Series Acceleration Through Precise Remainder Asymptotics

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Outline

- Brief review of series acceleration
- The new method for analytic series
  - Results about convergence/slowed divergence
- Examples and comparison
- Conclusions

Series Acceleration

Series acceleration methods of the Levin type are based on extrapolating the sequence of partial sums of the series. If we define those sums as:

$$s_n = \sum_{k=0}^{n-1} t_k$$

then we can partition the partial sums into the (anti-) limit and the remainders:

$$s_n = s + \rho_n$$

An acceleration method estimates the remainders from the terms of the series.

Series Acceleration

In particular, Levin-type methods proceed by an asymptotic expansion of the remainder into a dominant *remainder estimate* and a (usually asymptotic) series with free parameters:

$$\rho_n \sim \omega_n \sum_{k=0}^{\infty} c_k \, g_k(n), \qquad n \to \infty$$

Different choices of the remainder estimate ω_n and the functions $g_k(n)$ lead to different series acceleration methods.   Often, the remainder estimate depends on the terms of the sequence, so the method is nonlinear.

Analytic Series

When the terms of the series are known only numerically, or are very expensive to compute, a remainder estimate based only on the terms of the series can be very powerful. But if the series is known analytically, then it is worth asking if this analytic knowledge can be leveraged to improve the performance of standard methods.

In this talk we will show that they can, when the ratio of successive terms can be expanded in an asymptotic series of inverse powers of n.

Main result

If $\dfrac{t_{n+1}}{t_n} \sim z\, n^{\ell} \left(1 + \sum_{k=1}^{\infty} \dfrac{r_k}{n^k}\right), \quad n \to \infty, \quad \ell \in \mathbb{Z}$

then if $\ell \leq 0$, we can write:

$$s_n \sim s + \mu\, \omega_n \sum_{k=0}^{\infty} \frac{c_k}{n^k}, \quad n \to \infty$$

where we can calculate the remainder estimate ω_n and all of the $\{c_k\}$ in terms of z, ℓ, and the $\{r_k\}$.

Leading Remainder

Specifically, we can show that the leading remainder is:

$$\omega_n = \begin{cases} z^n n^{r_1} & \text{if } \ell = 0 \text{ and } z \neq 1 \\ n^{r_1+1} & \text{if } \ell = 0,\ z = 1, \text{ and } r_1 \notin \{-1, 0, 1, \ldots\} \\ [(n-2)!]^\ell z^n n^{r_1+\ell} & \text{if } \ell \leq -1 \end{cases}$$

If $\ell > 0$ the original series has zero radius of convergence, and the analytic method with inverse powers fails; we do *not* slow the divergence of these series. If $\ell = 0,\ z = 1,$ and $r_1 \in \{-1, 0, 1, \ldots\}$, then we do not know if the divergence is slowed.

Outline of Proof

The key to proving this result is to recognize that the partial sums satisfy a homogeneous, second-order linear difference equation:

$$s_{n+2} - (1 + zn^{\ell} r(n)) s_{n+1} + zn^{\ell} r(n) s_n = 0$$

where:

$$r(n) \sim 1 + \sum_{k=1}^{\infty} \frac{r_k}{n^k}, \quad n \to \infty$$

R. Wong and H. Li studied the asymptotic solutions of homgeneous, linear, second order equations of the form:

$$w_{n+2} + n^p a(n) w_{n+1} + n^q b(n) w_n = 0$$

in two 1992 papers.

Outline of Proof

Those papers consider several cases, and give the recursive formulas for the asymptotic coefficients. Because of the special form of our recurrence, we can show (in the cases of the theorem) that one of the two solutions is in fact a constant, and determine the form of the leading remainder for the non-constant solution. The recursion for the asymptotic coefficients then completes the method.

