Evaluation of minors for weighing matrices W(n, n - 1)having zeros on the diagonal

Anna Karapiperi, M. Mitrouli, University of Athens, Greece J. Seberry University of Wollongong, Australia

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Determinants and minors are required in

¹orthogonal matrices with elements ±1 satisfying $HH^{T} = nI_{\overline{n}} \rightarrow \langle \overline{a} \rangle \rightarrow \langle \overline{a} \rangle \rightarrow \overline{a} \rightarrow \overline{a} \rightarrow \langle \overline{a} \rangle$

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Determinants and minors are required in

- specification of pivot patterns
- the detection of P matrices
- self validating algorithms
- interval matrix analysis

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High complexity for their computation.

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High complexity for their computation.

Analytical formulas only for specially structured matrices such as Hadamard ¹ or Weighing matrices.

¹orthogonal matrices with elements ±1 satisfying $HH^{T} = nI_{\overline{n}} \rightarrow \langle \overline{a} \rangle \rightarrow \langle \overline{a} \rangle \rightarrow \langle \overline{a} \rangle \rightarrow \langle \overline{a} \rangle \rightarrow \langle \overline{a} \rangle$

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Minors of Hadamard matrices

Proposition ² Let *H* be a Hadamard matrix of order *n*. Then all possible $(n-1) \times (n-1) \text{ minors of } H \text{ are } 0 \text{ and } n^{\frac{n}{2}-1},$ $(n-2) \times (n-2) \text{ minors of } H \text{ are } 0 \text{ and } 2n^{\frac{n}{2}-2},$ $(n-3) \times (n-3) \text{ minors of } H \text{ are } 0 \text{ and } 4n^{\frac{n}{2}-3},$ $(n-4) \times (n-4) \text{ minors of } H \text{ are } 0, 8n^{\frac{n}{2}-4} \text{ and } 16n^{\frac{n}{2}-4}.$

²C. Koukouvinos, M. Mitrouli and J. Seberry, An algorithm to find formulae and values of minors for Hadamard matrices, W(n, n − 1), Linear Algebra and its Appl., **330** (2001), 129-147.

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Weighing Matrices

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A (0, 1, -1) matrix W = W(n, n - k), k = 1, 2, ..., of dimension $n \times n$, satisfying

$$W^{T}W = WW^{T} = (n-k)I_{n},$$

is called a **weighing matrix** of order *n* and weight n - k or simply a weighing matrix.

A (0, 1, -1) matrix W = W(n, n - k), k = 1, 2, ..., of dimension $n \times n$, satisfying

$$W^{T}W = WW^{T} = (n-k)I_{n},$$

is called a **weighing matrix** of order *n* and weight n - k or simply a weighing matrix.

A **conference** matrix *C* of order *n* is a $n \times n$ matrix with diagonal entries 0 and all other elements ± 1 , satisfying

$$CC^T = (n-1)I_n.$$

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Two matrices are said to be *Hadamard equivalent* or **H** - **equivalent** if one can be obtained from the other by a sequence of the operations :

- 1. interchange any pairs of rows and / or columns
- 2. multiply any rows and / or columns through by -1.

We will denote the above relation of equivalence by \sim_{H} .

³ J.M. Goethals and J.J. Seidel, Orthogonal matrices with zero diagonal, Canad. J. Math; Vol. 192(1967), pp. 1001-1010. 🚊 🔗 9, C

Two matrices are said to be *Hadamard equivalent* or **H** - **equivalent** if one can be obtained from the other by a sequence of the operations :

- 1. interchange any pairs of rows and / or columns
- 2. multiply any rows and / or columns through by -1.

We will denote the above relation of equivalence by \sim_{H} .

Lemma

Every weighing matrix W(n, n - 1), with *n* even, is H-equivalent to a conference matrix.

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³ J.M. Goethals and J.J. Seidel, Orthogonal matrices with zero diagonal, Canad. J. Math; Vol. 197(1967), pp. 1001-1010. 🚊 🔗 9, C

Properties

- Every row and column of a W(n, n k) contains exactly k zeros;
- 2 Every two distinct rows and columns of a W(n, n k) are orthogonal to each other, which means that their inner product is zero.

