Evaluation of minors for weighing matrices $W(n, n-1)$ having zeros on the diagonal

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Motivation of the problem

Determinants and minors are required in

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[1] orthogonal matrices with elements ±1 satisfying $HH^T = nI_n$.

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Determinants and minors are required in

- specification of pivot patterns
- the detection of P matrices
- self validating algorithms
- interval matrix analysis

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High complexity for their computation.

Analytical formulas only for specially structured matrices such as Hadamard [1]or Weighing matrices.

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Minors of Hadamard matrices

Proposition [2]

Let H be a Hadamard matrix of order n.

Then all possible

- $(n-1) \times (n-1)$ minors of H are 0 and $n^{\frac{n}{2}-1}$,
- $(n-2) \times (n-2)$ minors of H are 0 and $2n^{\frac{n}{2}-2}$,
- $(n-3) \times (n-3)$ minors of H are 0 and $4n^{\frac{n}{2}-3}$,
- $(n-4) \times (n-4)$ minors of H are 0, $8n^{\frac{n}{2}-4}$ and $16n^{\frac{n}{2}-4}$.

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[2] C. Koukouvinos, M. Mitrouli and J. Seberry, An algorithm to find formulae and values of minors for Hadamard matrices, $W(n, n-1)$, *Linear Algebra and its Appl.*, **330** (2001), 129-147.

Weighing Matrices

Definitions

A $(0, 1, -1)$ matrix $W = W(n, n-k)$, $k = 1, 2, \ldots,$ of dimension $n \times n$, satisfying

$$W^T W = WW^T = (n-k)I_n,$$

is called a **weighing matrix** of order n and weight $n-k$ or simply a *weighing matrix*.

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$$W^T W = WW^T = (n-k)I_n,$$

is called a **weighing matrix** of order n and weight $n-k$ or simply a *weighing matrix*.

A **conference** matrix C of order n is a $n \times n$ matrix with diagonal entries 0 and all other elements ± 1, satisfying

$$CC^T = (n-1)I_n.$$

Definitions

Two matrices are said to be *Hadamard equivalent* or **H - equivalent** if one can be obtained from the other by a sequence of the operations :

1. interchange any pairs of rows and / or columns
2. multiply any rows and / or columns through by -1.

We will denote the above relation of equivalence by $\sim_H$.

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[3] J.M. Goethals and J.J. Seidel, Orthogonal matrices with zero diagonal, *Canad. J. Math,* Vol. 19 (1967), pp. 1001-1010.

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1. interchange any pairs of rows and / or columns
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We will denote the above relation of equivalence by $\sim_H$.

Lemma [3]

Every weighing matrix $W(n, n-1)$, with n even, is H - equivalent to a conference matrix.

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Properties

1. Every row and column of a $W(n, n-k)$ contains exactly k zeros;
2. Every two distinct rows and columns of a $W(n, n-k)$ are orthogonal to each other, which means that their inner product is zero.

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If $W_{n \times n}$ is a conference matrix, then n is even.

- If $n \equiv 2 (\bmod 4)$, then W is H - equivalent to a symmetric matrix.
- If $n \equiv 0 (\bmod 4)$, then W is H - equivalent to a skew - symmetric matrix.

On the evaluation of minors for weighing matrices $W(n, n-1)$

Minors of Weighing matrices

Proposition [4]

Let W be a weighing matrix $W(n, n-1)$, where n is even.

Then all possible

- $(n-1) \times (n-1)$ minors of W are 0 and $(n-1)^{\frac{n}{2}-1}$,
- $(n-2) \times (n-2)$ minors of W are 0, $(n-1)^{\frac{n}{2}-2}$ and $2(n-1)^{\frac{n}{2}-2}$,
- $(n-3) \times (n-3)$ minors of W are
    - 0, $2(n-1)^{\frac{n}{2}-3}$ or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 0 \pmod 4$ and
    - $2(n-1)^{\frac{n}{2}-3}$ or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 2 \pmod 4$.

