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A radial basis function based partition of unity method for solving PDE problems

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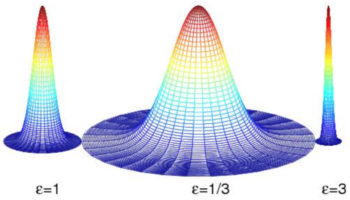
Outline

RBF approximation

Basis functions: $\phi_j(\underline{x}) = \phi(\varepsilon\|\underline{x} - \underline{x}_j\|)$. Translates of one single function rotated around a center point.

Example: Gaussians
$\phi(\varepsilon r) = \exp(-\varepsilon^2 r^2)$

Approximations:
$u_\varepsilon(\underline{x}) = \sum_{j=1}^{N} \lambda_j \phi_j(\underline{x})$



Why is it a good idea to use RBFs?

- RBF methods are meshfree.
- Spectral accuracy for smooth solutions.
- Only distances between points are needed.

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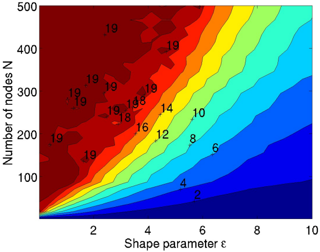
Practical challenges in RBF approximation

Conditioning for small ε and large N

- Spectral convergence with N requires fixed ε.
- For smooth solutions, the best ε is small.
- We need to compute in the red region.

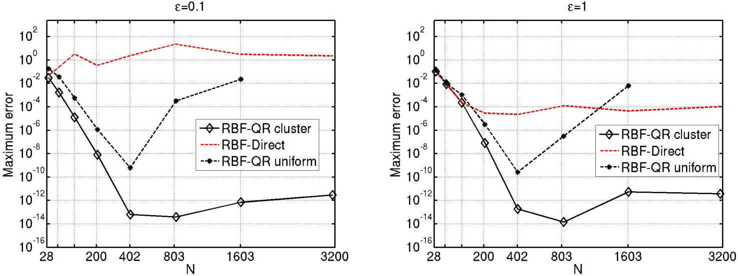
$\log_{10}(\mathrm{cond}(A))$



Computational cost

- Coefficient matrices are typically dense (for infinitely smooth RBFs).
- Direct methods are $\mathcal{O}(N^3)$ and known fast methods are most efficient for larger ε.

Stable computation as $\varepsilon \to 0$ and $N \to \infty$



The RBF-QR method allows stable computations for small ε. *(Fornberg, Larsson, Flyer 2011)*

Consider a finite non-periodic domain.

Theorem (Platte, Trefethen, and Kuijlaars 2010):

Exponential convergence on equispaced nodes $\Rightarrow$ exponential ill-conditioning.

Solution #1:

Cluster nodes towards the domain boundaries.

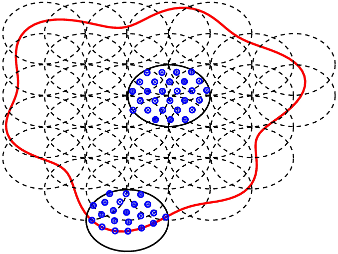
Solution #2: A partition of unity RBF collocation approach for PDEs

(RBF-PU for interpolation Wendland 2002)

Global approximant:

$$\tilde{u}(\underline{x}) = \sum_{i=1}^{M} w_i(\underline{x}) u_i(\underline{x}),$$

where $w_i(\underline{x})$ are weight functions.



Local RBF approximants:

$$u_i(\underline{x}) = \sum_{j=1}^{N_i} \lambda_j^i \phi_j(\underline{x}).$$

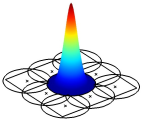
Applying operators:

$$\Delta \tilde{u}(\underline{x}) = \sum_{i=1}^{M} \Delta w_i u_i + 2\nabla w_i \cdot \nabla u_i + w_i \Delta u_i$$

- Sparsity reduces memory and computational cost.
- Subdomain approach introduces parallelism.

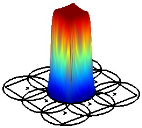
Approximation details

Weight functions



Generate weight functions from compactly supported C^2 Wendland functions

$$\psi(\rho) = (4\rho + 1)(1 - \rho)_+^4$$



using Shepard's method $w_i(\underline{x}) = \frac{\psi_i(\underline{x})}{\sum_{j=1}^{M} \psi_j(\underline{x})}$.