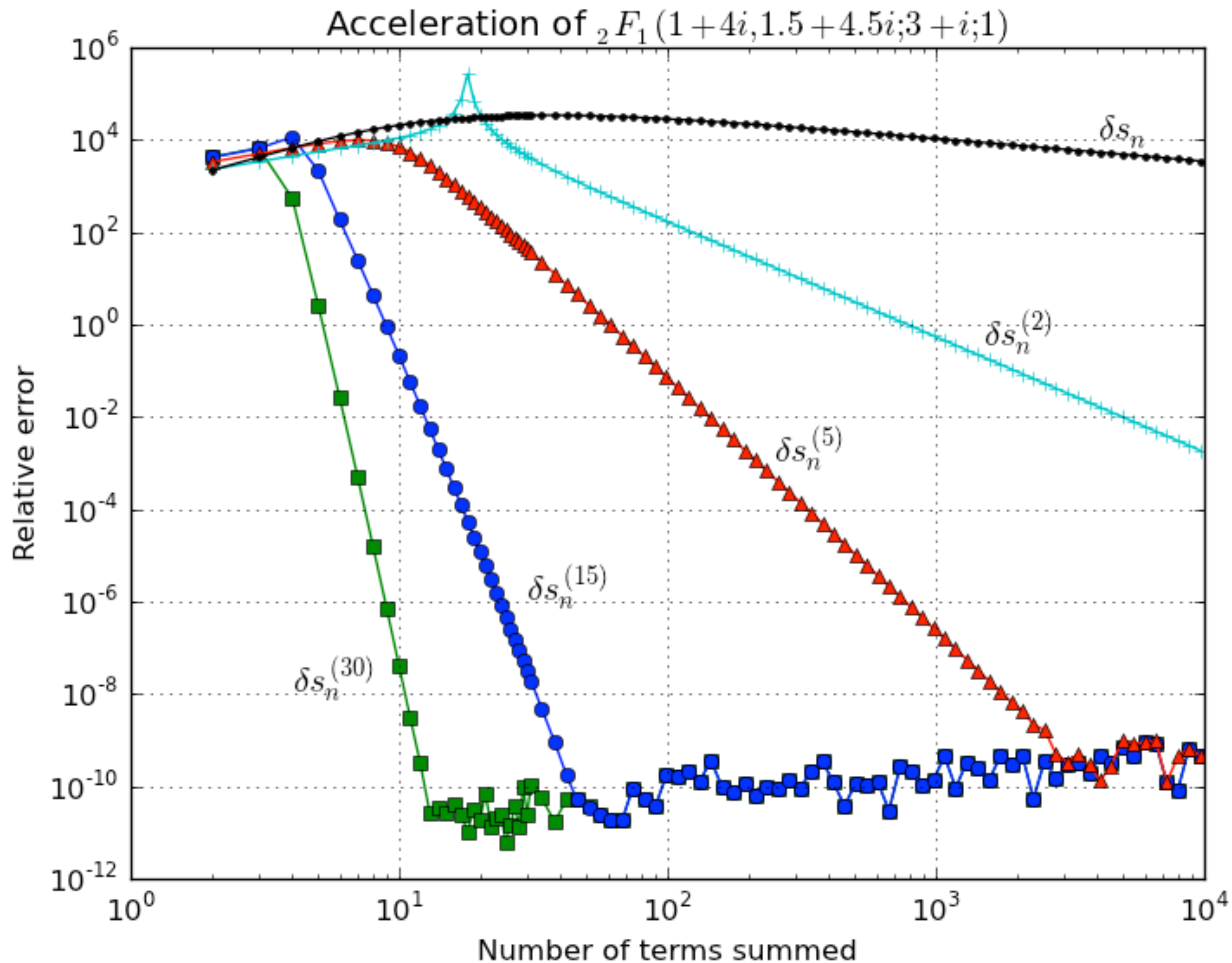
If we truncate the asymptotic series at some fixed order, then from any two successive partial sums we may calculate s (as a hopefully improved estimate of the sum) and μ.