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[4] C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, *Linear Algebra Appl.*, **426** (2007), 774-809.

Evaluation of minors for weighing matrices $W(n, n-1)$ having zeros on the diagonal

Notation

$$(0, +, -) \rightarrow (0, 1, -1)$$

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I_n : the identity matrix of order n

$J_{m \times n}$: $m \times n$ matrix with ones

$O_{m \times n}$: $m \times n$ matrix with zeros

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$W(j)$: the absolute value of the minor of any $j \times j$ submatrix of the matrix W.

Notation

We write U_r for all the matrices with r rows and the appropriate number of columns, in which the vector $\tilde{u}_k$ occurs u_k times, $k = 1, 2, \ldots, 2^{r-1}$. So,

$$U_r = \begin{array}{ccccc} \overbrace{+\ldots+}^{u_1} & \overbrace{+\ldots+}^{u_2} & \cdots & \overbrace{+\ldots+}^{u_{2^{r-1}-1}} & \overbrace{+\ldots+}^{u_{2^{r-1}}} \\ +\ldots+ & +\ldots+ & \cdots & -\ldots- & -\ldots- \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ +\ldots+ & +\ldots+ & \cdots & +\ldots+ & -\ldots- \\ +\ldots+ & -\ldots- & \cdots & +\ldots+ & -\ldots- \end{array} = \begin{array}{ccccc} \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_{2^{r-1}-1} & \tilde{u}_{2^{r-1}} \\ + & + & \cdots & + & + \\ + & + & \cdots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ + & + & \cdots & - & - \\ + & - & \cdots & + & - \end{array}$$

Notation

We write U_r for all the matrices with r rows and the appropriate number of columns, in which the vector $\tilde{u}_k$ occurs u_k times, $k = 1, 2, \dots, 2^{r-1}$. So,

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Example:

$$U_3 = \begin{array}{cccc} \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 \\ + & + & + & + \\ + & + & - & - \\ + & - & + & - \end{array} \text{ and } U_4 = \begin{array}{cccccccc} \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 & \tilde{u}_5 & \tilde{u}_6 & \tilde{u}_7 & \tilde{u}_8 \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{array}$$

Minor $W(n-1)$

Proposition

Let W be a weighing matrix $W(n, n-1)$ of order $n > 6$, where n is even, with zeros on the diagonal.
Then,

$$W(n-1) = 0.$$

Minor $W(n-1)$

Proof.

$$W \sim_H \left[\begin{array}{c|ccc} 0 & + & \dots & + \\ \hline + & & & \\ \vdots & & C & \\ + & & & \end{array}\right].$$

Minor $W(n-1)$

Proof.

$$W \sim_H \left[\begin{array}{c|ccc} 0 & + & \dots & + \\ \hline + & & & \\ \vdots & & C & \\ + & & & \end{array}\right].$$

$$CC^T = \begin{bmatrix} n-2 & -1 & \dots & -1 \\ -1 & n-2 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-2 \end{bmatrix}$$

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that is

$$CC^T = (n-1)I_{n-1} - J_{n-1}.$$

Lemma [5]

Suppose A is a $m \times m$ matrix satisfying $A = (k - \lambda)I_m + \lambda J_m$, then

$$det\, A = [k + (m-1)\lambda](k - \lambda)^{m-1}$$

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[5] C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, *Linear Algebra Appl.*, **426** (2007), 774-809.

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Suppose A is a $m \times m$ matrix satisfying $A = (k - \lambda)I_m + \lambda J_m$, then

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So,

$$det\, CC^T = (n-2) - [(n-1) - 1](n-1)^{(n-1)-1} = 0 \Rightarrow det\, C = 0.$$

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Proposition [6]

All possible $(n-1)\times(n-1)$ minors of a weighing matrix $W(n, n-1)$, where n is even, are

$$0 \text{ and } (n-1)^{\frac{n}{2}-1}.$$

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[6] C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, *Linear Algebra Appl.*, **426** (2007), 774-809.

