Electrical impedance imaging using nonlinear Fourier transform

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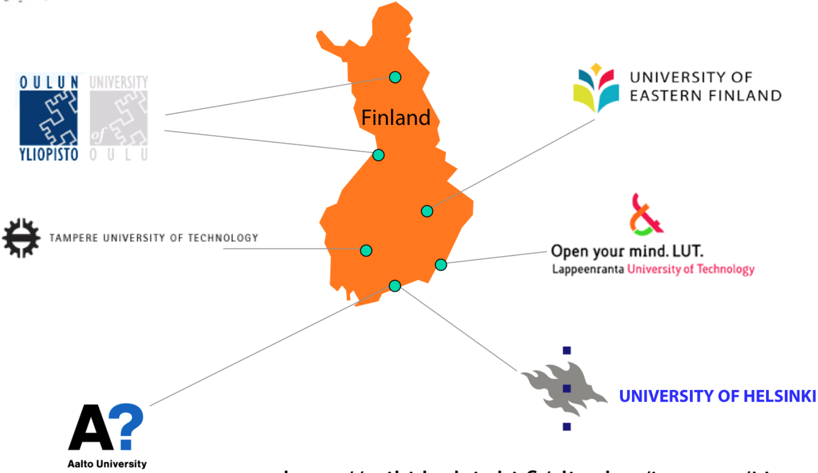

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This is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA

Kim Knudsen, Technical University of Denmark

Matti Lassas, University of Helsinki, Finland

Jon Newell, Rensselaer Polytechnic Institute, USA

Jennifer Mueller, Colorado State University, USA

Outline

Electrical impedance tomography

Regularization of nonlinear inverse problems

D-bar method for infinite-precision data

Regularization using non-linear low-pass filtering

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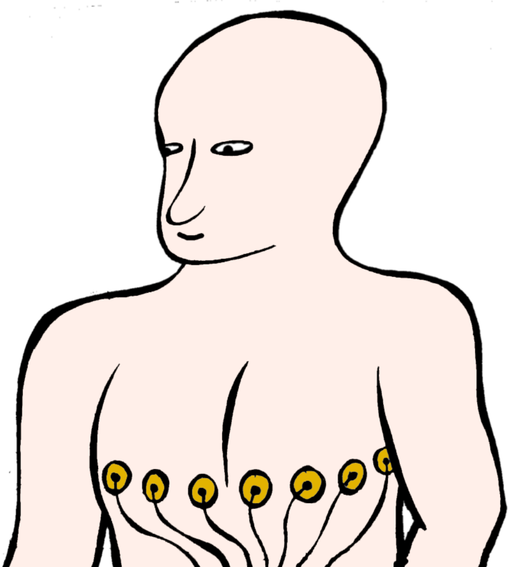
Electrical impedance tomography (EIT) is an emerging medical imaging technique

Feed electric currents through electrodes. **Measure** the resulting voltages. Repeat the measurement for several current patterns.

Reconstruct distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient's inner structure.

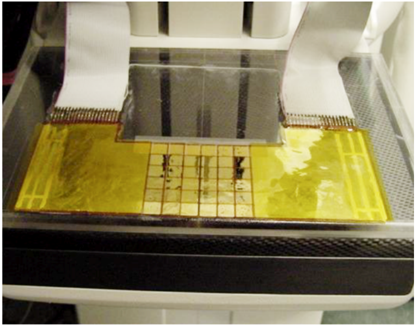
EIT is a harmless and painless imaging method suitable for long-term monitoring.



The most promising use of EIT is detection of breast cancer in combination with mammography