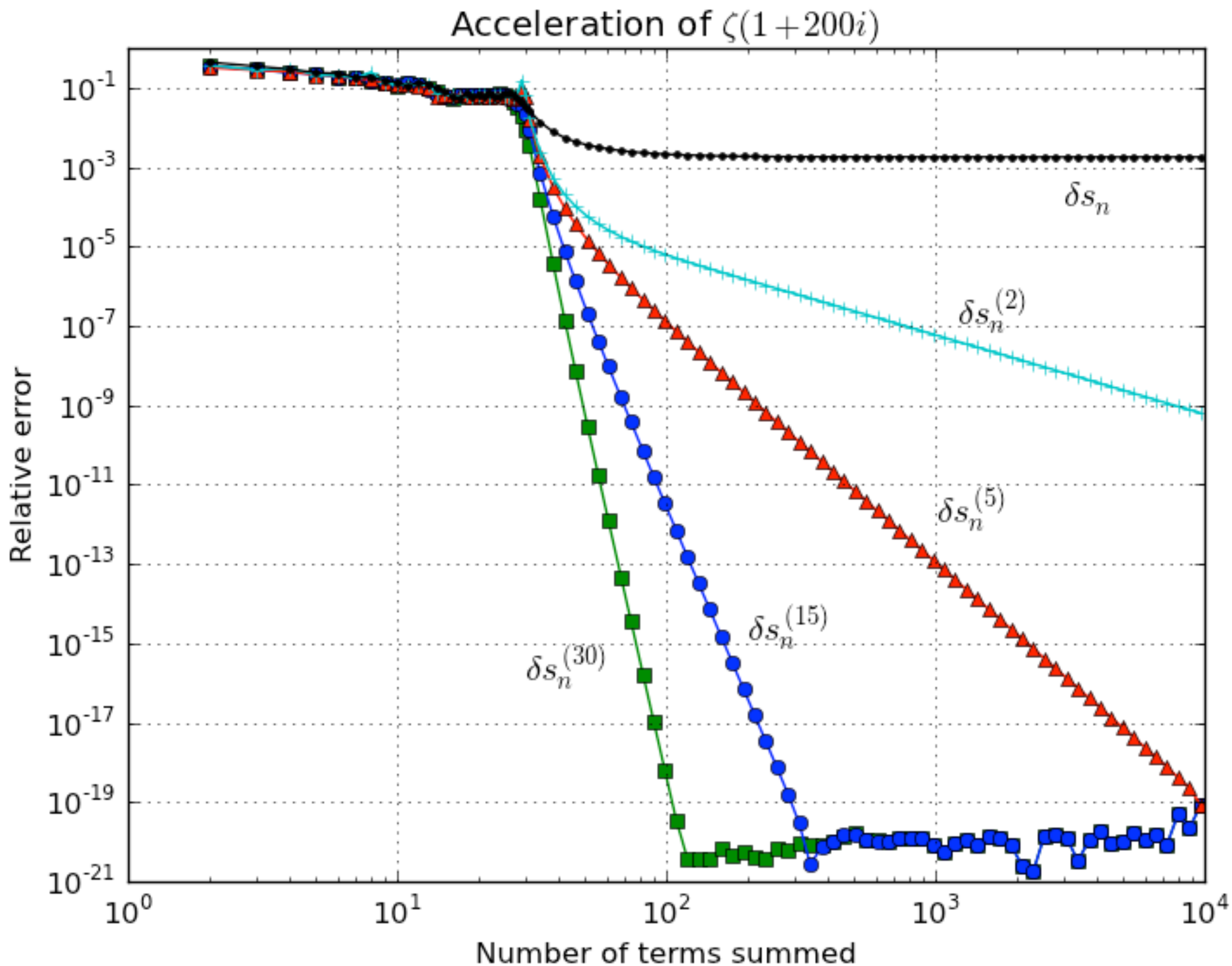
Calculating r_k

To use these formulas, we must be able to calculate the asymptotic expansions of the term ratio. But for many classes of special functions, this is tractable:

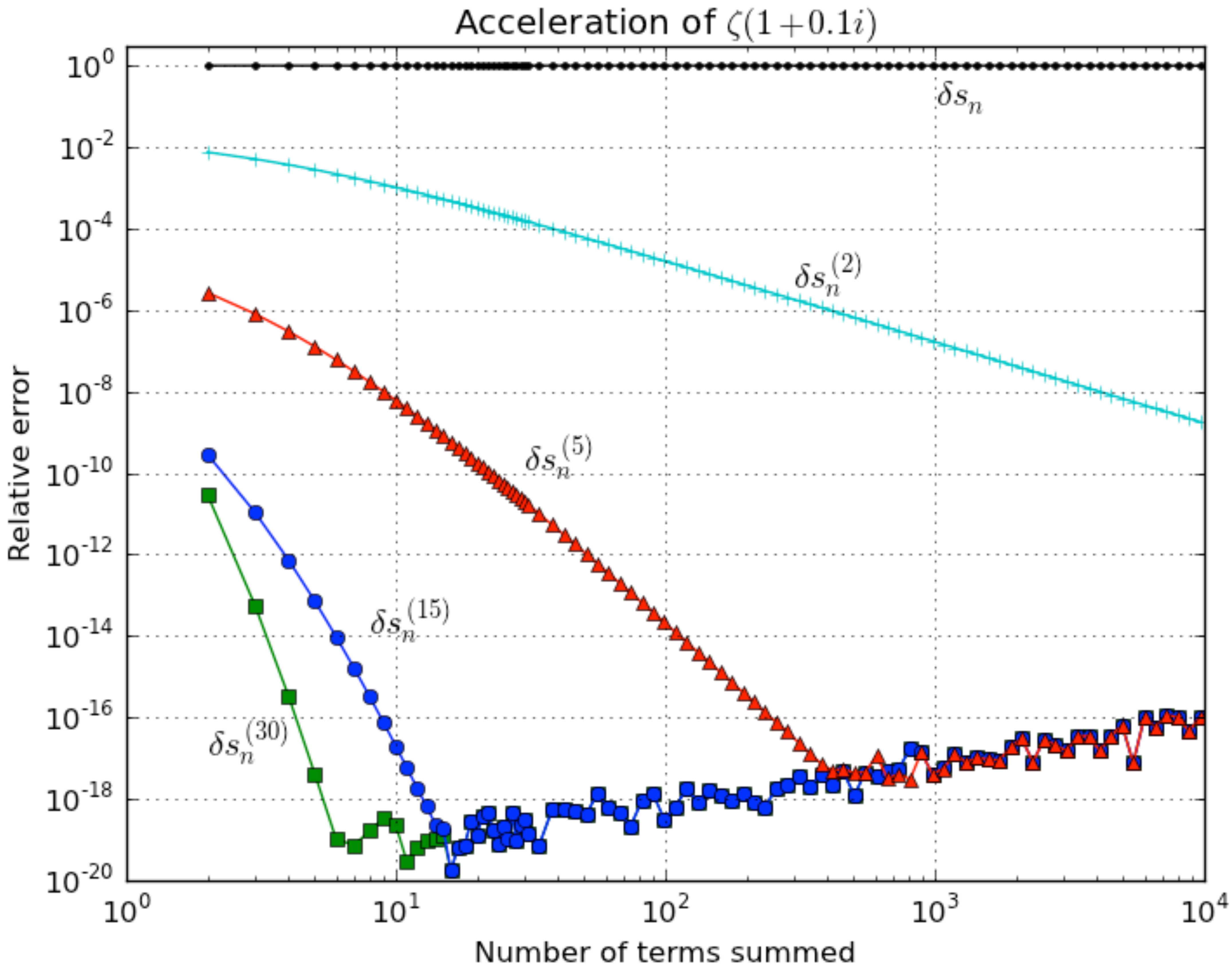
- **Generalized Hypergeometric Functions** Term ratio is a rational function, and Taylor coefficients can be found using complete homogeneous and elementary symmetric polynomials
- **Zeta and Hurwitz Functions** Coefficients may be found from binomial expansion

