Integrability and solvability of nonlocal wave interaction models

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BACKGROUND

- Integrability via Lax pair (differential geometry)

$$\Psi_x = V\Psi \quad , \quad \Psi_t = A\Psi \quad , \quad [V\,,\,A] + V_t - A_x = 0$$

$$V = ik\sigma + Q(x,t)$$

- Solvability of the initial value problem (classical dynamics)

$$Q(x,t) \rightarrow_{CT} ST(k,t) \text{ together with } ST(k,t) \rightarrow Q(x,t)$$

$$Q(x,0) \rightarrow ST(k,0) \rightarrow_L ST(k,t) \rightarrow Q(x,t)$$

$$Q(x,t) \rightarrow 0 \text{ as } x \rightarrow \pm\infty$$

$\rightarrow_{CT}$ means "canonical transformation to action–angle variables"
$\rightarrow_L$ means " linear evolution"

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BOOMERONIC EQUATIONS 1

$$\Psi_x = [ik\sigma + Q(x,t)]\Psi \quad , \quad N \times N \;\; matrix \;\; equation$$

assume σ has multiple (i.e. non simple) eigenvalues (at least one)

$$\sigma = \begin{pmatrix} \sigma_1 \mathbf{1}_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma_M \mathbf{1}_M \end{pmatrix} \qquad \begin{array}{lcl} \sigma_j \neq \sigma_m & , & \text{if } j \neq m \\ \mathbf{1}_j = N_j x N_j \ \text{ identity} & , & \sum_{j=1}^{M} N_j = N \end{array}$$

hierarchy of evolutions : $Q_t = \sigma \sum_{n=0}^{\nu} L^n [C_n \, , \, \sigma Q]$

$$C_n = \begin{pmatrix} C_{n1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & C_{nM} \end{pmatrix}$$

non boomeronic equations (such as vector NLS) correspond to

$$C_n = c_n \, \sigma$$

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BOOMERONIC EQUATIONS 2

consequences:

1. noncommuting flows
2. nonlocal (integro–differential) evolution equations
3. solitons have a time–dependent velocity

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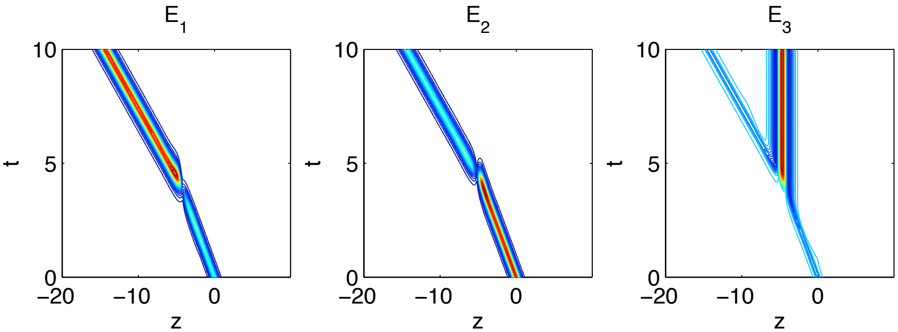
OPTICAL BOOMERON



REFERENCE

A. Degasperis, M. Conforti, F. Baronio and S. Wabnitz
Stable control of pulse speed in parametric three-wave solitons
Phys. Rev. Lett., vol. 97, p. 093901, 2006.

BOOMERONIC EQUATIONS 3

$$\Psi_x = ik\sigma + Q(x,t)\Psi \quad , \quad \Psi_t = A(x,t,k)\Psi - \Psi A(-\infty,t,k)$$

$$\begin{cases} \Psi \;\rightarrow\; \exp(ik\sigma x) & x \rightarrow -\infty \;, \\ \Psi \;\rightarrow\; \exp(ik\sigma x)S(t,k) & x \rightarrow +\infty \end{cases}$$

$$S_t = A(+\infty,t,k)S - SA(-\infty,t,k)$$

FOR BOOMERONIC EQUATIONS:

- $A(-\infty,t,k) \neq A(+\infty,t,k)$
- generically $A(-\infty,t,k)$ and $A(+\infty,t,k)$ depend on $Q(x,t)$

CONSEQUENCE :
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EXAMPLE: NON LOCAL WAVE INTERACTION

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \;,\quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & f^{(1)*} & f^{(2)*} \\ f^{(1)} & 0 & 0 \\ f^{(2)} & 0 & 0 \end{pmatrix}$$

$$\Psi_t = (2ikC - \sigma W + \sigma[C, Q(x,t)])\,\Psi \;,$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;,\quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -g \\ 0 & g^* & 0 \end{pmatrix} \;.$$