Minor $W(n-2)$

Proposition

Let W be a weighing matrix $W(n, n-1)$ of order $n > 6$, where n is even, with zeros on the diagonal.

Then,

$$W(n-2) = (n-1)^{\frac{n}{2}-2}.$$

Minor $W(n-2)$

Proof.

If $n \equiv 0 \pmod 4$, then

$$W \sim_H \left[\begin{array}{c|c}
\begin{array}{cc} 0 & + \\ + & 0 \end{array} & \begin{array}{cc} \overbrace{+\cdots+}^{u_1} & \overbrace{+\cdots+}^{u_2} \\ +\cdots+ & -\cdots- \end{array} \\ \hline
\begin{array}{r} u_1\left\{\begin{array}{cc} + & - \\ \vdots & \vdots \\ + & - \end{array}\right. \\ u_2\left\{\begin{array}{cc} + & + \\ \vdots & \vdots \\ + & + \end{array}\right. \end{array} & C
\end{array}\right],$$

where $u_1 = u_2 = \frac{n-2}{2}$.

while, if $n \equiv 2 \pmod 4$, then

$$W \sim_H \left[\begin{array}{cc|cc} 0 & + & \overbrace{+\cdots+}^{u_1} & \overbrace{+\cdots+}^{u_2} \\ + & 0 & +\cdots+ & -\cdots- \\ \hline \multicolumn{2}{c|}{u_1\left\{\begin{array}{cc} + & + \\ \vdots & \vdots \\ + & + \end{array}\right.} & \multicolumn{2}{c}{} \\ \multicolumn{2}{c|}{u_2\left\{\begin{array}{cc} + & - \\ \vdots & \vdots \\ + & - \end{array}\right.} & \multicolumn{2}{c}{C} \end{array}\right],$$

where $u_1 = u_2 = \frac{n-2}{2}$.

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In both cases

$$CC^T = \begin{bmatrix} D & O \\ O & D \end{bmatrix},$$

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$$D = \begin{bmatrix} n-3 & -2 & \dots & -2 \\ -2 & n-3 & \dots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \dots & n-3 \end{bmatrix} = (n-1)I_m - 2J_m, \quad m = \frac{n}{2} - 1.$$

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Then,

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Then,

$$det\, D = (n-1)^{\frac{n}{2}-2}.$$

Thus,

$$det\, CC^T = (det\, D)^2 \Rightarrow det\, C = (det\, CC^T)^{\frac{1}{2}} = det\, D = (n-1)^{\frac{n}{2}-2}.$$

Proposition

All possible $(n-2) \times (n-2)$ minors of a weighing matrix $W(n, n-1)$, where n is even, are

$$0, (n-1)^{\frac{n}{2}-2} \text{ and } 2(n-1)^{\frac{n}{2}-2}.$$

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[6] C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, *Linear Algebra Appl.*, **426** (2007), 774-809.

Minor $W(n-3)$

Proposition

Let W be a weighing matrix $W(n, n-1)$ of order $n \geq 8$, where n is even, with zeros on the diagonal.
Then,

$$W(n-3) = 0\,, \quad \text{for } n \equiv 0 (\mathrm{mod}\ 4)$$

and

$$W(n-3) = 2(n-1)^{\frac{n}{2}-3}\,, \quad \text{for } n \equiv 2 (\mathrm{mod}\ 4).$$

Minor $W(n-3)$

Proof for $n \equiv 0 (\bmod 4)$.