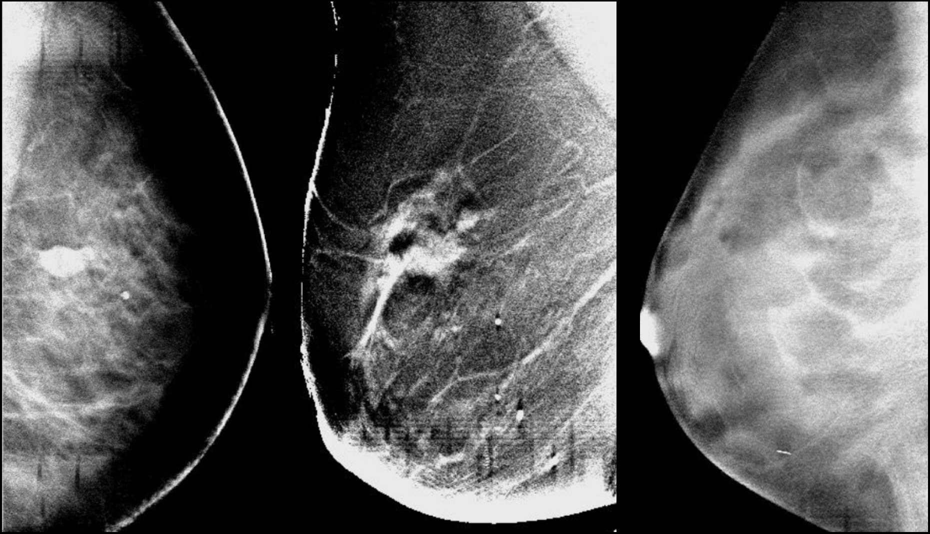
ACT4 and mammography devices

Radiolucent electrodes

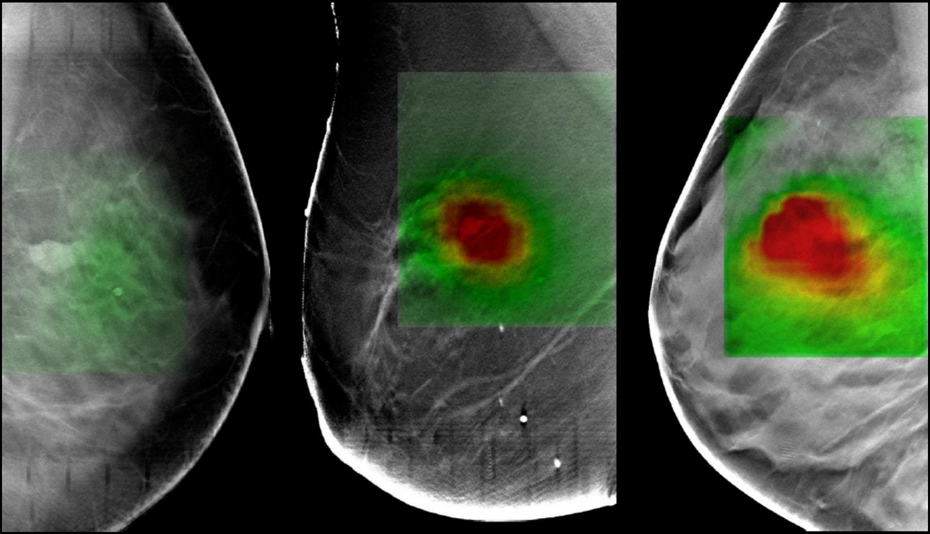


Cancerous tissue is up to four times more conductive than healthy breast tissue [Jossinet 1998]. The above setup of David Isaacson's team measures 3D X-ray mammograms and EIT data at the same time.

Which of these three breasts have cancer?



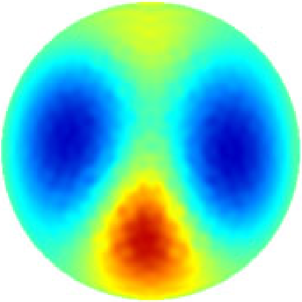
Spectral EIT can detect cancerous tissue

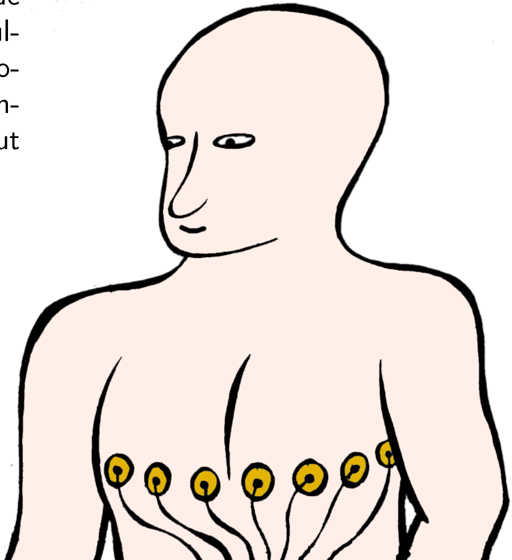


[Kim, Isaacson, Xia, Kao, Newell & Saulnier 2007]

This talk concentrates on applications of EIT to chest imaging

Applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.