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Properties

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- Every row and column of a W(n, n k) contains exactly k zeros;
- 2 Every two distinct rows and columns of a W(n, n k) are orthogonal to each other, which means that their inner product is zero.
- If $W_{n \times n}$ is a conference matrix, then *n* is even.
 - If $n \equiv 2 \pmod{4}$, then W is H-equivalent to a symmetric matrix.
 - If $n \equiv 0 \pmod{4}$, then W is H-equivalent to a skew-symmetric matrix.

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On the evaluation of minors for weighing matrices W(n, n-1)

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Minors of Weighing matrices

Proposition⁴

Let W be a weighing matrix W(n, n-1), where n is even.

Then all possible

■
$$(n-1) \times (n-1)$$
 minors of W are 0 and $(n-1)^{\frac{n}{2}-1}$,
■ $(n-2) \times (n-2)$ minors of W are 0, $(n-1)^{\frac{n}{2}-2}$ and $2(n-1)^{\frac{n}{2}-2}$,
■ $(n-3) \times (n-3)$ minors of W are
■ $0, 2(n-1)^{\frac{n}{2}-3}$ or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 0 \pmod{4}$ and
■ $2(n-1)^{\frac{n}{2}-3}$ or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 2 \pmod{4}$.

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⁴C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, Linear Algébra Appli; 426 (2007), 774-809. 🕗 🧠 🔿

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$$(0,+,-) \rightarrow (0,1,-1)$$

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$$(0,+,-) \rightarrow (0,1,-1)$$

- I_n : the identity matrix of order n
- $J_{m \times n}$: $m \times n$ matrix with ones
- $O_{m \times n}$: $m \times n$ matrix with zeros

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$$(0,+,-) \rightarrow (0,1,-1)$$

- I_n : the identity matrix of order n
- $J_{m \times n}$: $m \times n$ matrix with ones
- $O_{m \times n}$: $m \times n$ matrix with zeros

W(j): the absolute value of the minor of any $j \times j$ submatrix of the matrix W.

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We write U_r for all the matrices with r rows and the appropriate number of columns, in which the vector \tilde{u}_k occurs u_k times, $k = 1, 2, ..., 2^{r-1}$. So,

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We write U_r for all the matrices with r rows and the appropriate number of columns, in which the vector \tilde{u}_k occurs u_k times, $k = 1, 2, ..., 2^{r-1}$. So,

Example:

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Proposition

Let W be a weighing matrix W(n, n-1) of order n > 6, where n is even, with zeros on the diagonal. Then,

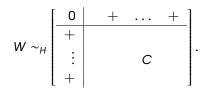
$$W(n-1)=0.$$

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Proof.

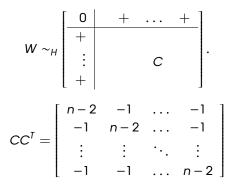


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Proof.

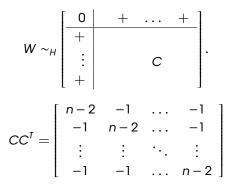


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Proof.



that is

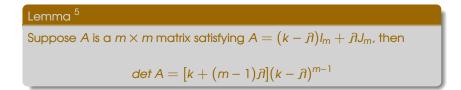
$$CC^{T} = (n-1)I_{n-1} - J_{n-1}.$$

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⁵C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, Linear Algebra Apple 426 (2007), 774-809. 🕗 🔍 🔿

Lemma

Suppose A is a $m \times m$ matrix satisfying $A = (k - j)I_m + jJ_m$, then

det A =
$$[k + (m-1)\beta](k - \beta)^{m-1}$$

So,

det
$$CC^{T} = (n-2) - [(n-1) - 1](n-1)^{(n-1)-1} = 0 \Rightarrow det C = 0.$$

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⁵C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, Linear Algébra Apple 426 (2007), 774-809. 🕗 🔍 🔿

Proposition ⁶

All possible $(n-1) \times (n-1)$ minors of a weighing matrix W(n, n-1), where *n* is even, are

0 and $(n-1)^{\frac{n}{2}-1}$.

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⁶C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, Linear Algebra Apple, 426 (2007), 774-809. 🕗 🔍 🔿

Proposition

Let W be a weighing matrix W(n, n-1) of order n > 6, where n is even, with zeros on the diagonal.