$$W \sim_H \left[\begin{array}{cccc|cccc}
 & 0 & + & + & \overbrace{+\cdots+}^{u_1} & \overbrace{+\cdots+}^{u_2} & \overbrace{+\cdots+}^{u_3} & \overbrace{+\cdots+}^{u_4} \\
 & + & 0 & + & +\cdots+ & +\cdots+ & -\cdots- & -\cdots- \\
 & + & - & 0 & +\cdots+ & -\cdots- & +\cdots+ & -\cdots- \\
\hline
u_1\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c}+\\ \vdots\\ +\end{array} & \begin{array}{c}-\\ \vdots\\ -\end{array} & \begin{array}{c}-\\ \vdots\\ -\end{array} & & & & \\
u_2\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c}+\\ \vdots\\ +\end{array} & \begin{array}{c}-\\ \vdots\\ -\end{array} & \begin{array}{c}+\\ \vdots\\ +\end{array} & & & & \\
u_3\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c}+\\ \vdots\\ +\end{array} & \begin{array}{c}+\\ \vdots\\ +\end{array} & \begin{array}{c}-\\ \vdots\\ -\end{array} & & \multicolumn{2}{c}{C} & \\
u_4\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c}+\\ \vdots\\ +\end{array} & \begin{array}{c}+\\ \vdots\\ +\end{array} & \begin{array}{c}+\\ \vdots\\ +\end{array} & & & &
\end{array}\right],$$

where $u_1 = u_2 = u_4 = u$ and $u_3 = u + 1$.

Minor $W(n-3)$

$$CC^T \sim_H \begin{bmatrix} B_{u\times u} & -J_{u\times u} & -J_{u\times(u+1)} & -J_{u\times u} \\ -J_{u\times u} & B_{u\times u} & J_{u\times(u+1)} & J_{u\times u} \\ -J_{(u+1)\times u} & J_{(u+1)\times u} & B_{(u+1)\times(u+1)} & J_{(u+1)\times u} \\ -J_{u\times u} & J_{u\times u} & J_{u\times(u+1)} & B_{u\times u} \end{bmatrix},$$

where $B_{m\times m} = (n-1)I_{m\times m} - 3J_{m\times m}$, with $m = u, u+1$.

Minor $W(n-3)$

$$CC^T \sim_H \begin{bmatrix} B_{u\times u} & -J_{u\times u} & -J_{u\times(u+1)} & -J_{u\times u} \\ -J_{u\times u} & B_{u\times u} & J_{u\times(u+1)} & J_{u\times u} \\ -J_{(u+1)\times u} & J_{(u+1)\times u} & B_{(u+1)\times(u+1)} & J_{(u+1)\times u} \\ -J_{u\times u} & J_{u\times u} & J_{u\times(u+1)} & B_{u\times u} \end{bmatrix},$$

where $B_{m\times m} = (n-1)I_{m\times m} - 3J_{m\times m}$, with $m = u, u+1$.

$$CC^T \sim_H \begin{bmatrix} B_{u\times u} & -J_{u\times u} & -J_{u\times(u+1)} & -J_{u\times u} \\ P_{u\times u} & P_{u\times u} & O_{u\times(u+1)} & O_{u\times u} \\ P_{(u+1)\times u} & O_{(u+1)\times u} & P_{(u+1)\times(u+1)} & O_{(u+1)\times u} \\ P_{u\times u} & O_{u\times u} & O_{u\times(u+1)} & P_{u\times u} \end{bmatrix},$$

where $P_{m\times l} = (n-1)I_{m\times l} - 4J_{m\times l}$, with $m, l = u, u+1$ and

$$P_{(u+1)\times u} = \left[\begin{array}{cccc} \multicolumn{4}{c}{P_{u\times u}} \\ \hline n-5 & -4 & \ldots & -4 \end{array}\right].$$

Schur's Formula [7]

Let us consider the partitioned matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where the submatrix D is assumed to be square and nonsingular.

The Schur complement of D in M, denoted by (M/D), is the matrix

$$(M/D) = A - BD^{-1}C.$$

If M is square, then

$$det\, M = det\, D \cdot det\,(M/D).$$

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[7] C. Brezinski, The Schur Complement in Numerical Analysis, *The Schur Complement and its Applications*, (Editor F. Zhang), Springer, 2004, 227-228.

