The Lagrange interpolation for bounded variation functions

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The case of continuous functions

Let

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1$$

be a Jacobi weight.
If $f \in \mathbb{C}[-1,1]$

$$L_m(w, f, x)$$

denotes the Lagrange interpolating polynomial based on the zeros of the orthonormal system $\{p_m(w)\}$ w.r.t. w, i.e.

$$L_m(w, f, x_i) = f(x_i), \quad i = 1, \ldots, m$$

where $p_m(w, x_i) = 0, \quad i = 1 \ldots, m$

Let $\mathcal{BV}$ denote the set of all functions of bounded variation on $[-1, 1]$.
Consider $f \in C([-1, 1]) \cap \mathcal{BV}$

- In 1963 Geronimus proved pointwise convergence in compact subintervals of $(-1, 1)$ for arbitrary $\alpha, \beta > -1$
- In $1980, 1983$ P. Vértesi proved that for $-1 < \alpha, \beta < 1/2$

$$\lim_m \max_{x\in[-1,1]} |L_m(w, f, x) - f(x)| = 0 \tag{1}$$

  and that (1) usually does not hold if $\max(\alpha, \beta) \geq 1/2$.
- Nevai (1974), Kelzon $(1979, 1984)$, Sun (1989), Kvernadze (1996)

We will consider continuous functions with $f^{(r)} \in \mathcal{BV}$, $r \geq 1$.

Main results

Now let $Q = \{q_m(x)\}_{m=0,1,\dots}$ be a sequence of polynomials such that:

a) for any m the zeros $\{z_k\}_{k=1,m}$ of $q_m \in \mathbb{P}_m$ belong to $[-1,1]$

b) for any k, $\left|\sum_{i=1}^{k} \frac{1}{q_m'(z_i)}\right| \leq \frac{C}{m}, \quad C \neq C(m,k)$

and consider the Lagrange interpolation process based on the $z_k, k = 1 \dots, m$, i.e.

$$L_m(Q, f, z_k) = f(z_k), \qquad k = 1, \dots, m$$

Examples of polynomial sequences satisfying properties a)-b)

- orthonormal polynomials w.r.t. a Jacobi weight $w(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -1$
- orthonormal polynomials w.r.t. a Generalized Jacobi weight of the type $w(x) = (1-x)^\alpha (1+x)^\beta |x|^\eta$, $\alpha, \beta > -1$, $\eta \geq 0$

Theorem

Let f be a continuos and convex function on $[-1,1]$ and $L_m(Q,f)$ be the interpolating operator constructed on the nodes of the polynomials $Q=\{q_m\}_m$ defined by properties a)-b). The following estimate holds

$$|f(x)-L_m(Q,f,x)| \leq \frac{\mathcal{C}}{m}|q_m(x)|\,[z_1,x,z_m;f], \quad |x|\leq 1 \qquad (2)$$

where $\mathcal{C}$ is a positive constant independent of m, x ad f.

Now if $f' \in \mathcal{BV}$, denoting by $\Gamma_{t,1}(x)$ the first truncated power i.e.

$$\Gamma_{t,1}(x) := \left[\begin{array}{ll} (x-t), & x > t \\ \\ 0, & x \le t \end{array} \right.$$

it results

$$|f(x) - L_m(Q,f,x)| \le \int_{-1}^{1} |\Gamma_{t,1}(x) - L_m(Q,\Gamma_{t,1},x)|\ |df'(t)|$$

$$\le \quad \frac{\mathcal{C}}{m}|q_m(x)| \int_{-1}^{1} [z_1, x, z_m; \Gamma_{t,1}]\ |df'(t)| \le \frac{\mathcal{C}}{m}|q_m(x)| \int_{-1}^{1}\ |df'(t)|$$

By iteration on r, using the Peano formula and the estimate

$$E_m(\Gamma_{t,0})_1 \le \frac{\mathcal{C}}{m}\sqrt{1-t^2}, \qquad \mathcal{C} \ne \mathcal{C}(m,t)$$

we get the following

Theorem

Let $f \in C([-1,1])$ be such that $f^{(r)}$, $r \geq 1$, (eventually discontinuous) is of bounded variation. Moreover let $L_m(Q,f)$ be the interpolating operator defined above with $m > r$. Then for all $x \in [-1,1]$

$$|f(x) - L_m(Q,f,x)| \leq \frac{\mathcal{C}}{m^r} |q_m(x)| \int_{-1}^{1} \left(\sqrt{1-t^2}\right)^{r-1} |df^{(r)}(t)| \tag{3}$$

where $\mathcal{C}$ is a positive constant independent of m, x ad f.

Remark In the Timan book the following Nikolskii result (1947) is proved

$$\lim_m m^r E_m(f) = \mathcal{C} \max_{x \in (-1,1)} |f^{(r)}(x)^+ - f^{(r)}(x)^-| \left(\sqrt{1-x^2}\right)^r$$

where $\mathcal{C}$ depends only on $r \geq 1$

Corollary

If u, w are two Jacobi weights then for all $x \in [-1, 1]$

$$|f(x) - L_m(w, f, x)|u(x) \leq \frac{\mathcal{C}}{m^r}|p_m(w, x)u(x)| \int_{-1}^{1} \left(\sqrt{1 - t^2}\right)^{r-1} |df^{(r)}(t)|$$

where $\mathcal{C}$ is a positive constant independent of m, x ad f.