MODEL EQUATIONS

$$f_t^{(1)} - f_x^{(1)} = -g f^{(2)} \quad , \qquad f_t^{(2)} + f_x^{(2)} = g^* f^{(1)} \,,$$

$$g(x,t) = g_0(t) + \int_a^x dy\; f^{(2)*}(y,t)\; f^{(1)}(y,t)$$

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$$\begin{cases} \Psi_L \to \exp(ikx\sigma) \ , \quad x \to -\infty \\ \Psi_L \to \exp(ikx\sigma) S_L(k,t) \ , \quad x \to +\infty \ . \end{cases}$$

$$\begin{cases} \Psi_R \to \exp(ikx\sigma) \ , \quad x \to +\infty \\ \Psi_R \to \exp(ikx\sigma) S_R(k,t) \ , \quad x \to -\infty \ . \end{cases}$$

$$S_L = (1 + R_L) T_L^{-1} \quad , \quad S_R = (1 + R_R) T_R^{-1} \quad ,$$

$$R_L = \begin{pmatrix} 0 & R_L^{(1)*} & R_L^{(2)*} \\ R_L^{(1)} & 0 & 0 \\ R_L^{(2)} & 0 & 0 \end{pmatrix} \ , \ R_R = \begin{pmatrix} 0 & R_R^{(1)*} & R_R^{(2)*} \\ R_R^{(1)} & 0 & 0 \\ R_R^{(2)} & 0 & 0 \end{pmatrix}$$

$$T_L = \begin{pmatrix} T_{L11} & 0 & 0 \\ 0 & T_{L22} & T_{L23} \\ 0 & T_{L32} & T_{L33} \end{pmatrix} \ , \ T_R = \begin{pmatrix} T_{R11} & 0 & 0 \\ 0 & T_{R22} & T_{R23} \\ 0 & T_{R32} & T_{R33} \end{pmatrix}$$

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INITIAL VALUE PROBLEM

$f^{(1)}(x,0)$ and $f^{(2)}(x,0)$ are given, and also $g(a,t) = g_0(t)$ is given

$$R_{Lt} = [A_+ \, , \, R_L] \quad , \quad R_{Rt} = [A_- \, , \, R_R] \quad ,$$

$$A_\pm = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2ik & -g_\pm \\ 0 & g_\pm^* & 2ik \end{pmatrix}$$

$$g_\pm(t) = g_0(t) + \int_a^{\pm\infty} dx f^{(2)*}(x,t) f^{(1)}(x,t)$$

$$A_- = A_+ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (f^{(2)} \, , \, f^{(1)}) \\ 0 & -(f^{(1)} \, , \, f^{(2)}) & 0 \end{pmatrix}$$

$$(f^{(2)} \, , \, f^{(1)}) = \int_{-\infty}^{+\infty} dx \; f^{(2)*}(x,t) f^{(1)}(x,t)$$

SOLVABLE INITIAL VALUE PROBLEMS

$$a = +\infty \quad , \quad g_+(t) = g_0(t)$$

$$\begin{pmatrix} R_L^{(2)} \\ R_L^{(1)} \end{pmatrix}_t = \begin{pmatrix} 2ik & g_0^*(t) \\ -g_0(t) & -2ik \end{pmatrix} \begin{pmatrix} R_L^{(2)} \\ R_L^{(1)} \end{pmatrix}$$

$$a = -\infty \quad , \quad g_-(t) = g_0(t)$$

$$\begin{pmatrix} R_R^{(2)} \\ R_R^{(1)} \end{pmatrix}_t = \begin{pmatrix} 2ik & g_0^*(t) \\ -g_0(t) & -2ik \end{pmatrix} \begin{pmatrix} R_R^{(2)} \\ R_R^{(1)} \end{pmatrix}$$

A. D. "Integrable nonlocal wave interaction models", Journal of Physics A: Math. Theor. **44** (January 2011) 052002 (7pp)

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THREE REMARKS

1. there exists one family of infinitely (but not sufficiently) many conservation laws
2. this dynamical system is Lagrangian and Hamiltonian
3. the motion of one soliton can be investigated also in the case $|a| < \infty$ and $g_0(t) = 0$ in terms of elliptic integrals