Minor $W(n-3)$

If we write the matrix CC^T in the form

$$CC^T \equiv M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$A = B_{u\times u}, \quad B = \begin{bmatrix} -J_{u\times u} & -J_{u\times(u+1)} & -J_{u\times u} \end{bmatrix},$$
$$C = \begin{bmatrix} P_{u\times u} \\ P_{(u+1)\times u} \\ P_{u\times u} \end{bmatrix}, D = \begin{bmatrix} P_{u\times u} & O_{u\times(u+1)} & O_{u\times u} \\ O_{(u+1)\times u} & P_{(u+1)\times(u+1)} & O_{(u+1)\times u} \\ O_{u\times u} & O_{u\times(u+1)} & P_{u\times u} \end{bmatrix},$$

Minor $W(n-3)$

If we write the matrix CC^T in the form

$$CC^T \equiv M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$A = B_{u\times u}, \quad B = \begin{bmatrix} -J_{u\times u} & -J_{u\times(u+1)} & -J_{u\times u} \end{bmatrix},$$

$$C = \begin{bmatrix} P_{u\times u} \\ P_{(u+1)\times u} \\ P_{u\times u} \end{bmatrix}, D = \begin{bmatrix} P_{u\times u} & O_{u\times(u+1)} & O_{u\times u} \\ O_{(u+1)\times u} & P_{(u+1)\times(u+1)} & O_{(u+1)\times u} \\ O_{u\times u} & O_{u\times(u+1)} & P_{u\times u} \end{bmatrix},$$

then,

$$BD^{-1}C = \begin{bmatrix} n-4 & -3 & \dots & -3 \\ n-4 & -3 & \dots & -3 \\ \vdots & \vdots & \ddots & \vdots \\ n-4 & -3 & \dots & -3 \end{bmatrix}$$

Minor $W(n-3)$

and $$(M/D) = A - BD^{-1}C = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -(n-7) & n-7 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(n-7) & 0 & \dots & n-7 \end{bmatrix}$$

Minor $W(n-3)$

and $$(M/D) = A - BD^{-1}C = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -(n-7) & n-7 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(n-7) & 0 & \dots & n-7 \end{bmatrix}$$

Hence,

$$det\, M = det\, D \cdot det\,(M/D) = det\, D \cdot 0 = 0$$

Minor $W(n-3)$

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Hence,

$$det\, M = det\, D \cdot det\,(M/D) = det\, D \cdot 0 = 0$$

that is,

$$det\, C = det\,(CC^T)^{\frac{1}{2}} = 0.$$

Proposition [8]

All possible $(n-3) \times (n-3)$ minors of a weighing matrix $W(n, n-1)$, where n is even, are

- $0, 2(n-1)^{\frac{n}{2}-3}$ or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 0 (\mathrm{mod}\ 4)$ and
- $2(n-1)^{\frac{n}{2}-3}$ or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 2 (\mathrm{mod}\ 4)$.

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[8] C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, *Linear Algebra Appl.*, **426** (2007), 774-809.

Minors of higher order

Theorem [9]

Let W be a weighing matrix $W(n, n-1)$ of order $n > 6$, where n is even, with zeros on the diagonal. Then, the $(n-r) \times (n-r)$, $r \geq 1$, minor of W is

$$W(n-r) = [(n-1)^{n-r-2^{r-1}} det\, M]^{1/2},$$

where

$$M = \begin{bmatrix} n-1-ru_1 & u_1 c_{1,2} & u_1 c_{1,3} & \cdots & u_1 c_{1,2^{r-1}} \\ u_2 c_{1,2} & n-1-ru_2 & u_2 c_{2,3} & \cdots & u_2 c_{2,2^{r-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{2^{r-1}} c_{1,2^{r-1}} & u_{2^{r-1}} c_{2,2^{r-1}} & u_{2^{r-1}} c_{3,2^{r-1}} & \cdots & n-1-ru_{2^{r-1}} \end{bmatrix}_{2^{r-1} \times 2^{r-1}},$$

$$c_{i,j} = -\tilde{u}_i^{\,T} \cdot \tilde{u}_j, \quad i, j = 1, \ldots, 2^{r-1}.$$