Hence if u is chosen such that $\frac{u}{\sqrt{w\varphi}} \in L^\infty$, where $\varphi(t) = \sqrt{1 - t^2}$, (i.e. s.t. $\{p_m(w)u\}_m$ is uniformly bounded w.r.t. m) then we get an optimal interpolation process.

The L^p_u case

Let $u(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma, \delta > -1$ be a Jacobi weight and L^p_u denote the usual L^p weighted space.

Let $\varphi(x) = \sqrt{1-x^2}$, $r \geq 1$, $1 \leq p \leq \infty$

$$W^p_r(u) = \left\{ f \in L^p_u : f^{(r-1)} \in \mathcal{AC}(-1,1) \text{ and } \|f^{(r)}\varphi^r u\|_p < \infty \right\}$$

$$\|f\|_{W^p_r} := \|fu\|_p + \|f^{(r)}\varphi^r u\|_p$$

Moreover let
$w(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, be another Jacobi weight
If $f \in \mathbb{C}[-1,1]$

$$L_m(w, f, x)$$

denote the Lagrange interpolating polynomial based on the zeros of the orthonormal system w.r.t. w

$$\|L_m(w,f)\|_{W^p_r(u)} \le \mathcal{C}\|f\|_{W^p_r(u)}, \qquad r\ge 1, \qquad 1<p<\infty \tag{4}$$

and

$$0\le s\le r, \qquad \|[f-L_m(w,f)]\|_{W^p_s(u)} \le \mathcal{C}\frac{\|f\|_{W^p_r(u)}}{m^{r-s}} \tag{5}$$

hold if and only if

$$\frac{u}{\sqrt{w\varphi}}\in L^p, \qquad \frac{\sqrt{w\varphi}}{u}\in L^q, \qquad q=\frac{p}{p-1} \tag{6}$$

(Mastroianni, R., 1999)

Now let $f \in \mathcal{BV}$.

If y is a jump point we will define the value of f in y as follows

$$f(y) := \frac{f(y)^- + f(y)^+}{2} \tag{7}$$

where $f(y)^\pm = \lim_{x\to y^\pm} f(x)$.

The previous result can be completed by the following

Theorem

Let $f^{(r)} \in \mathcal{BV}$, $r \geq 0$. Then for all $1 < p < \infty$ and with $\mathcal{C} \neq \mathcal{C}(m, f)$ it results

$$\|[f - L_m(w, f)]u\|_p \leq \frac{\mathcal{C}}{m^{r+\frac{1}{p}}} \int_{-1}^{1} \left(\sqrt{1-t^2}\right)^{r+\frac{1}{p}} u(t)|df^{(r)}(t)| \quad (8)$$

if and only if (6) hold true.

Remark

Hence if $f^{(r)} \in \mathcal{BV}$, $r \geq 1$ we get an extra $\frac{1}{p}$ in the order of convergence, as in the error of best approximation.

In 1993 Prestin proved for $1 \leq p < \infty$ and for arbitrary $\alpha, \beta, \gamma, \delta > -1$, $\epsilon > 0$, that for $f \in \mathcal{BV}$

$$\left(\int_{-1+\epsilon}^{1-\epsilon} |f(x) - L_m(w, f, x)|^p u^p(x)\, dx\right)^{\frac{1}{p}} \leq \frac{\mathcal{C}}{m^{\frac{1}{p}}} V(f) \left[\begin{array}{ll} \log m, & p = 1 \\ 1, & 1 < p < \infty \end{array} \right.$$

where $V(f)$ denotes the total variation of f on $[-1, 1]$.

Remark

By the stated Theorem it results

$$\|[f - L_m(w, f)]u\|_p = \mathcal{O}\left(m^{-r-\frac{1}{p}}\right).$$

The arising question is: when the "$\mathcal{O}$" can be replaced by "o"?
A simple L^p condition is

$$\int_0^1 \frac{\Omega_\varphi^s(f^{(r)}, t)_{u,p}}{t^{1+\frac{1}{p}}}\, dt < +\infty, \quad s > r, \tag{9}$$

since by the Theorem in [Mastroianni,R.] it easily follows that

$$\|[f - L_m(w, f)]u\|_p = o\left(m^{-r-\frac{1}{p}}\right).$$

Note that (9) implies that $f^{(r)} \in C^0(-1, 1)$.

More in general we get the following Corollary.

Corollary

Under the assumptions (6), if $f^{(r)} \in \mathcal{BV}$, $r \geq 0$, is continuous on $[-1, 1]$, then

$$\|[f - L_m(w, f)]u\|_p = o\left(m^{-r-\frac{1}{p}}\right).$$

and the constants in "o" are independent of m.

That's all.

Thank you for your attention!