A radial basis function based partition of unity method for solving PDE problems

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Outline

An introduction to RBFs

The RBF-PU method

Theory

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Summary
RBF approximation

**Basis functions:** $\phi_j(x) = \phi(\varepsilon \|x - x_j\|)$. Translates of one single function rotated around a center point.

**Example:** Gaussians

$\phi(\varepsilon r) = \exp(-\varepsilon^2 r^2)$

**Approximations:**

$u_{\varepsilon}(x) = \sum_{j=1}^{N} \lambda_j \phi_j(x)$

**Why is it a good idea to use RBFs?**

- RBF methods are meshfree.
- Spectral accuracy for smooth solutions.
- Only distances between points are needed.
Practical challenges in RBF approximation

Conditioning for small $\varepsilon$ and large $N$

- Spectral convergence with $N$ requires fixed $\varepsilon$.
- For smooth solutions, the best $\varepsilon$ is small.
- We need to compute in the red region.

Computational cost

- Coefficient matrices are typically dense (for infinitely smooth RBFs).
- Direct methods are $O(N^3)$ and known fast methods are most efficient for larger $\varepsilon$. 

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Stable computation as $\varepsilon \to 0$ and $N \to \infty$

The RBF-QR method allows stable computations for small $\varepsilon$. \textit{(Fornberg, Larsson, Flyer 2011)}

Consider a finite non-periodic domain.

\textbf{Theorem (Platte, Trefethen, and Kuijlaars 2010):}

Exponential convergence on equispaced nodes $\Rightarrow$ exponential ill-conditioning.

\textbf{Solution #1:}

Cluster nodes towards the domain boundaries.

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Solution #2: A partition of unity RBF collocation approach for PDEs

(RBF-PU for interpolation Wendland 2002)

Global approximant:

\[ \tilde{u}(x) = \sum_{i=1}^{M} w_i(x) u_i(x), \]

where \( w_i(x) \) are weight functions.

Local RBF approximants:

\[ u_i(x) = \sum_{j=1}^{N_i} \lambda_{ij} \phi_j(x). \]

Applying operators:

\[ \Delta \tilde{u}(x) = \sum_{i=1}^{M} \Delta w_i u_i + 2 \nabla w_i \cdot \nabla u_i + w_i \Delta u_i \]

- Sparsity reduces memory and computational cost.
- Subdomain approach introduces parallelism.

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Approximation details

Weight functions

Generate weight functions from compactly supported $C^2$ Wendland functions

$$
\psi(\rho) = (4\rho + 1)(1 - \rho)^4
$$

using Shepard’s method

$$
w_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^{M} \psi_j(x)}.
$$

Local RBF differentiation matrices

Define the vector of local nodal solution values $u_i$, then

$$
u_i = A\lambda_i^i, \text{ where } A_{ij} = \phi_j(x_i).
$$

Applying a linear operator yields

$$
\mathcal{L}u_i = B\lambda_i = BA^{-1}u_i, \text{ where } B_{ij} = \mathcal{L}\phi_j(x_i).
$$
Covering the domain

Cover $\Omega$ with boxes (practical).
Retain boxes with center inside $\Omega$.

Enlarge boxes near boundary to cover all of $\Omega$.

Create overlapping partitions based on the boxes.
Dealing with cost: Solving the linear system

Plan: Parallel software, exploiting partition structure.
Done: Weights and local RBF-matrices in parallel.

Original sparse unstructured matrix
After approximate minimal degree reordering
Theoretical convergence results: Definitions

Let $H_j = \text{diam}(\Omega_j)$ and define the fill distance $h_j = \sup_{x \in \Omega_j} \min_{y \in X_j} \|x - y\|_2 = C \frac{H_j}{N_j^{1/d}}$.

Write the exact solution as $u(x) \equiv \sum_{j=1}^{M} w_j(x)u(x)$ and introduce the partition of unity interpolant $I_h u(x) \equiv \sum_{j=1}^{M} w_j(x)I_h u_j(x)$.

If we have

- Ellipticity $\|u\|_U \leq C_\Omega \|L(u)\|_F$, $\forall u \in U$.
- Stability of trial space and test discretization.
- Small discrete residual $\sim \kappa(A)\epsilon_M$.