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[9] A. K., M. Mitrouli, J. Seberry and M. G. Neubauer, An eigenvalue approach evaluating minors for weighing matrices $W(n, n-1)$, under revision in LAA.

Proof.

$$W \sim_H \left[\begin{array}{c|c}
A_{r\times r} & \overbrace{\tilde{u}_1 \ \dots \ \tilde{u}_1}^{u_1} \ \dots \ \overbrace{\tilde{u}_i \ \dots \ \tilde{u}_i}^{u_i} \ \dots \ \overbrace{\tilde{u}_{2^r-1} \ \dots \ \tilde{u}_{2^r-1}}^{u_{2^r-1}} \\
\hline
u_1 \left\{\begin{array}{c} \tilde{u}_1^T \\ \vdots \\ \tilde{u}_1^T \end{array}\right. & \\
\dots & \\
u_i \left\{\begin{array}{c} \tilde{u}_i^T \\ \vdots \\ \tilde{u}_i^T \end{array}\right. & C_{(n-r)\times(n-r)} \\
\dots & \\
u_{2^r-1} \left\{\begin{array}{c} \tilde{u}_{2^r-1}^T \\ \vdots \\ \tilde{u}_{2^r-1}^T \end{array}\right. &
\end{array}\right]$$

$$CC^T = \begin{bmatrix} D_1 & c_{1,2}J & c_{1,3}J & \dots & c_{1,2^{r-1}}J \\ c_{1,2}J^T & D_2 & c_{2,3}J & \dots & c_{2,2^{r-1}}J \\ c_{1,3}J^T & c_{2,3}J^T & D_3 & \dots & c_{3,2^{r-1}}J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1,2^{r-1}}J^T & c_{2,2^{r-1}}J^T & c_{3,2^{r-1}}J^T & \dots & D_{2^{r-1}} \end{bmatrix},$$

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where

$$D_i = \begin{bmatrix} n-r-1 & -r & \dots & -r & -r \\ -r & n-r-1 & \dots & -r & -r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -r & -r & \dots & -r & n-r-1 \end{bmatrix}_{u_i \times u_i}$$

and $c_{i,j} = -\tilde{u}_i{}^T \cdot \tilde{u}_j, \quad i, j = 1, \dots, 2^{r-1}$.

1st Step : Write $v_i = \sum_{j=1}^{i} u_j$. We take column v_i from columns $v_{i-1} + 1$, $v_{i-1} + 2, \ldots, v_{i-1} + u_i - 1$, $i = 1, 2, \ldots, 2^{k-1}$. (we consider $v_0 = 0$).

1st Step : Write $v_i = \sum_{j=1}^{i} u_j$. We take column v_i from columns $v_{i-1}+1$, $v_{i-1}+2, \ldots, v_{i-1}+u_i-1$, $i = 1, 2, \ldots, 2^{k-1}$. (we consider $v_0 = 0$).

Then,

$$D_i \sim \begin{bmatrix} n-1 & 0 & \ldots & 0 & -r \\ 0 & n-1 & \ldots & 0 & -r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & n-1 & -r \\ -n+1 & -n+1 & \ldots & -n+1 & n-r-1 \end{bmatrix}_{u_i \times u_i},$$

$$c_{i,j}J \sim \begin{bmatrix} 0 & 0 & \ldots & c_{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{i,j} \\ 0 & 0 & \ldots & c_{i,j} \end{bmatrix}_{u_i \times u_j}$$

2nd Step : We add rows $v_{i-1}+1$, $v_{i-1}+2$, . . .,$v_{i-1}+u_i-1$ to row v_i, $i=1,2,\ldots,2^{r-1}$.