The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

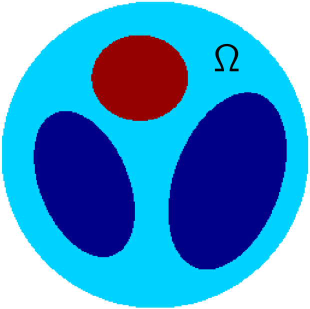
$$0 < M^{-1} \leq \sigma(z) \leq M.$$

Applying voltage f at the boundary $\partial\Omega$ leads to the elliptic PDE

$$\begin{cases} \nabla \cdot \sigma \nabla u & = & 0 \text{ in } \Omega, \\ u|_{\partial\Omega} & = & f. \end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega}.$$



Calderón's problem is to recover σ from the knowledge of Λ_σ. It is a nonlinear and ill-posed inverse problem.

Many different types of reconstruction methods have been suggested for EIT in the literature

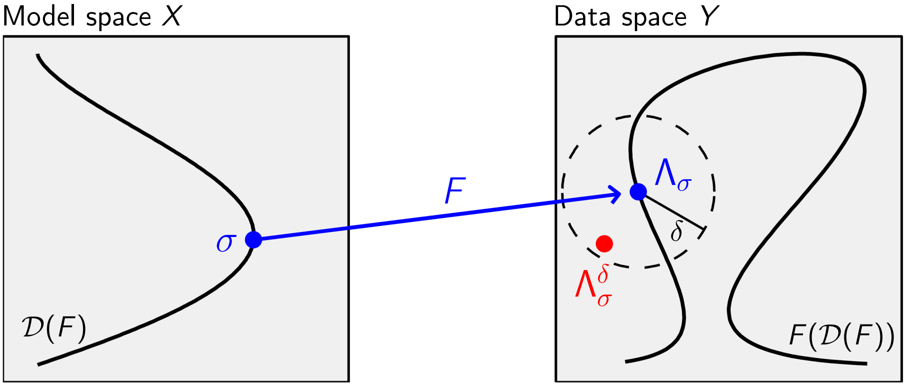
- **Linearization:** Barber, Bikowski, Brown, Cheney, Isaacson, Mueller, Newell
- **Iterative regularization:** Dobson, Hua, Kindermann, Leitão, Lechleiter, Neubauer, Rieder, Rondi, Santosa, Tompkins, Webster, Woo
- **Bayesian inversion:** Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen
- **Resistor network methods:** Borcea, Druskin, Mamonov, Vasquez
- **Layer stripping:** Cheney, Isaacson, Isaacson, Somersalo
- **D-bar methods:** Astala, Bikowski, Bowerman, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Saulnier, S, Tamasan
- **Teichmüller space methods:** Kolehmainen, Lassas, Ola
- **Methods for partial information:** Alessandrini, Ammari, Bilotta, Brühl, Erhard, Gebauer, Hanke, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kwon, Lechleiter, Lim, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others

