Approximation of smooth functions by weighted means of N-point Padé approximants

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Let f be an analytic function having the power expansions at N different real points

$$-R < x_1 < x_2 < \ldots < x_N < \infty, \quad j = 1, \ldots, N : \qquad \sum_{k=0}^{p_j-1} c_k(x_j)(x-x_j)^k + O\Big((x-x_j)^{p_j}\Big)$$

$$[m/n]^{p_1 p_2 \ldots p_N}_{x_1 x_2 \ldots x_N}(x) = \frac{a_0 + a_1 x + \ldots + a_m x^m}{1 + b_1 x + \ldots + b_n x^n}, \qquad m+n+1 = p = p_1 + p_2 + \ldots + p_N$$

$$j = 1, 2, \ldots N : \qquad f(x) - [m/n](x) = O\Big((x-x_j)^{p_j}\Big),$$

where each p_j represents the number of coefficients $c_k(x_j)$ of expansion actually used for the computation of NPA.

$$L(x) = \sum_{j=1}^{N} p_j H(x-x_j), \quad L(x_k) = p_1 + p_2 + \ldots + p_k, \quad L(x_N) = p = \sum_{j=1}^{N} p_j.$$

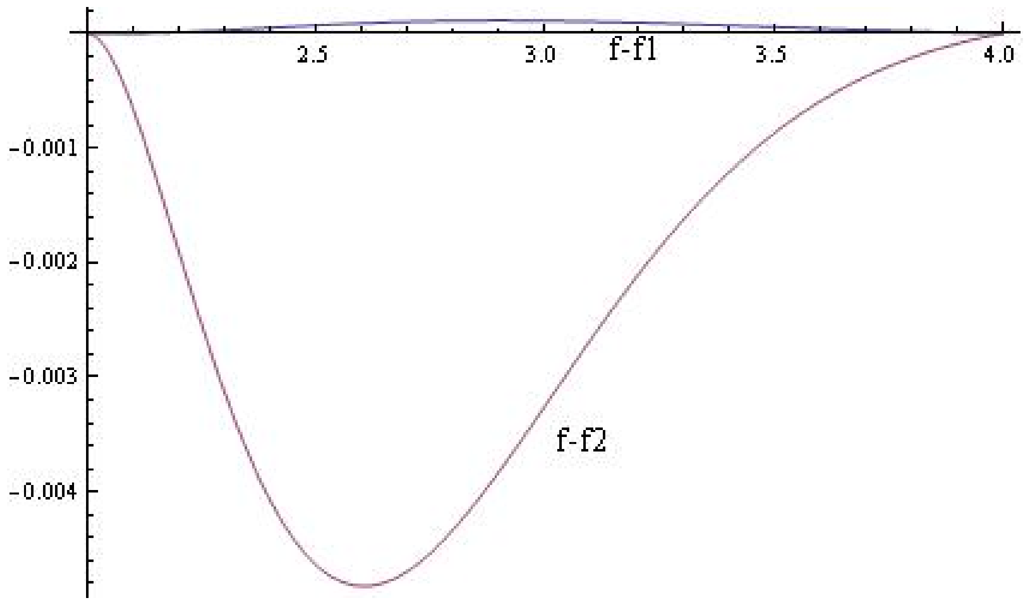
The value $L(x)$ denotes the total number of given coefficients of power series expansions of f

at all points $x_j \le x$.

Michael Barnsley theorem:

The NPA $[k-1/k]$ and $[k/k]$ to the Stieltjes function s obey the following inequality:

$$x \in]-R, \infty[: \qquad (-1)^{L(x)}[m/n](x) \le (-1)^{L(x)} s(x).$$



Two-sided estimates of s: Barnsley theorem

The weighted means must satisfy in each interval $[x_i, x_{i+1}]$ the following condition:

$$i = 1, 2, \ldots, N-1 \quad \forall x \in [x_i, x_{i+1}] : \quad |s - (\alpha_i s_1 + (1 - \alpha_i) s_2)| < |s - s_1|.$$

Suppose we wish to approximate some smooth function f on the interval $[x_1, x_N]$, such as the function s, knowing $p_1 > 1, p_2, \ldots, p_N$ coefficients of expansion of f at the points $x_1, x_2, \ldots, x_N$. Additionally we want to rescale the auxiliary known function s such that

$$s(x_1) = f(x_1) \quad \text{and} \quad s(x_N) = f(x_N).$$

The NPA f_1 and f_2 of f behave like s_1 and s_2, e.g. they are located on the opposite sides of f in each subinterval. Using the weights α_i of auxiliary function s we compute in each interval the means:

$$i = 1, 2, \ldots, N-1 : \quad m_i(x) = \alpha_i f_1(x) + (1 - \alpha_i) f_2(x), \quad x \in [x_i, x_{i+1}]$$

or, we compute the mean of all weights and the global approximation:

$$\alpha = (\alpha_1 + \alpha_2 + \ldots + \alpha_{N-1})/(N-1), \quad m = \alpha f_1 + (1 - \alpha) f_2.$$