Then,

$$W(n-2) = (n-1)^{\frac{n}{2}-2}.$$

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Proof. If $n \equiv 0 \pmod{4}$, then

where
$$u_1 = u_2 = \frac{n-2}{2}$$

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while, if $n \equiv 2 \pmod{4}$, then

$$W \sim_{H} \left[\begin{array}{c|cccc} 0 & + & & \stackrel{u_{1}}{\longrightarrow} & \stackrel{u_{2}}{\longrightarrow} \\ + & 0 & & \stackrel{+\dots+}{\longrightarrow} & \stackrel{+\dots+}{\longrightarrow} \\ + & 0 & & \stackrel{+\dots+}{\longrightarrow} & \stackrel{-}{\longrightarrow} \\ \\ u_{1} \left\{ \begin{array}{c} + & + & & & \\ \vdots & \vdots & & & \\ + & + & & & \\ & & C & & \\ u_{2} \left\{ \begin{array}{c} + & - & & & \\ \vdots & \vdots & & & \\ + & - & & & \\ \end{array} \right\} \right]$$

where
$$u_1 = u_2 = \frac{n-2}{2}$$
.

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In both cases

$$CC^{\mathsf{T}} = \left[\begin{array}{cc} D & O \\ O & D \end{array} \right],$$

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In both cases

$$CC^{\mathsf{T}} = \left[\begin{array}{cc} D & O \\ O & D \end{array} \right],$$

where

$$D = \begin{bmatrix} n-3 & -2 & \dots & -2 \\ -2 & n-3 & \dots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \dots & n-3 \end{bmatrix} = (n-1)I_m - 2J_m, \quad m = \frac{n}{2} - 1.$$

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In both cases

$$CC^{\mathsf{T}} = \left[\begin{array}{cc} D & O \\ O & D \end{array} \right],$$

where

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Then,

det D =
$$(n-1)^{\frac{n}{2}-2}$$
.

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In both cases

$$CC^{\mathsf{T}} = \left[\begin{array}{cc} D & O \\ O & D \end{array} \right],$$

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Then,

det
$$D = (n-1)^{\frac{n}{2}-2}$$
.

Thus,

$$\det CC^{\mathsf{T}} = (\det D)^2 \Rightarrow \det C = (\det CC^{\mathsf{T}})^{\frac{1}{2}} = \det D = (n-1)^{\frac{n}{2}-2}.$$

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Proposition

All possible $(n-2) \times (n-2)$ minors of a weighing matrix W(n, n-1), where n is even, are

0,
$$(n-1)^{\frac{n}{2}-2}$$
 and $2(n-1)^{\frac{n}{2}-2}$.

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⁶C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, Linear Algebra Apple, 426 (2007), 774-809. 🕗 🔍 🔿

Proposition

Let W be a weighing matrix W(n, n-1) of order $n \ge 8$, where n is even, with zeros on the diagonal. Then,

$$W(n-3) = 0$$
, for $n \equiv 0 \pmod{4}$

and

$$W(n-3) = 2(n-1)^{\frac{n}{2}-3}$$
, for $n \equiv 2 \pmod{4}$.

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Proof for $n \equiv 0 \pmod{4}$.

	[0 + +	+ 0 -	+ + 0	$\overbrace{+\cdots+}^{u_1}\\+\cdots+\\+\cdots+$	$\overbrace{+\cdots+}^{u_2}\\+\cdots+\\+\cdots+\\-\cdots-$	$\overbrace{+\cdots+}^{u_3}\\+\cdots+\\+\cdots+$	$\overbrace{+\cdots+}^{u_4}$	
W ∼ _H	$u_1 \begin{cases} u_2 \\ u_2 \\ u_3 \\ u_4 \end{cases}$	$+ \cdots + + + + + \cdots + + + + + \cdots +$	+++			(2		,

where $u_1 = u_2 = u_4 = u$ and $u_3 = u + 1$.

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$$CC^{T} \sim_{H} \begin{bmatrix} B_{u \times u} & -J_{u \times u} & -J_{u \times (u+1)} & -J_{u \times u} \\ -J_{u \times u} & B_{u \times u} & J_{u \times (u+1)} & J_{u \times u} \\ -J_{(u+1) \times u} & J_{(u+1) \times u} & B_{(u+1) \times (u+1)} & J_{(u+1) \times u} \\ -J_{u \times u} & J_{u \times u} & J_{u \times (u+1)} & B_{u \times u} \end{bmatrix},$$

where $B_{m \times m} = (n-1)I_{m \times m} - 3J_{m \times m}$, with m = u, u + 1.