Examples

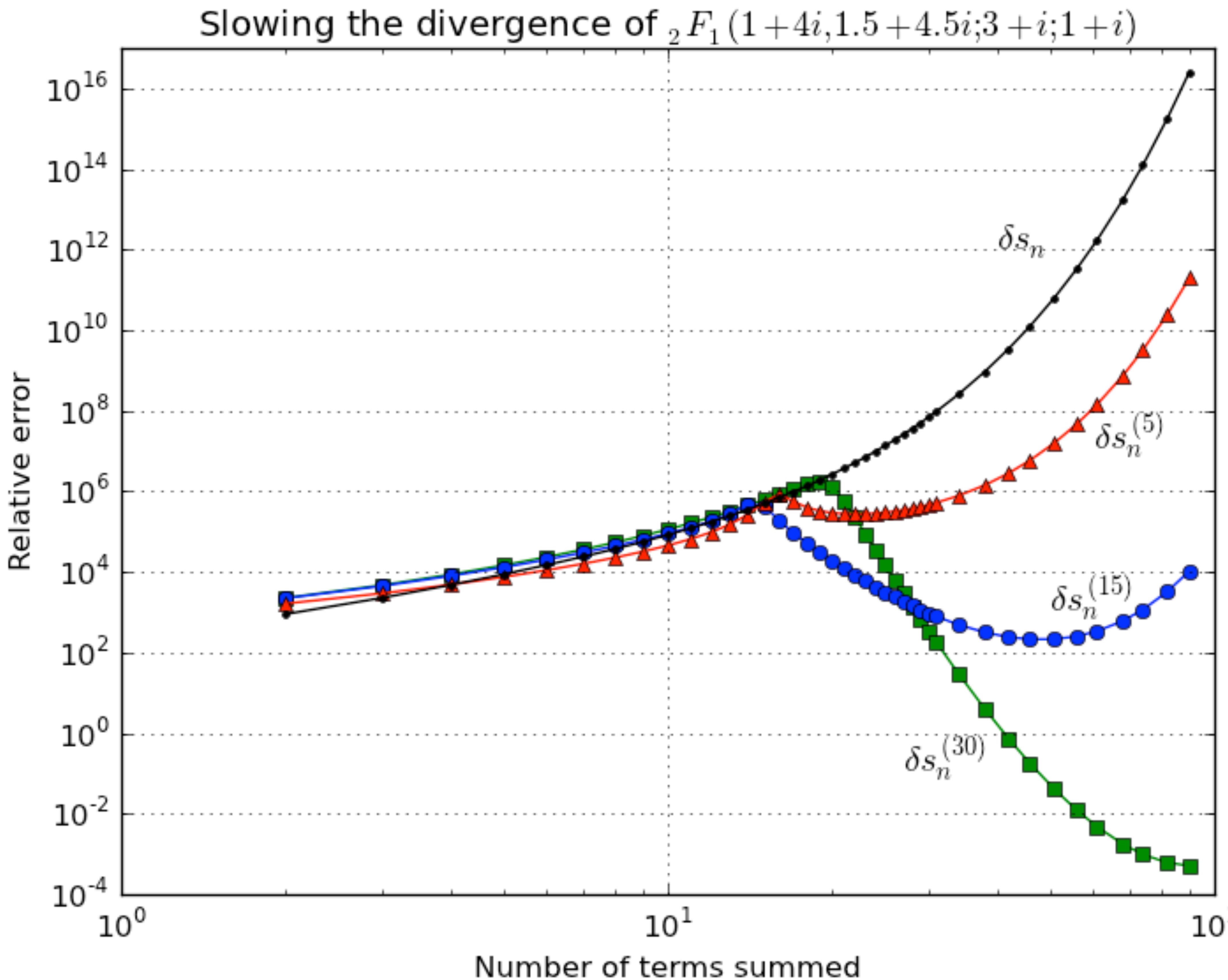

Acceleration of $_2F_1(1+4i, 1.5+4.5i; 3+i; 1)$
Relative error
Number of terms summed
10^6
10^4
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^0
10^1
10^2
10^3
10^4
δs_n
$\delta s_n^{(2)}$
$\delta s_n^{(5)}$
$\delta s_n^{(15)}$
$\delta s_n^{(30)}$



Acceleration of $\zeta(1+200i)$
Relative error
Number of terms summed
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{-13}
10^{-15}
10^{-17}
10^{-19}
10^{-21}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
δs_n
$\delta s_n^{(2)}$
$\delta s_n^{(5)}$
$\delta s_n^{(15)}$
$\delta s_n^{(30)}$



Acceleration of $\zeta(1+0.1i)$
Relative error
Number of terms summed
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
δs_n
$\delta s_n^{(2)}$
$\delta s_n^{(5)}$
$\delta s_n^{(15)}$
$\delta s_n^{(30)}$



Slowing the divergence of $_2F_1(1+4i, 1.5+4.5i; 3+i; 1+i)$
Relative error
Number of terms summed
10^{16}
10^{14}
10^{12}
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{0}
10^{1}
10^{2}
δs_n
$\delta s_n^{(5)}$
$\delta s_n^{(15)}$
$\delta s_n^{(30)}$

Random Tests

For generalized hypergeometric functions, some more exhaustive testing has been done

${}_2F_1(1)$ in double precision, target relative tolerance of 1×10^{-12}

R	C	FP	NC	FN	$n_{\max}$
1	100.0%	0.0%	0.0%	0.0%	0.0%
5	93.82%	0.046%	5.91%	0.23%	5.21%
10	78.89%	0.073%	20.24%	0.80%	13.18%
50	36.40%	0.014%	62.57%	1.02%	11.90%
100	22.62%	0.006%	76.56%	0.81%	9.13%