Then the crucial component affecting convergence is the approximation properties of the trial space in the form $\|\mathcal{L}(u - I_h(u))\| = \| \sum_{j=1}^{M} \Delta w_j e_j + \nabla w_j \cdot \nabla e_j + w_j \Delta e_j \|$

Algebraic convergence in partition size

Keeping $N_i$ fixed while varying $H$ leads to $h = CH$. We have the following estimates

$$\|w_j\| \leq C_0, \quad \|e_j\|_\infty \leq Ch_j^{m_j - 0 - \frac{d}{2}} \|u\|_{\mathcal{N}(\Omega)},$$

$$\|\nabla w_j\| \leq \frac{C_1}{H_j}, \quad \|\nabla e_j\|_\infty \leq Ch_j^{m_j - 1 - \frac{d}{2}} \|u\|_{\mathcal{N}(\Omega)},$$

$$\|\Delta w_j\| \leq \frac{C_2}{H_j^2}, \quad \|\Delta e_j\|_\infty \leq Ch_j^{m_j - 2 - \frac{d}{2}} \|u\|_{\mathcal{N}(\Omega)},$$

leading to

$$\|\mathcal{L}(u - I_h(u))\| \leq qCH^{m - 2 - \frac{d}{2}} \|u\|_{\mathcal{N}(\Omega)},$$

where $q$ is max # partitions that overlap at a given point.

$$\dim(\mathcal{P}_K) \leq N_j < \dim(\mathcal{P}_{K+1}) \Rightarrow m_j = K + 1$$

- A single small $N_i$ reduces the global convergence order.

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Spectral convergence with fill distance

Now, we keep $H_i$ fixed while increasing $N_i$, which reduces $h_i$. We use another form for the estimates \((\text{Rieger and Zwickenagl 2008})\)

\[
\|w_j\| \leq \bar{C}_0, \quad \|e_j\| \leq 2e^{c_0 \log(h_j) / \sqrt{h_j}} \|u\|_{\mathcal{N}(\Omega)},
\]
\[
\|\nabla w_j\| \leq \bar{C}_1, \quad \|\nabla e_j\| \leq 2e^{c_1 \log(h_j) / \sqrt{h_j}} \|u\|_{\mathcal{N}(\Omega)},
\]
\[
\|\Delta w_j\| \leq \bar{C}_2, \quad \|\Delta e_j\| \leq 2e^{c_2 \log(h_j) / \sqrt{h_j}} \|u\|_{\mathcal{N}(\Omega)},
\]

leading to

\[
\|\mathcal{L}(u - I_h(u))\| \leq qCe^{c \log(h) / \sqrt{\bar{h}}} \|u\|_{\mathcal{N}(\Omega)}.\]

- In both cases, the estimates require fixed shape, $\varepsilon = \varepsilon_0$, leading to ill-conditioning either due to effective shape parameter $\varepsilon H \to 0$ or $N \to \infty$.
- The RBF-QR method or an alternative is required.
Simple Poisson test problem

Domain defined by: \( r_b(\theta) = 1 + \frac{1}{10} (\sin(6\theta) + \sin(3\theta)) \).

PDE: \[ \begin{align*}
\Delta u &= f(x), \quad x \in \Omega, \\
u &= g(x), \quad x \text{ on } \partial \Omega,
\end{align*} \]

with \( u(r, \theta) = \frac{1}{0.25r^2+1} \).

\[ \log_{10}(\text{error}) \]
Numerical convergence results with RBF-PU

Increasing the number of local points for fixed number of partitions ⇒ Spectral convergence.

Increasing the number of partitions for fixed \( n_{\text{loc}} \) (21, 28, 45, 66) ⇒ Algebraic convergence.
Example in 3-D

N = 1120

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Summary

- RBF methods are attractive because they are easy to implement, meshfree, and highly accurate.
- The shape parameter $\varepsilon$ has a significant influence on the accuracy of the methods.
- The RBF-QR method removes the ill-conditioning associated with small $\varepsilon$ in up to three dimensions.
- Partition of unity RBF methods provide a promising way to reduce computational cost while maintaining high order of accuracy.