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$$CC^{T} \sim_{H} \begin{bmatrix} B_{u \times u} & -J_{u \times u} & -J_{u \times (u+1)} & -J_{u \times u} \\ -J_{u \times u} & B_{u \times u} & J_{u \times (u+1)} & J_{u \times u} \\ -J_{(u+1) \times u} & J_{(u+1) \times u} & B_{(u+1) \times (u+1)} & J_{(u+1) \times u} \\ -J_{u \times u} & J_{u \times u} & J_{u \times (u+1)} & B_{u \times u} \end{bmatrix},$$

where $B_{m \times m} = (n-1)I_{m \times m} - 3J_{m \times m}$, with m = u, u+1.

$$CC^{T} \sim_{H} \begin{bmatrix} B_{u \times u} & -J_{u \times u} & -J_{u \times (u+1)} & -J_{u \times u} \\ P_{u \times u} & P_{u \times u} & O_{u \times (u+1)} & O_{u \times u} \\ P_{(u+1) \times u} & O_{(u+1) \times u} & P_{(u+1) \times (u+1)} & O_{(u+1) \times u} \\ P_{u \times u} & O_{u \times u} & O_{u \times (u+1)} & P_{u \times u} \end{bmatrix},$$

where
$$P_{m \times l} = (n-1)l_{m \times l} - 4J_{m \times l}$$
, with $m, l = u, u+1$ and
 $P_{(u+1) \times u} = \left[\frac{P_{u \times u}}{n-5 - 4 \dots - 4} \right].$

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Schur's Formula⁷

Let us consider the partitioned matrix
$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
, where the submatrix

D is assumed to be square and nonsingular.

The Schur complement of D in M, denoted by (M/D), is the matrix

$$(M/D) = A - BD^{-1}C.$$

If M is square, then

det
$$M = \det D \cdot \det (M/D)$$
.

⁷C. Brezinski, The Schur Complement in Numerical Analysis, The Schur Complement and its Applications, (Editor F. Zhang), Springer, 2004, 227-228.

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If we write the matrix CC^{T} in the form

$$CC^T \equiv M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$\begin{split} A &= B_{u \times u}, \quad B = \begin{bmatrix} -J_{u \times u} & -J_{u \times (u+1)} & -J_{u \times u} \end{bmatrix}, \\ C &= \begin{bmatrix} P_{u \times u} \\ P_{(u+1) \times u} \\ P_{u \times u} \end{bmatrix}, \quad D = \begin{bmatrix} P_{u \times u} & O_{u \times (u+1)} & O_{u \times u} \\ O_{(u+1) \times u} & P_{(u+1) \times (u+1)} & O_{(u+1) \times u} \\ O_{u \times u} & O_{u \times (u+1)} & P_{u \times u} \end{bmatrix}, \end{split}$$

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If we write the matrix CC^{T} in the form

$$CC^T \equiv M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$\begin{split} A &= B_{u \times u}, \quad B = \begin{bmatrix} -J_{u \times u} & -J_{u \times (u+1)} & -J_{u \times u} \end{bmatrix}, \\ C &= \begin{bmatrix} P_{u \times u} & P_{u \times u} & O_{u \times (u+1)} & O_{u \times u} \\ P_{(u+1) \times u} & P_{(u+1) \times (u+1)} & O_{(u+1) \times u} \\ P_{u \times u} & O_{u \times u} & O_{u \times (u+1)} & P_{u \times u} \end{bmatrix}, \end{split}$$

then,

$$BD^{-1}C = \begin{bmatrix} n-4 & -3 & \dots & -3 \\ n-4 & -3 & \dots & -3 \\ \vdots & \vdots & \ddots & \vdots \\ n-4 & -3 & \dots & -3 \end{bmatrix}$$

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and
$$(M/D) = A - BD^{-1}C = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -(n-7) & n-7 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(n-7) & 0 & \dots & n-7 \end{bmatrix}$$

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and
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Hence,

$$\det M = \det D \cdot \det (M/D) = \det D \cdot 0 = 0$$

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and
$$(M/D) = A - BD^{-1}C = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -(n-7) & n-7 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(n-7) & 0 & \dots & n-7 \end{bmatrix}$$

Hence,

$$\det M = \det D \cdot \det (M/D) = \det D \cdot 0 = 0$$

that is,

det
$$C = det (CC^{T})^{\frac{1}{2}} = 0.$$

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Proposition⁸

All possible $(n-3) \times (n-3)$ minors of a weighing matrix W(n, n-1), where *n* is even, are

• $0, 2(n-1)^{\frac{n}{2}-3}$ or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 0 \pmod{4}$ and

■
$$2(n-1)^{\frac{n}{2}-3}$$
 or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 2 \pmod{4}$.