${}_{q+1}F_q(z)$ in double precision, target relative tolerance of 2×10^{-14}

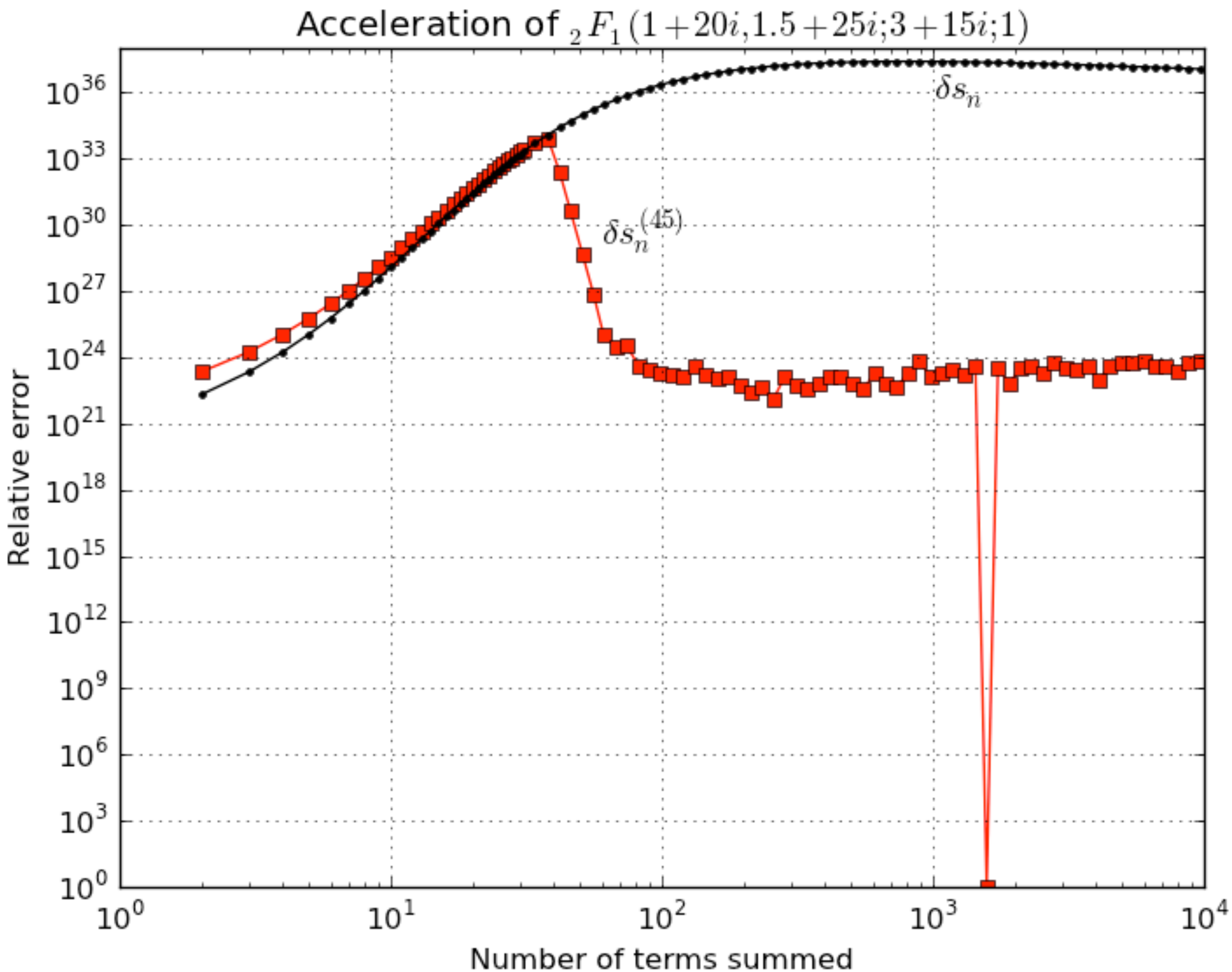
	R	C	FP	NC	FN	$n_{\max}$
${}_2F_1$	1	99.86%	0.002%	0.14%	0.0%	0.0%
	5	94.63%	0.15%	5.10%	0.12%	0.011%
	10	85.98%	0.13%	13.70%	0.19%	0.036%
	50	49.32%	0.12%	50.36%	0.20%	0.17%
	100	31.73%	0.11%	68.03%	0.13%	0.11%
${}_3F_2$	1	99.76%	0.03%	0.21%	0.0%	0.0%
	5	92.85%	0.20%	6.78%	0.10%	0.01%
	10	84.88%	0.13%	14.83%	0.16%	0.03%
${}_4F_3$	1	99.74%	0.0%	0.25%	0.010%	0.0%
	5	91.35%	0.12%	8.44%	0.09%	0.03%