Local RBF differentiation matrices

Define the vector of local nodal solution values $\underline{u}_i$, then

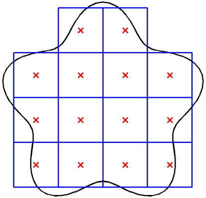
$$\underline{u}_i = A\underline{\lambda}^i, \text{ where } A_{ij} = \phi_j(\underline{x}_i).$$

Applying a linear operator yields

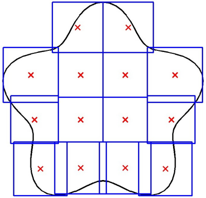
$$\mathcal{L}\underline{u}_i = B\lambda_i = BA^{-1}\underline{u}_i, \text{ where } B_{ij} = \mathcal{L}\phi_j(\underline{x}_i).$$

Covering the domain

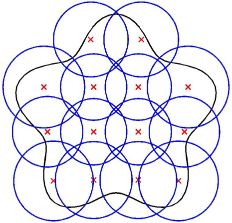
Cover Ω with boxes (practical).
Retain boxes with center inside Ω.



Enlarge boxes near boundary to cover all of Ω.



Create overlapping partitions based on the boxes.

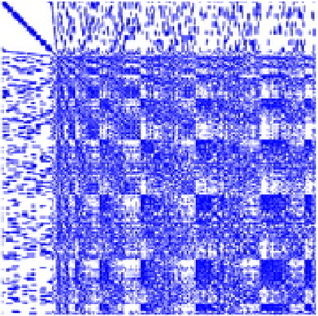


Dealing with cost: Solving the linear system

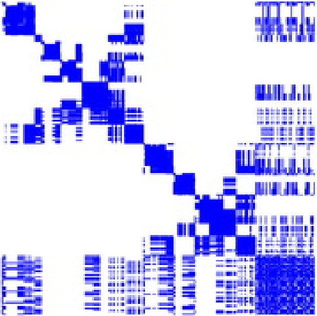
Plan: Parallel software, exploiting partition structure.

Done: Weights and local RBF-matrices in parallel.

Ongoing work: Domain decomposition & iterative solver with A. Ramage and L. von Sydow.



Original sparse unstructured matrix



After approximate minimal degree reordering

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Theoretical convergence results: Definitions

Let $H_j = \mathrm{diam}(\Omega_j)$ and define the *fill distance* $h_j = \sup_{x\in\Omega_j} \min_{y\in X_j} \|x-y\|_2 = C\frac{H_j}{N_j^{1/d}}$.

Write the exact solution as $u(\underline{x}) \equiv \sum_{j=1}^{M} w_j(\underline{x})u(\underline{x})$ and introduce the partition of unity interpolant

$$I_h u(\underline{x}) \equiv \sum_{j=1}^{M} w_j(\underline{x}) I_h u_j(\underline{x}).$$

If we have

- Ellipticity $\|u\|_U \le C_\Omega \|L(u)\|_F, \quad \forall u \in U$.
- Stability of trial space and test discretization.
- Small discrete residual $\sim \kappa(A)\epsilon_M$.

Then the crucial component affecting convergence is the approximation properties of the trial space in the form $\|\mathcal{L}(u - I_h(u))\| = \|\sum_{j=1}^{M} \Delta w_j e_j + \nabla w_j \cdot \nabla e_j + w_j \Delta e_j\|$

R. Schaback, Convergence of unsymmetric kernel-based meshless collocation methods, SINUM, 2007.

Algebraic convergence in partition size

Keeping N_i fixed while varying H leads to $h = CH$. We have the following estimates

$$\|w_j\| \le C_0, \qquad \|e_j\|_\infty \le Ch_j^{m_j-0-\frac{d}{2}}\|u\|_{\mathcal{N}(\Omega)},$$
$$\|\nabla w_j\| \le \frac{C_1}{H_j}, \qquad \|\nabla e_j\|_\infty \le Ch_j^{m_j-1-\frac{d}{2}}\|u\|_{\mathcal{N}(\Omega)},$$
$$\|\Delta w_j\| \le \frac{C_2}{H_j^2}, \qquad \|\Delta e_j\|_\infty \le Ch_j^{m_j-2-\frac{d}{2}}\|u\|_{\mathcal{N}(\Omega)},$$

leading to

$$\|\mathcal{L}(u - I_h(u))\| \le qCH^{m-2-\frac{d}{2}}\|u\|_{\mathcal{N}(\Omega)},$$

where q is max # partitions that overlap at a given point.

$$\dim(\mathcal{P}_K) \le N_j < \dim(\mathcal{P}_{K+1}) \quad \Rightarrow m_j = K + 1$$

- A single small N_i reduces the global convergence order.