Rescaling

To conserve the Stieltjes character of s one must restrict the transformation of a Stieltjes function h to the following:

$$s(x) = c \times h(ax + b).$$

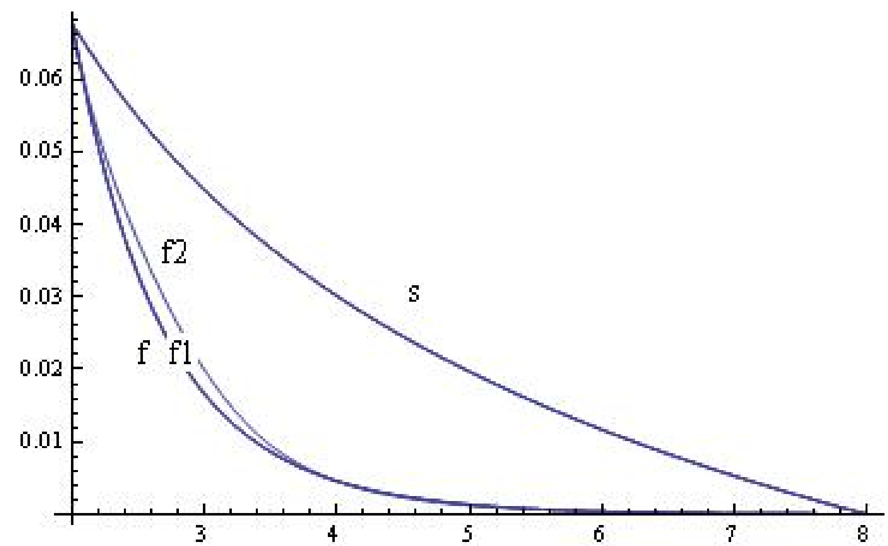
Here we have a degree of flexibility: the three parameters subject to only two conditions $s(x_1) = f(x_1)$ and $s(x_N) = f(x_N)$. Unfortunately in some cases the calculated parameters a, b, c are extremely large. In such cases, we may select only one condition listed previously, or give up the Stieltjes character of s. In particular we have used the following rescaling:

$$s(x) = (ax + b) \times h(cx + d).$$

To obtain a good approximation of f using the weights of s we require that s be as close as possible to f. For example, we can use some global condition minimizing the distances between s and f. We can also use a simple condition in one point x^* of interval $[x_1, x_N]$:

$$s(x^*) = f(x^*)$$

as long as we know the value $f(x^*)$. In all such cases we also require that function s be relatively simple.

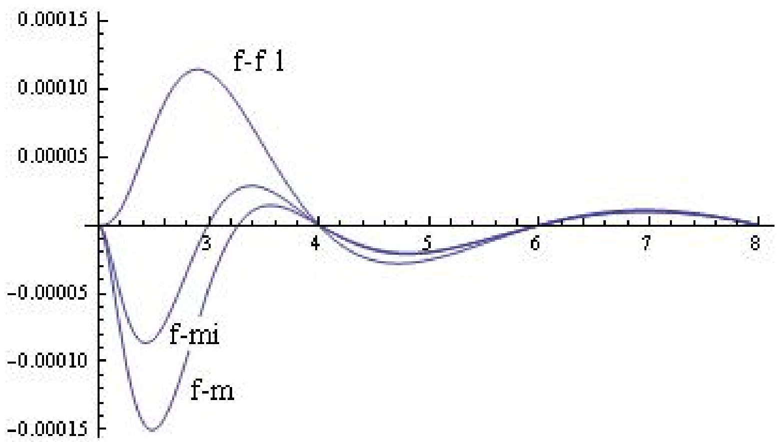


$$f(x) = \frac{e^{-x}}{x} \qquad f_1 = [2/3]^{3111}_{2468} \qquad f_2 = [1/3]^{2111}_{2468}$$

$$s(x) = (-.016247x + .130118)\frac{log(x)}{x-1}$$

Respecting the Stieltjes property the auxiliary function becomes:

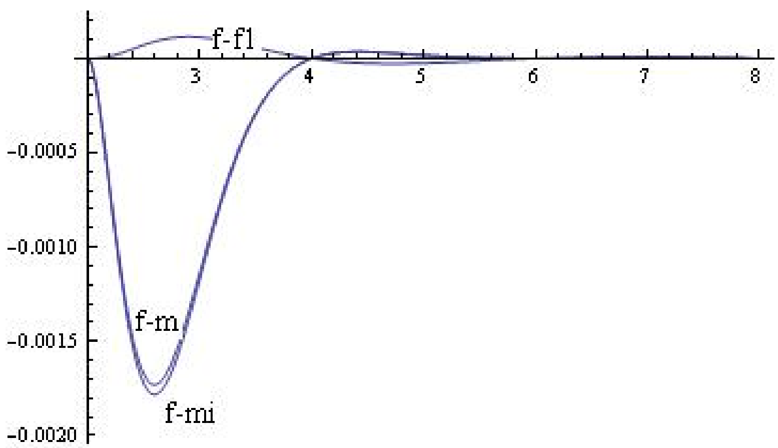
$$s^*(x) = \frac{log(50127x - 100191)}{50127x - 100192}$$