Timing

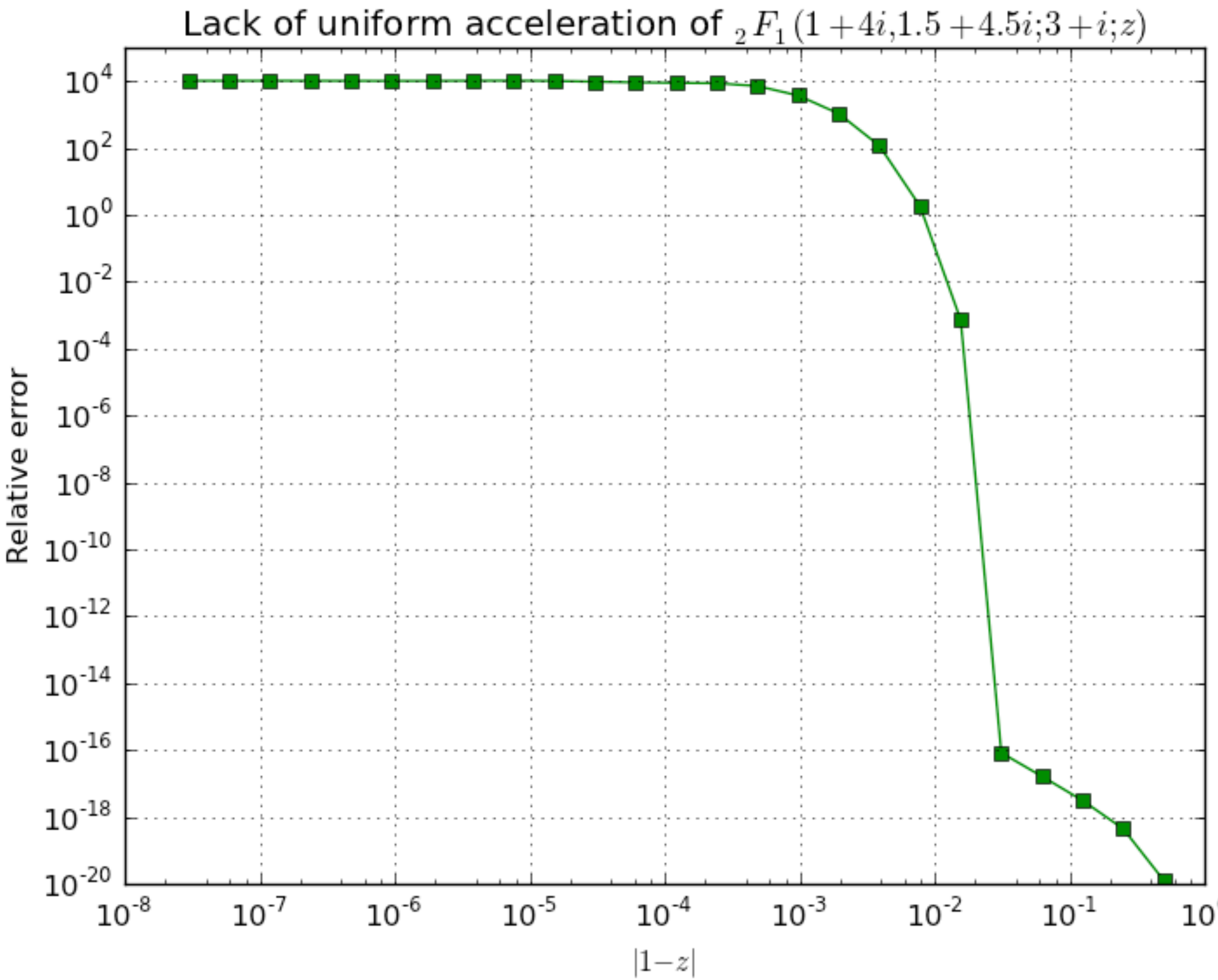
		$m = 30$		$m = 45$	
Function	R	All	Only converged	All	Only converged
${}_2F_1(1)$	1	0.154 ms	0.0951 ms	0.307 ms	0.250 ms
	5	0.675 ms	0.115 ms	0.858 ms	0.292 ms
	10	0.911 ms	0.145 ms	1.15 ms	0.339 ms
	50	0.982 ms	0.332 ms	1.25 ms	0.505 ms
	100	1.11 ms	0.539 ms	1.38 ms	0.728 ms
${}_3F_2(1)$	1	0.119 ms	0.109 ms	0.338 ms	0.330 ms
	5	0.680 ms	0.124 ms	0.843 ms	0.307 ms
	10	0.966 ms	0.157 ms	1.19 ms	0.323 ms
	50	1.15 ms	0.330 ms	1.39 ms	0.485 ms
	100	1.21 ms	0.406 ms	1.46 ms	0.546 ms
${}_4F_3(1)$	1	0.118 ms	0.0981 ms	0.322 ms	0.310 ms
	5	0.726 ms	0.118 ms	0.855 ms	0.258 ms
	10	1.05 ms	0.148 ms	1.29 ms	0.308 ms
	50	1.21 ms	0.286 ms	1.46 ms	0.438 ms
	100	1.33 ms	0.381 ms	1.58 ms	0.546 ms

Average of 1000 evaluations, target tolerance of 2×10^{-14}

Caveats



Acceleration of $_2F_1(1+20i, 1.5+25i; 3+15i; 1)$
Relative error
Number of terms summed
δs_n
$\delta s_n^{(45)}$
10^{36}
10^{33}
10^{30}
10^{27}
10^{24}
10^{21}
10^{18}
10^{15}
10^{12}
10^{9}
10^{6}
10^{3}
10^{0}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}


Lack of uniform acceleration of ${}_2F_1(1+4i, 1.5+4.5i; 3+i; z)$
Relative error
$|1-z|$
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}