History of CGO-based methods for real 2D EIT

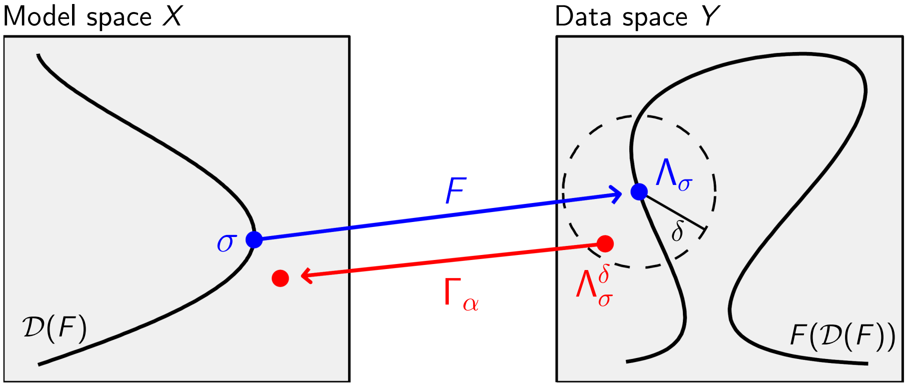
Infinite-precision data	Practical data
1980 Calderón	**2008 Bikowski & Mueller**
1987 Sylvester & Uhlmann ($d \geq 3$) 1988 Nachman 1988 R G Novikov	2008 Boverman, Isaacson, Kao, Saulnier & Newell 2010 Bikowski, Knudsen & Mueller
1996 Nachman ($\sigma \in C^2(\Omega)$) 1997 Liu	2000 S, Mueller & Isaacson 2003 Mueller & S 2004 Isaacson, Mueller, Newell & S 2006 Isaacson, Mueller, Newell & S 2007 Murphy & Mueller 2008 Knudsen, Lassas, Mueller & S 2009 Knudsen, Lassas, Mueller & S 2009 S & Tamminen
1997 Brown & Uhlmann ($\sigma \in C^1(\Omega)$) 2001 Barceló, Barceló & Ruiz	2001 Knudsen & Tamasan 2003 Knudsen
2000 Francini	
2003 Astala & Päivärinta ($\sigma \in L^\infty(\Omega)$) 2005 Astala, Lassas & Päivärinta 2007 Barceló, Faraco & Ruiz 2008 Clop, Faraco & Ruiz	2009 Astala, Mueller, Päivärinta & S 2011 Astala, Mueller, Päivärinta, Perämäki & S

Outline

The forward map $F : X \supset \mathcal{D}(F) \to Y$ of an ill-posed problem does not have a continuous inverse



Regularization means constructing a continuous map $\Gamma_\alpha : Y \to X$ that inverts F approximately



The regularization strategy need to be constructed so that these assumptions are satisfied

A family $\Gamma_\alpha : Y \to X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a *regularization strategy* for F if

$$\lim_{\alpha \to 0} \|\Gamma_\alpha(\Lambda_\sigma) - \sigma\|_X = 0$$

for each fixed $\sigma \in \mathcal{D}(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called *admissible* if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $\sigma \in \mathcal{D}(F)$ the following holds:

$$\sup_{\Lambda_\sigma^\delta} \left\{ \|\Gamma_{\alpha(\delta)}(\Lambda_\sigma^\delta) - \sigma\|_X \ : \ \|\Lambda_\sigma^\delta - \Lambda_\sigma\|_Y \leq \delta \right\} \to 0 \text{ as } \delta \to 0.$$

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Nachman's 1996 uniqueness proof for 2D inverse conductivity problem relies on CGO solutions

Define a potential q by setting $q(z) \equiv 0$ for z outside Ω and

$$q(z) = \frac{\Delta\sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for } z \in \Omega.$$

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(-\Delta + q)\psi(\,\cdot\,, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz}\psi(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2), \quad \tilde{p} > 2,$$

where $ikz = i(k_1 + ik_2)(x + iy)$. By [Nachman 1996] we know that there exists a unique solution $\psi(\,\cdot\,, k)$ for any fixed $k \neq 0$.

The crucial intermediate object in the proof is the non-physical scattering transform $\mathbf{t}(k)$

We denote $z = x + iy \in \mathbb{C}$ or $z = (x, y) \in \mathbb{R}^2$ whenever needed.
The scattering transform $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ is defined by

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\overline{k}\overline{z}} q(z)\psi(z,k)\,dxdy. \tag{1}$$

Sometimes (1) is called the nonlinear Fourier transform of q.
This is because asymptotically $\psi(z,k) \sim e^{ikz}$ as $|z| \to \infty$,
and substituting e^{ikz} in place of $\psi(z,k)$ into (1) results in

$$\begin{aligned}
\int_{\mathbb{R}^2} e^{i(kz+\overline{k}\overline{z})} q(z)dxdy &= \int_{\mathbb{R}^2} e^{-i(-2k_1,2k_2)\cdot(x,y)} q(z)dxdy \\
&= \widehat{q}(-2k_1, 2k_2).
\end{aligned}$$

Another convenient trick in the proof is to make use of the functions $\mu(z,k) = e^{-ikz}\psi(z,k)$

Define $\mu(z,k) = e^{-ikz}\psi(z,k)$. Then $(-\Delta + q)\psi = 0$ implies

$$(-\Delta - 4ik\overline{\partial}_z + q)\mu(\,\cdot\,,k) = 0, \tag{2}$$

where the D-bar operator is defined by $\overline{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.
The asymptotic properties of ψ imply that

$$\mu(z,k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2), \quad \tilde{p} > 2. \tag{3}$$