Spectral convergence with fill distance

Now, we keep H_i fixed while increasing N_i, which reduces h_i. We use another form for the estimates *(Rieger and Zwicknagl 2008)*

$$\|w_j\| \leq \tilde{C}_0, \qquad \|e_j\| \leq 2e^{c_0 \log(h_j)/\sqrt{h_j}} \|u\|_{\mathcal{N}(\Omega)},$$
$$\|\nabla w_j\| \leq \tilde{C}_1, \qquad \|\nabla e_j\| \leq 2e^{c_1 \log(h_j)/\sqrt{h_j}} \|u\|_{\mathcal{N}(\Omega)},$$
$$\|\Delta w_j\| \leq \tilde{C}_2, \qquad \|\Delta e_j\| \leq 2e^{c_2 \log(h_j)/\sqrt{h_j}} \|u\|_{\mathcal{N}(\Omega)},$$

leading to

$$\|\mathcal{L}(u - I_h(u))\| \leq qCe^{c \log(h)/\sqrt{h}} \|u\|_{\mathcal{N}(\Omega)}.$$

- In both cases, the estimates require fixed shape, $\varepsilon = \varepsilon_0$, leading to ill-conditioning either due to effective shape parameter $\varepsilon H \to 0$ or $N \to \infty$.
- The RBF-QR method or an alternative is required.

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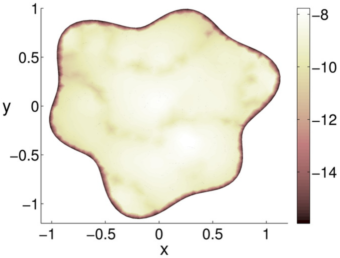
Simple Poisson test problem

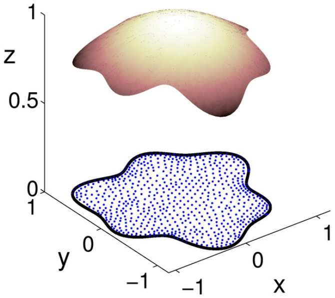
Domain defined by: $r_b(\theta) = 1 + \frac{1}{10}(\sin(6\theta) + \sin(3\theta))$.

PDE: $\begin{cases} \Delta u = f(\underline{x}), & \underline{x} \in \Omega, \\ u = g(\underline{x}), & \underline{x} \text{ on } \partial\Omega, \end{cases}$ with $u(r,\theta) = \frac{1}{0.25r^2+1}$.

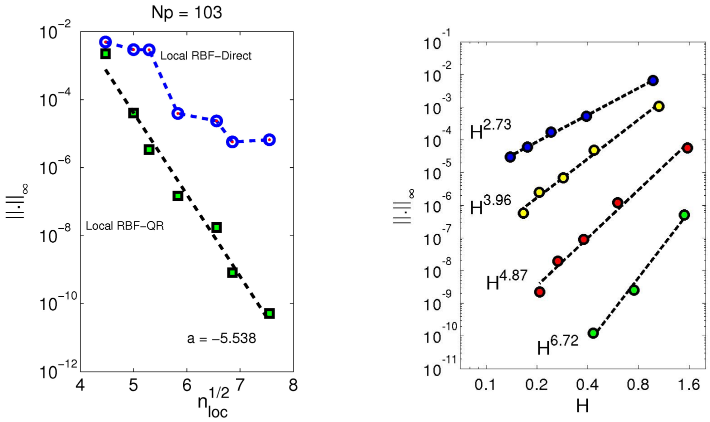
log$_{10}$(error)



RBF-PU solution



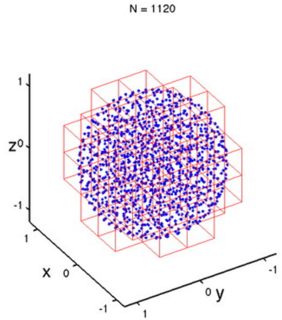
Numerical convergence results with RBF-PU

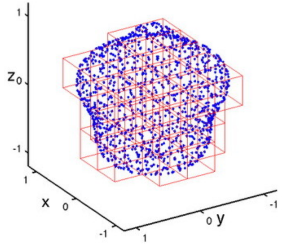


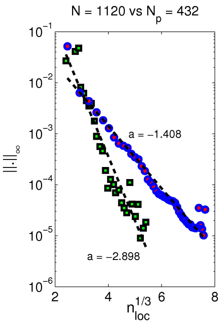
Increasing the number of local points for fixed number of partitions ⇒ Spectral convergence.

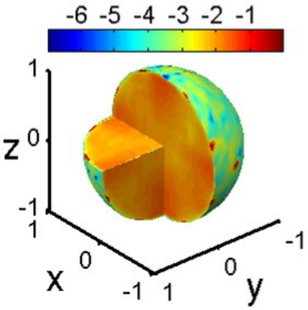
Increasing the number of partitions for fixed n_{loc} (21, 28, 45, 66) ⇒ Algebraic convergence.

Example in 3-D









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Summary

- RBF methods are attractive because they are easy to implement, meshfree, and highly accurate.
- The shape parameter ε has a significant influence on the accuracy of the methods.
- The RBF-QR method removes the ill-conditioning associated with small ε in up to three dimensions.
- Partition of unity RBF methods provide a promising way to reduce computational cost while maintaining high order of accuracy.