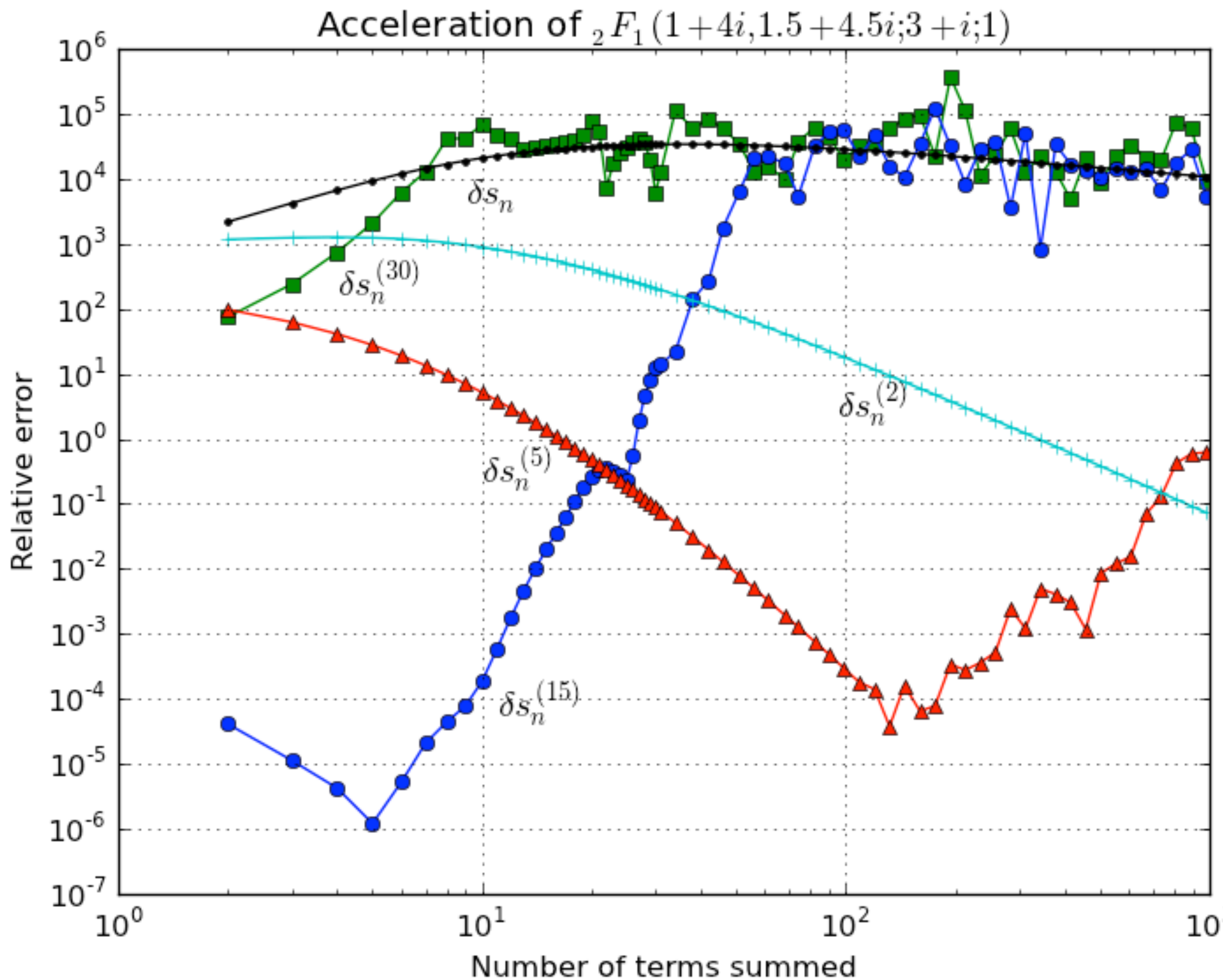
Summary of Problems

- The failure to converge due to finite precision is unavoidable at any fixed precision in summation based methods. But it can be circumvented with higher precision.
- The lack of uniform convergence is more serious; no easy way around it is known.
- But the precise analytic knowledge of the remainders is useful, as we can very accurately estimate truncation error, and use that and roundoff error to determine success or failure.

Comparison

Comparison to E-method

- We mentioned at the beginning that there are a wide variety of acceleration methods; how does ours compare?
- We compare to the E-method, where we do not calculate the asymptotic coefficients analytically, but determine them from several successive partial sums.


Acceleration of $_2F_1(1+4i, 1.5+4.5i; 3+i; 1)$
Relative error
Number of terms summed
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{0}
10^{1}
10^{2}
10^{3}
δs_n
$\delta s_n^{(30)}$
$\delta s_n^{(2)}$
$\delta s_n^{(5)}$
$\delta s_n^{(15)}$

Why the instability?

We can at least heuristically understand the instability of the *E*-method in this case, as it is governed by a linear system:

$$\begin{bmatrix} 1 & z^n/n^{-\lambda} & z^n/n^{1-\lambda} & \dots & z^n/n^{m-1-\lambda} \\ 1 & z^{n+1}/(n+1)^{-\lambda} & z^{n+1}/(n+1)^{1-\lambda} & \dots & z^{n+1}/(n+1)^{m-1-\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z^{n+m}/(n+m)^{-\lambda} & z^{n+m}/(n+m)^{1-\lambda} & \dots & z^{n+m}/(n+m)^{m-1-\lambda} \end{bmatrix} \begin{bmatrix} s \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_m \end{bmatrix} = \begin{bmatrix} s_n \\ s_{n+1} \\ \vdots \\ s_{n+m} \end{bmatrix}$$

Since the coefficients are asymptotic and grow without bound, this linear system *must* be unstable. We can confirm this in examples by calculating condition numbers, which grow rapidly with both n and m.

Conclusions

- For special functions, it is possible to get detailed asymptotic knowledge about the remainders and from this construct a new acceleration method.
- This method provably accelerates convergence or slows divergence in almost all cases where the original series had non-zero radius of convergence.
- For ${}_{q+1}F_q$ at the branch point, this method seems the most effective of any in the literature.
- Future directions: expansions in inverse factorials; can method be adapted to series without known expansion of term ratio?