Substituting $k = 0$ into (2) gives

$$(-\Delta + \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}})\mu(\,\cdot\,,0) = 0, \tag{4}$$

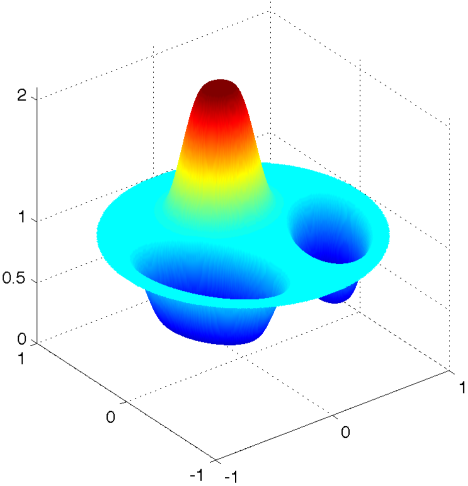
and $\mu(z,0) = \sqrt{\sigma(z)}$ gives the unique solution of (3) and (4).

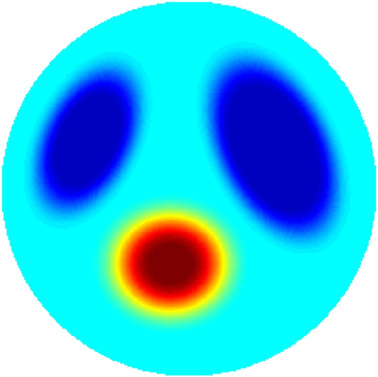
These are the steps of Nachman's 1996 proof:

Solve boundary integral equation $\psi(\,\cdot\,, k)\vert_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi$ for every complex number $k \in \mathbb{C}$.	**Fredholm equation of 2nd kind, ill-posedness shows up here.**
Evaluate the scattering transform: $\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k)\, ds.$	**Simple integration.**
Fix $z \in \Omega$. Solve D-bar equation $\frac{\partial}{\partial \bar{k}}\mu(z,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}} e^{-i(kz+\bar{k}\bar{z})}\overline{\mu(z,k)}$ with $\mu(z,\,\cdot\,) - 1 \in L^r \cap L^\infty(\mathbb{C})$.	**Well-posed problem, can be analyzed by scattering theory.**
Reconstruct: $\sigma(z) = (\mu(z,0))^2$.	**Trivial step.**

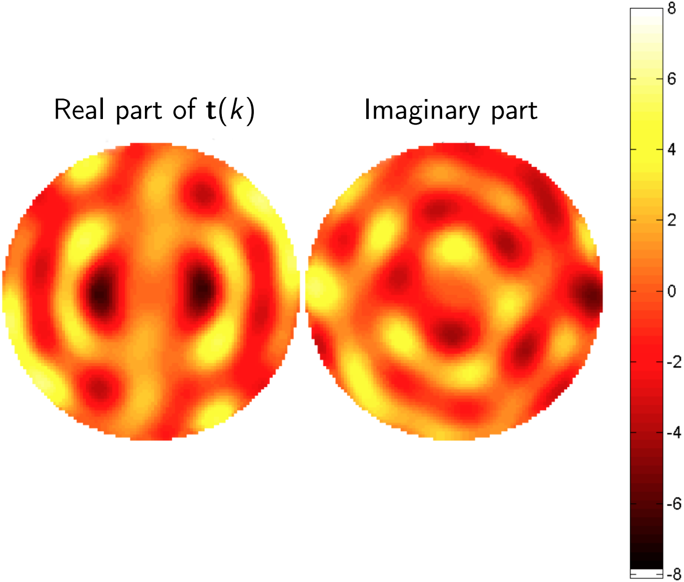
Outline

Let us analyze how the regularization works using a simulated heart-and-lungs phantom