$$\text{Errors } f - f_1, \quad f - m_i, \quad f - m$$

corresponding to the auxiliary function s

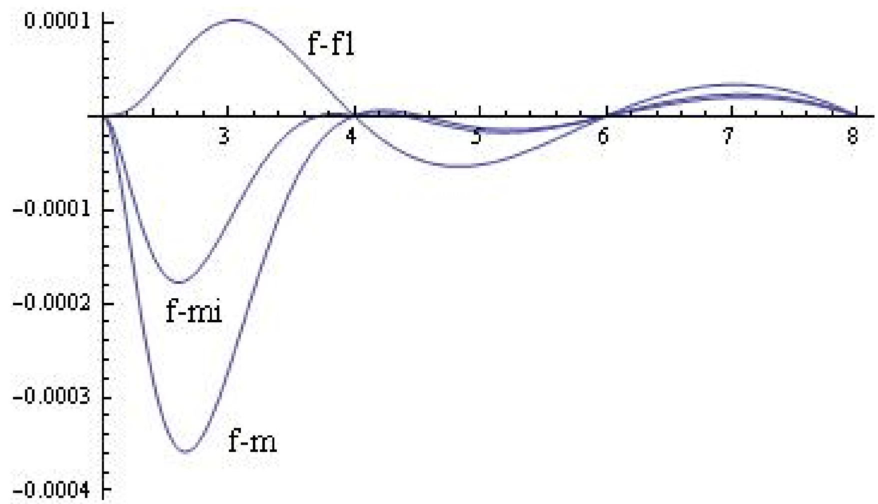
$$m = .967015 f_1 + .032985 f_2$$



$$\text{Errors } f - f_1, \quad f - m_i^*, \quad f - m^*$$

corresponding to the auxiliary function s^*

$$m^* = .629645 f_1 + .370356 f_2$$



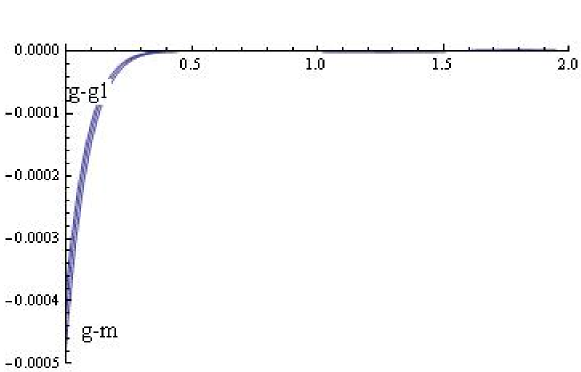
Approximation of $f(x) = e^{-x}$

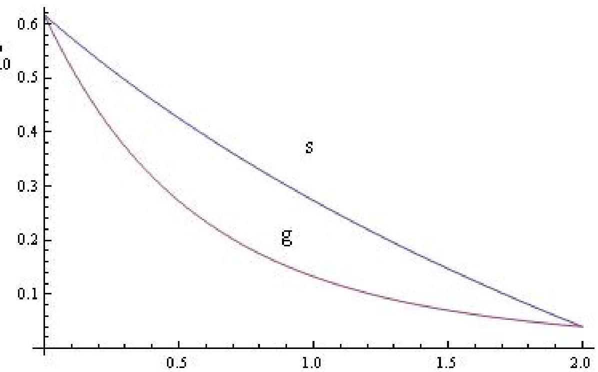
The enormous coefficients in the last rescaled function s suggest the need to restrict the adjustment of s and f to one point $x = 2$ and to determine three coefficients a, b, c in the rescaling $s(x) = c \times h(ax + b)$. The auxiliary function becomes simpler, namely:

$$s(x) = .303 \frac{log(2x)}{2x - 1}.$$

$$f_1 = [2/3], \quad f_2 = [1/3]$$

$$m = .875895 f_1 + .1241045 f_2$$





Approximation of gaussian distribution

The integral of the gaussian distribution of asperity heights x in the tribology model of contact of two surfaces :

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \int_x^\infty (s-x)^{\frac{5}{2}} e^{-\frac{s^2}{2}} ds$$

A number of authors proposed different simple analytical formulas for g based on the tabulated values of g. In the first paper, we demonstrated that the simple Padé approximation approach gives better results than all previously proposed formulas. More, the accuracy of this result was still improved using the presented here method.

Conclusion

Although at the moment the mathematical background of the presented method is incomplete, we are convinced that the proposed idea deserves serious consideration and constitutes an interesting new open problem concerning the methods of the numerical approximation of functions.