2nd Step : We add rows $v_{i-1}+1$, $v_{i-1}+2, \ldots, v_{i-1}+u_i-1$ to row v_i, $i = 1, 2, \ldots, 2^{r-1}$.

Then,

$$D_i \sim \begin{bmatrix} n-1 & 0 & \ldots & 0 & -r \\ 0 & n-1 & \ldots & 0 & -r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & n-1 & -r \\ 0 & 0 & \ldots & 0 & n-u_i r-1 \end{bmatrix}_{u_i \times u_i},$$

$$c_{i,j} J \sim \begin{bmatrix} 0 & 0 & \ldots & c_{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{i,j} \\ 0 & 0 & \ldots & u_i c_{i,j} \end{bmatrix}_{u_i \times u_j}$$

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where

$$M = \begin{bmatrix} n-1-ru_1 & u_1 c_{1,2} & u_1 c_{1,3} & \cdots & u_1 c_{1,2^{r-1}} \\ u_2 c_{1,2} & n-1-ru_2 & u_2 c_{2,3} & \cdots & u_2 c_{2,2^{r-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{2^{r-1}} c_{1,2^{r-1}} & u_{2^{r-1}} c_{2,2^{r-1}} & u_{2^{r-1}} c_{3,2^{r-1}} & \cdots & n-1-ru_{2^{r-1}} \end{bmatrix}.$$

Implementation of the algorithm

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requires the computation of inner products of the form $c_{i,j} = -\tilde{u}_i^T \cdot \tilde{u}_j$.

Experimental Results

n	$W(n-1)$	$W(n-2)$	$\mathbf{W(n-3)}$	$W(n-4)$	$W(n-5)$
14	0	$(n-1)^{\frac{n}{2}-2}$	$[(n-1)^7 M_{4\times4}]^{\frac{1}{2}}$	$[(n-1)^3 M_{7\times7}]^{\frac{1}{2}}$	-
16	0	$(n-1)^{\frac{n}{2}-2}$	$0 = [(n-1)^9 M_{4\times4}]^{\frac{1}{2}}$	$[(n-1)^4 M_{8\times8}]^{\frac{1}{2}}$	-
18	0	$(n-1)^{\frac{n}{2}-2}$	$[(n-1)^{11} M_{4\times4}]^{\frac{1}{2}}$	$[(n-1)^7 M_{7\times7}]^{\frac{1}{2}}$	-
20	0	$(n-1)^{\frac{n}{2}-2}$	$0 = [(n-1)^{13} M_{4\times4}]^{\frac{1}{2}}$	$[(n-1)^8 M_{8\times8}]^{\frac{1}{2}}$	$0 = [(n-1)^3 M_{12\times12}]^{\frac{1}{2}}$
24	0	$(n-1)^{\frac{n}{2}-2}$	$0 = \mathbf{[(n-1)^{17} M_{4\times4}]^{\frac{1}{2}}}$	$[(n-1)^{12} M_{8\times8}]^{\frac{1}{2}}$	$0 = [(n-1)^7 M_{12\times12}]^{\frac{1}{2}}$
32	0	$(n-1)^{\frac{n}{2}-2}$	$[(n-1)^{25} M_{4\times4}]^{\frac{1}{2}}$	$[(n-1)^{20} M_{8\times8}]^{\frac{1}{2}}$	$[(n-1)^{12} M_{15\times15}]^{\frac{1}{2}}$

Open research problems

- improvement of the algorithm in order to achieve an even lower complexity
- possible application of the new algorithm in other orthogonal matrices (i.e. binary Hadamard matrices)
- introduction of other methods for the evaluation of minors of matrices (i.e using eigenvalues).