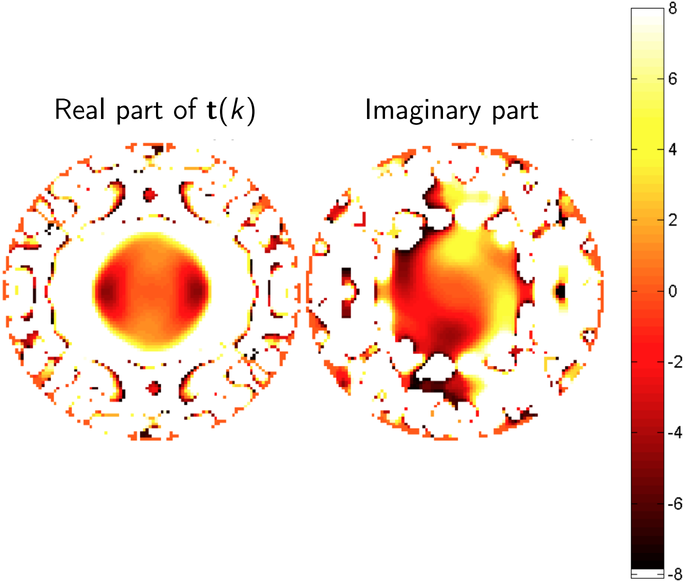




This is how the actual scattering transform looks like in the disc $|k| < 10$, computed by knowing σ



Scattering transform in the disc $|k| < 10$, here computed from noisy measurement Λ_σ^δ



Infinite-precision data:

Solve boundary integral equation

$$\psi(\,\cdot\,,k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi$$

for every complex number $k \in \mathbb{C}$.

Evaluate the scattering transform:

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\sigma - \Lambda_1)\psi(\cdot,k)\,ds.$$

Fix $z \in \Omega$. Solve D-bar equation

$$\frac{\partial}{\partial \bar{k}}\mu(z,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}}e^{-i(kz+\bar{k}\bar{z})}\overline{\mu(z,k)}$$

with $\mu(z,\,\cdot\,) - 1 \in L^r \cap L^\infty(\mathbb{C})$.

Reconstruct: $\sigma(z) = (\mu(z,0))^2$.

Practical data:

Solve boundary integral equation

$$\psi^\delta(\,\cdot\,,k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta$$

for all $|k| < R = R(\delta)$.

For $|k| \geq R$ set $\mathbf{t}_R^\delta(k) = 0$. For $|k| < R$

$$\mathbf{t}_R^\delta(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta(\cdot,k)\,ds.$$

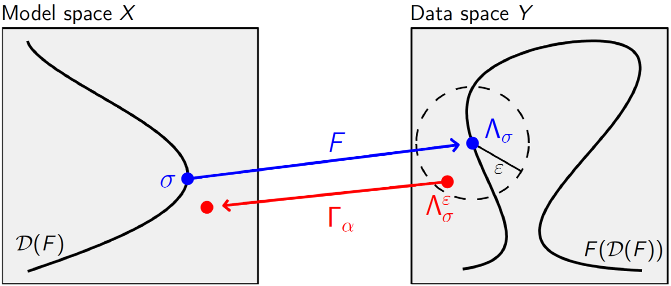
Fix $z \in \Omega$. Solve D-bar equation

$$\frac{\partial}{\partial \bar{k}}\mu_R^\delta(z,k) = \frac{\mathbf{t}_R^\delta(k)}{4\pi\bar{k}}e^{-i(kz+\bar{k}\bar{z})}\overline{\mu_R^\delta(z,k)}$$

with $\mu_R^\delta(z,\,\cdot\,) - 1 \in L^r \cap L^\infty(\mathbb{C})$.

Set $\Gamma_{\alpha(\delta)}(\Lambda_\sigma^\delta) := (\mu_R^\delta(z,0))^2$.

We define spaces for our regularization strategy



Consider $F : X \supset \mathcal{D}(F) \to Y$ with $X = L^\infty(\Omega)$. Let $M > 0$ and $0 < \rho < 1$. Now $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ satisfying

$$\|\sigma\|_{C^2(\overline{\Omega})} \leq M, \quad \sigma(z) \geq M^{-1}, \quad \text{and } \sigma(z) \equiv 1 \text{ for } \rho < |z| < 1.$$

Y comprises bounded linear operators $A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying $A(1) = 0$ and $\int_{\partial\Omega} A(f) d\sigma = 0$.

Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

Theorem (Knudsen, Lassas, Mueller & S 2009)

There exists a constant $0 < \delta_0 < 1$*, depending only on* M *and* ρ*, with the following properties. Let* $\sigma \in \mathcal{D}(F)$ *be arbitrary and assume given noisy data* Λ_σ^δ *satisfying*

$$\|\Lambda_\sigma^\delta - \Lambda_\sigma\|_Y \leq \delta < \delta_0.$$

Then Γ_α *with the choice*

$$R(\delta) = -\frac{1}{10}\log\delta, \qquad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\delta)}(\Lambda_\sigma^\delta) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log\delta)^{-1/14}.$$

Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

$$\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}.$$

We invert the linear operator appearing in the equation

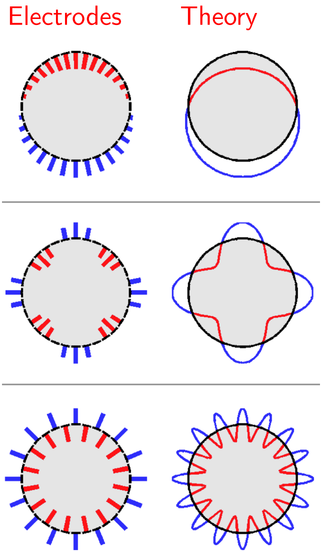
$$\psi^{\delta}(\,\cdot\,,k)|_{\partial\Omega} = [I + S_k(\Lambda_\sigma^{\delta} - \Lambda_1)]e^{ikz}|_{\partial\Omega}$$

as a matrix in $\mathrm{span}\big(\{\varphi_n\}_{n=-N}^{N}\big)$.

The single-layer operator

$$(S_k\phi)(z) = \int_{\partial\Omega} G_k(z-w)\phi(w)\,ds(w)$$

uses Faddeev's Green's function.



Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

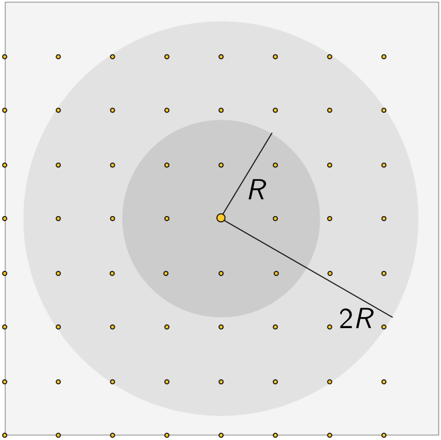
The generalization of Vainikko's method for the D-bar equation is described in [Knudsen, Mueller & S 2004].
The D-bar equation

$$\frac{\partial}{\partial \bar{k}}\mu_R^\delta = \frac{1}{4\pi \bar{k}}\mathbf{t}_R^\delta(k)e_{-z}(k)\overline{\mu_R^\delta}$$

together with the asymptotics

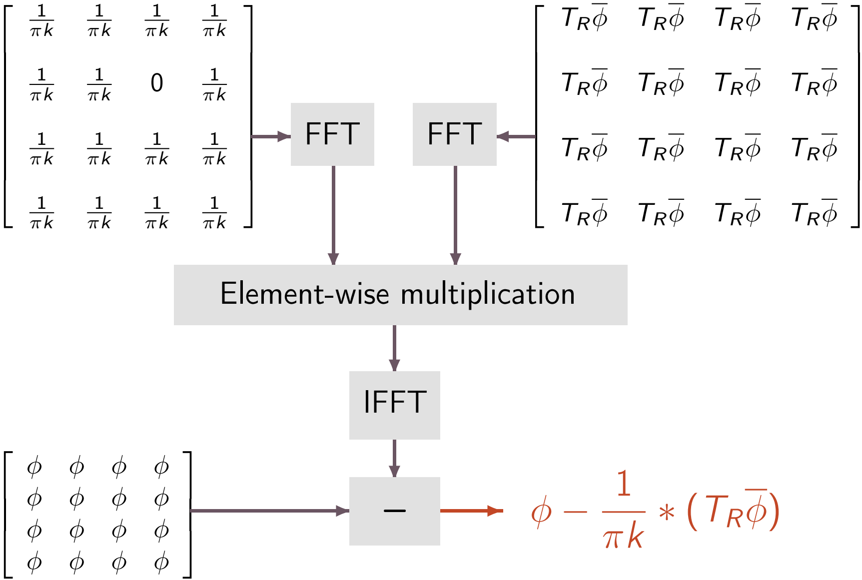
$$\mu_R^\delta(z,\,\cdot\,) - 1 \in L^r \cap L^\infty(\mathbb{C})$$

can be combined in a generalized Lippmann-Schwinger equation:

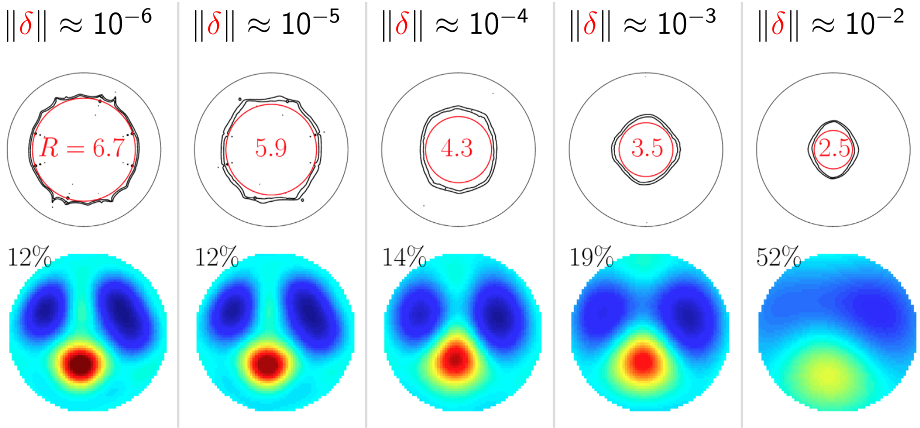


$$\mu_R^\delta(z,k) = 1 - \frac{1}{4\pi^2}\int_{\mathbb{C}} \frac{\mathbf{t}_R^\delta(k')}{(k-k')\bar{k'}} e_{-z}(k')\overline{\mu_R^\delta(z,k')}\, dk_1' dk_2'.$$

This is the real-linear operation given to GMRES

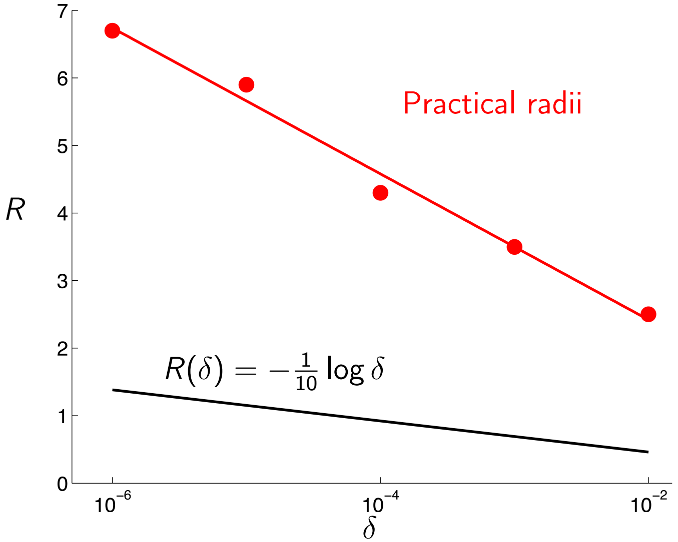


Regularized reconstructions from simulated data with noise amplitude $\|\delta\| = \|\Lambda_\sigma^\delta - \Lambda_\sigma\|_Y$



The percentages are the relative square norm errors in the reconstructions.

The observed radii are better (=larger) than those given by the theoretical formula $R(\delta) = -\frac{1}{10} \log \delta$



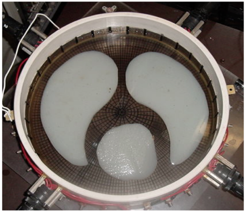
Conclusion

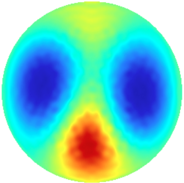
We have constructed the first direct (non-iterative) regularization strategy for a global nonlinear PDE coefficient recovery problem.

Efficient implementation available, based on Vainikko's method.

The nonlinear low-pass filter regularization approach has an explicit speed of convergence in a Banach space setting.

The method works with real data as well:





[Isaacson, Mueller, Newell & S 2006]

Thank you for your attention!

Preprints available at **www.siltanen-research.net.**

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