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⁸C. Kravvaritis, and M. Mitrouli, Evaluation of minors associated to weighing matrices, Linear Algebra Appl., 426 (2007), 774-809. 🤄 🔍

Minors of higher order

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Theorem ⁹

Let W be a weighing matrix W(n, n-1) of order n > 6, where n is even, with zeros on the diagonal. Then, the $(n-r) \times (n-r)$, $r \ge 1$, minor of W is

$$W(n-r) = [(n-1)^{n-r-2^{r-1}} \det M]^{1/2},$$

where

$$M = \begin{bmatrix} n-1-ru_1 & u_1c_{1,2} & u_1c_{1,3} & \cdots & u_1c_{1,2^{r-1}} \\ u_2c_{1,2} & n-1-ru_2 & u_2c_{2,3} & \cdots & u_2c_{2,2^{r-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{2^{r-1}}c_{1,2^{r-1}} & u_{2^{r-1}}c_{2,2^{r-1}} & u_{2^{r-1}}c_{3,2^{r-1}} & \cdots & n-1-ru_{2^{r-1}} \end{bmatrix}_{2^{r-1}\times 2^{r-1}},$$

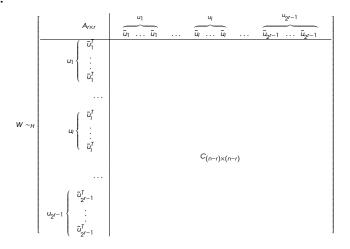
⁹A. K., M. Mitrouli, J. Seberry and M. G. Neubauer, An eigenvalue approach evaluating minors for weighing matrices *W*(*n*, *n* − 1), under revision in LAA.

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Proof.



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Evaluation of minors for weighing matrices W(n, n-1) having zeros on the diagonal

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$$CC^{T} = \begin{bmatrix} D_{1} & c_{1,2}J & c_{1,3}J & \dots & c_{1,2^{r-1}}J \\ c_{1,2}J^{T} & D_{2} & c_{2,3}J & \dots & c_{2,2^{r-1}}J \\ c_{1,3}J^{T} & c_{2,3}J^{T} & D_{3} & \dots & c_{3,2^{r-1}}J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1,2^{r-1}}J^{T} & c_{2,2^{r-1}}J^{T} & c_{3,2^{r-1}}J^{T} & \dots & D_{2^{r-1}} \end{bmatrix},$$

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$$CC^{T} = \begin{bmatrix} D_{1} & c_{1,2}J & c_{1,3}J & \dots & c_{1,2^{r-1}}J \\ c_{1,2}J^{T} & D_{2} & c_{2,3}J & \dots & c_{2,2^{r-1}}J \\ c_{1,3}J^{T} & c_{2,3}J^{T} & D_{3} & \dots & c_{3,2^{r-1}}J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1,2^{r-1}}J^{T} & c_{2,2^{r-1}}J^{T} & c_{3,2^{r-1}}J^{T} & \dots & D_{2^{r-1}} \end{bmatrix},$$

where

$$D_{i} = \begin{bmatrix} n-r-1 & -r & \dots & -r & -r \\ -r & n-r-1 & \dots & -r & -r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -r & -r & \dots & -r & n-r-1 \end{bmatrix}_{u_{i} \times u_{i}}$$

and $c_{i,i} = -\tilde{u}_i^T \cdot \tilde{u}_i$, $i, j = 1, ..., 2^{r-1}$.

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Evaluation of minors for weighing matrices W(n, n-1) having zeros on the diagonal

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1st Step : Write $v_i = \sum_{j=1}^{i} u_j$. We take column v_i from columns $v_{i-1} + 1$, $v_{i-1} + 2, \ldots, v_{i-1} + u_i - 1$, $i = 1, 2, \ldots, 2^{k-1}$. (we consider $v_0 = 0$).

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Evaluation of minors for weighing matrices W(n, n-1) having zeros on the diagonal

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. We take column v_i from columns $v_{i-1} + 1$, $v_{i-1} + 2, ..., v_{i-1} + u_i - 1, i = 1, 2, ..., 2^{k-1}$. (we consider $v_0 = 0$).

