlet (1976)

M. Bernadou, P.G. Ciarlet & B. Miara (1994)

A. Blouza & H. Le Dret (1999)

P.G. Ciarlet & S. Mardare (2001)

Pure traction problem

$$\boxed{j(\boldsymbol{\zeta}) = \inf_{\boldsymbol{\eta} \in \boldsymbol{V}(\omega)} j(\boldsymbol{\eta}), \quad \text{where } \boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)}$$

$$\begin{aligned} j(\boldsymbol{\eta}) &= \frac{\varepsilon}{2} \int_\omega A^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y \\ &+ \frac{\varepsilon^3}{6} \int_\omega A^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y \\ &- \int_\omega f^i \eta_i \sqrt{a}\, \mathrm{d}y \sqrt{a}\, \mathrm{d}y \end{aligned}$$

Applied forces must satisfy $\int_\Omega f^i \eta_i \sqrt{a}\, \mathrm{d}y = 0$ for all $\eta_i \boldsymbol{a}^i = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}$

Intrinsic approach: $c_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $r_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ become the **primary unknowns** instead of the covariant components $\eta_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$ of the displacement field.

THEOREM: $\omega \subset \mathbb{R}^2$: *bounded, simply-connected, connected, Lipschitz boundary*
Given $(\boldsymbol{c},\boldsymbol{r}) \in L^2(\omega;\mathbb{S}^2) \times L^2(\omega;\mathbb{S}^2)$, *there exists* $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i \in \boldsymbol{V}(\omega)$ *s.t.*

$$\boxed{(\boldsymbol{c},\boldsymbol{r}) = \big((\gamma_{\alpha\beta}(\boldsymbol{\eta}))\,,(\rho_{\alpha\beta}(\boldsymbol{\eta}))\big) \quad \Longleftrightarrow \quad \boldsymbol{R}(\boldsymbol{c},\boldsymbol{r}) = 0 \ \text{ in } \boldsymbol{H}^{-2}(\omega) \times \boldsymbol{H}^{-1}(\omega)}$$

Uniqueness of $\boldsymbol{\eta} = (\eta_i)$: *up to* $\eta_i \boldsymbol{a}^i = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}$

COROLLARY: *Existence and uniqueness of solution to the minimization problem of intrinsic linear shell theory*:

$$\boxed{\kappa(\boldsymbol{c}^*,\boldsymbol{r}^*) = \inf_{(\boldsymbol{c},\boldsymbol{r}) \in \boldsymbol{E}(\omega)} \kappa(\boldsymbol{c},\boldsymbol{r})}$$

$$\boxed{\begin{aligned} \boldsymbol{E}(\omega) &\stackrel{\text{def}}{=} \Big\{(\boldsymbol{c},\boldsymbol{r}) \in L^2_{\text{sym}}(\omega) \times L^2_{\text{sym}}(\omega); \boldsymbol{R}(\boldsymbol{c},\boldsymbol{r}) = 0 \ \text{ in } \boldsymbol{H}^{-2}(\omega) \times \boldsymbol{H}^{-1}(\omega)\Big\} \\ \kappa(\boldsymbol{c},\boldsymbol{r}) &\stackrel{\text{def}}{=} \frac{\varepsilon}{2}\int_\omega A^{\alpha\beta\sigma\tau} c_{\sigma\tau} c_{\alpha\beta}\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{6}\int_\omega A^{\alpha\beta\sigma\tau} r_{\sigma\tau} r_{\alpha\beta}\sqrt{a}\,\mathrm{d}y - \Lambda(c,r) \end{aligned}}$$

As expected: $\boldsymbol{c}^* = (\gamma_{\alpha\beta}(\boldsymbol{\zeta}))$ *and* $\boldsymbol{r}^* = (\rho_{\alpha\beta}(\boldsymbol{\zeta}))$.

Christoffel symbols:

$$\Gamma^{\tau}_{\alpha\beta} := \frac{1}{2} a^{\tau\sigma}(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta})$$

Mixed components of the Riemann curvature tensor:

$$R^{\nu}_{\cdot\alpha\sigma\tau} := \partial_\sigma \Gamma^{\nu}_{\alpha\tau} - \partial_\tau \Gamma^{\nu}_{\alpha\sigma} + \Gamma^{\mu}_{\alpha\tau}\Gamma^{\nu}_{\mu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\nu}_{\mu\tau}$$

COMPATIBILITY CONDITIONS OF SAINT–VENANT TYPE $\boldsymbol{R}(\boldsymbol{c},\boldsymbol{r}) = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\omega) \times \boldsymbol{H}^{-1}(\omega)$:

$$\boxed{\begin{aligned} c_{\sigma\alpha|\beta\tau} + c_{\tau\beta|\alpha\sigma} - c_{\tau\alpha|\beta\sigma} - c_{\sigma\beta|\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau} c_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau} c_{\alpha\nu} & \\ = b_{\tau\alpha} r_{\sigma\beta} + b_{\sigma\beta} r_{\tau\alpha} - b_{\sigma\alpha} r_{\tau\beta} - b_{\tau\beta} r_{\sigma\alpha} \quad & \text{in } H^{-2}(\omega), \\ r_{\sigma\alpha|\tau} - r_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(c_{\alpha\nu|\tau} + c_{\tau\nu|\alpha} - c_{\tau\alpha|\nu}) & \\ - b^{\nu}_{\tau}(c_{\alpha\nu|\sigma} + c_{\sigma\nu|\alpha} - c_{\sigma\alpha|\nu}) \quad & \text{in } H^{-1}(\omega) \end{aligned}}$$

CESÀRO–VOLTERRA PATH WITH INTEGRAL FORMULA ON A SURFACE

THEOREM: $\omega \subset \mathbb{R}^2$ *simply-connected. Let* $x_0 \in \omega$ *be fixed. If* $\boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \boldsymbol{0}$, *a particular solution to*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = c_{\alpha\beta} \quad \text{and} \quad \rho_{\alpha\beta}(\boldsymbol{\eta}) = r_{\alpha\beta}$$

is given at each $x \in \overline{\omega}$ *by*

$$\boxed{\begin{aligned} \eta_i(x)\boldsymbol{a}^i(x) = & \int_{\boldsymbol{\gamma}(x)} c_{\alpha\beta}(y)\boldsymbol{a}^\alpha(y)\,\mathrm{d}y^\beta \\ & + \int_{\boldsymbol{\gamma}(x)} (\boldsymbol{\theta}(x) - \boldsymbol{\theta}(y)) \wedge (\varepsilon^{\alpha\beta}(y) c_{\alpha\sigma|\beta}(y)\boldsymbol{a}_3(y)\,\mathrm{d}y^\sigma) \\ & + \int_{\boldsymbol{\gamma}(x)} (\boldsymbol{\theta}(x) - \boldsymbol{\theta}(y)) \wedge (\varepsilon^{\alpha\beta}(y)(r_{\alpha\sigma}(y) - b_\alpha^\tau(y)c_{\tau\sigma}(y) - b_\sigma^\tau(y)c_{\alpha\tau}(y))\boldsymbol{a}_\beta(y)\,\mathrm{d}y^\sigma, \end{aligned}}$$

where $\boldsymbol{\gamma}(x)$ *is any curve of class* $\mathcal{C}^1$ *joining* x_0 *to* x *in* ω, *and* $(\varepsilon^{\alpha\beta})$ *is the orientation tensor, defined by* $e^{11} = e^{22} = 0,\ e^{12} = -e^{21} = \dfrac{1}{\sqrt{a}}$.