Then,

$$D_{i} \sim \begin{bmatrix} n-1 & 0 & \dots & 0 & -r \\ 0 & n-1 & \dots & 0 & -r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n-1 & -r \\ -n+1 & -n+1 & \dots & -n+1 & n-r-1 \end{bmatrix}_{u_{i} \times u_{i}},$$

$$c_{i,j}J \sim \begin{bmatrix} 0 & 0 & \dots & c_{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{i,j} \\ 0 & 0 & \dots & c_{i,j} \\ 0 & 0 & \dots & c_{i,j} \end{bmatrix}_{u_{i} \times u_{i}}$$

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2nd Step: We add rows $v_{i-1} + 1$, $v_{i-1} + 2$, ..., $v_{i-1} + u_i - 1$ to row v_i , $i = 1, 2, ..., 2^{r-1}$.

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2nd Step: We add rows $v_{i-1} + 1$, $v_{i-1} + 2$, ..., $v_{i-1} + u_i - 1$ to row v_i , $i = 1, 2, ..., 2^{r-1}$.

Then,

$$D_{l} \sim \begin{bmatrix} n-1 & 0 & \dots & 0 & -r \\ 0 & n-1 & \dots & 0 & -r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n-1 & -r \\ 0 & 0 & \dots & 0 & n-u_{l}r-1 \end{bmatrix}_{u_{l} \times u_{l}},$$

$$c_{i,j}J \sim \begin{bmatrix} 0 & 0 & \dots & c_{l,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l}c_{i,j} \\ 0 & 0 & \dots & u_{l}c_{i,j} \end{bmatrix}_{u_{l} \times u_{j}}$$

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3nd Step: We expand this determinant, using the basic definition of the determinant, pivoting using the columns with a single non-zero entry.

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3nd Step: We expand this determinant, using the basic definition of the determinant, pivoting using the columns with a single non-zero entry.

So, we have

$$\det CC^{T} = (n-1)^{n-r-2^{r-1}} \det M$$

and

det
$$C = [(n-1)^{n-r-2^{r-1}} \det M]^{\frac{1}{2}}$$
,

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Implementation of the algorithm

This new algorithm

has lower complexity than an algebraic computing program

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Implementation of the algorithm

This new algorithm

has lower complexity than an algebraic computing program

requires the computation of inner products of the form $c_{i,j} = -\tilde{u}_i^T \cdot \tilde{u}_j$.

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Experimental Results

n	W(n-1)	W(n-2)	W(n – 3)	W(n-4)	W(n - 5)
14	0	$(n-1)^{\frac{n}{2}-2}$	$[(n-1)^7 M_{4\times 4}]^{\frac{1}{2}}$	$[(n-1)^3 M_{7\times 7}]^{\frac{1}{2}}$	-
16	0	$(n-1)^{\frac{n}{2}-2}$	$0 = [(n-1)^9 M_{4\times 4}]^{\frac{1}{2}}$	$[(n-1)^4 M_{8\times 8}]^{\frac{1}{2}}$	-
18	0	$(n-1)^{\frac{n}{2}-2}$	$[(n-1)^{11}M_{4\times 4}]^{\frac{1}{2}}$	$[(n-1)^7 M_{7\times7}]^{\frac{1}{2}}$	-
20	0	$(n-1)^{\frac{n}{2}-2}$	$0 = [(n-1)^{13}M_{4\times 4}]^{\frac{1}{2}}$	$[(n-1)^8 M_{8\times 8}]^{\frac{1}{2}}$	$0 = [(n-1)^3 M_{12 \times 12}]^{\frac{1}{2}}$
24	0	$(n-1)^{\frac{n}{2}-2}$	$0 = [(n-1)^{17}M_{4\times 4}]^{\frac{1}{2}}$	$[(n-1)^{12}M_{8\times 8}]^{\frac{1}{2}}$	$0 = [(n-1)^7 M_{12 \times 12}]^{\frac{1}{2}}$
32	0	$(n-1)^{\frac{n}{2}-2}$	$[(n-1)^{25}M_{4\times 4}]^{\frac{1}{2}}$	$[(n-1)^{20}M_{8\times 8}]^{\frac{1}{2}}$	$[(n-1)^{12}M_{15\times 15}]^{\frac{1}{2}}$

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Open research problems

- improvement of the algorithm in order to achieve an even lower complexity
- possible application of the new algorithm in other orthogonal matrices (i.e. binary Hadamard matrices)
- introduction of other methods for the evaluation of minors of matrices (i.e using eigenvalues).

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