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Properties and applications of the constrained dual Bernstein polynomials

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Part I. Definitions and properties

Dual Bernstein basis polynomials

- Bernstein basis polynomials of degree n

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$$D_0^n(x;\alpha,\beta),\ D_1^n(x;\alpha,\beta), \ldots, D_n^n(x;\alpha,\beta) \in \Pi_n$$

defined so that

$$\langle D_i^n, B_j^n \rangle_J = \delta_{ij} \qquad (i, j = 0, 1, \ldots, n),$$

where

$$\langle f, g \rangle_J := \int_0^1 (1-x)^\alpha x^\beta f(x)\, g(x)\, dx \qquad (\alpha, \beta > -1)$$

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$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (0 \le i \le n).$$

- Associated with the Bernstein basis, there is a unique dual basis $D_k^n(x; \alpha, \beta) \in \Pi_n$ $(0 \le k \le n)$ defined so that

$$\left\langle D_i^n,\, B_j^n \right\rangle_J = \delta_{ij} \qquad (i, j = 0, 1, \ldots, n),$$

where

$$\langle f, g \rangle_J := \int_0^1 (1-x)^\alpha x^\beta\, f(x)\, g(x)\, dx \qquad (\alpha, \beta > -1)$$

- Shifted Jacobi polynomials $R_k^{(\alpha,\beta)}(x)$ are orthogonal wrt the inner product $\langle f, g \rangle_J$, i.e.,

$$\left\langle R_k^{(\alpha,\beta)},\, R_l^{(\alpha,\beta)} \right\rangle_J = \delta_{kl} h_k \qquad (k, l = 0, 1, \ldots;\ h_k > 0).$$

Paweł Woźny, University of Wrocław, Poland

Constrained dual Bernstein basis polynomials

- Let us define

$$\Pi_n^{(k,l)} := \left\{ P \in \Pi_n \ : \ P^{(i)}(0) = 0 \quad (0 \le i < k), \quad P^{(j)}(1) = 0 \quad (0 \le j < l) \right\},$$

where $k + l \le n$. Certainly, $\Pi_n^{(k,l)} = \mathrm{lin}\left\{ B_k^n, B_{k+1}^n, \ldots, B_{n-l}^n \right\}$.

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- There is a unique constrained dual Bernstein basis of degree n

$$D_k^{(n,k,l)}(x;\alpha,\beta),\ D_{k+1}^{(n,k,l)}(x;\alpha,\beta), \ldots, D_{n-l}^{(n,k,l)}(x;\alpha,\beta) \in \Pi_n^{(k,l)},$$

satisfying the relation

$$\left\langle D_i^{(n,k,l)}, B_j^n \right\rangle_J = \delta_{ij} \quad (i, j = k, k+1, \ldots, n-l),$$

where

$$\langle f, g \rangle_J := \int_0^1 (1-x)^\alpha x^\beta\, f(x)\, g(x)\, dx \qquad (\alpha, \beta > -1)$$

Constrained and unconstrained dual Bernstein polynomials

- Constrained dual Bernstein polynomials $D_i^{(n,k,l)}(x;\alpha,\beta)$ can be expressed in terms of the unconstrained dual Bernstein polynomials of degree $n-k-l$, with parameters $\alpha+2l$ and $\beta+2k$ in the following way:

$$D_i^{(n,k,l)}(x;\alpha,\beta) = \binom{n-k-l}{i-k}\binom{n}{i}^{-1} x^k(1-x)^l\, D_{i-k}^{n-k-l}(x;\alpha+2l,\beta+2k)$$

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- Rababah and Al-Natour, 2007: extented Jüttler's results to the case of arbitrary $\alpha,\ \beta > -1$.
- W&L, 2009 ($\alpha,\ \beta > -1$, and $k, l \in \mathbb{N}$): recurrence relation, orthogonal expansion, "short" representation, relation beetwen constrained and unconstrained dual Bernstein polynomials

Constrained dual Bernstein basis polynomials. Applications

- Least-squares approximation by Bézier curves (Jüttler, 1998)

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- **Problem.** Given a function f. Find the Bézier form of the polynomial $P_n \in \Pi_n^{(k,l)}$ which gives the minimum value of the norm

$$\|f - P_n\|_{L_2} := \sqrt{\langle f - P_n, f - P_n \rangle_J}$$

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- **Solution:**

$$P_n(t) = \sum_{j=k}^{l} a_j B_j^n(t), \qquad a_j := \left\langle f, D_j^{(n,k,l)}(\cdot;\alpha,\beta) \right\rangle_J$$

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Dual Bernstein polynomials: explicit formulae (L&W, 2006)

- Recurrence relation

$$D_i^{n+1}(x;\alpha,\beta) = \left(1-\frac{i}{n+1}\right)D_i^n(x;\alpha,\beta)+\frac{i}{n+1}D_{i-1}^n(x;\alpha,\beta)+\vartheta_i^n R_{n+1}^{(\alpha,\beta)}(x),$$

where

$$\vartheta_i^n := (-1)^{n-i+1}\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{(2n+\alpha+\beta+3)(\alpha+\beta+2)_n}{(\beta+1)_i(\alpha+1)_{n+1-i}}$$

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- Orthogonal expansion

$$D_i^n(x;\alpha,\beta) = K \sum_{k=0}^{n} (-1)^k \frac{(2k/\sigma + 1)(\sigma)_k}{(\alpha+1)_k} Q_k(i;\beta,\alpha,n)\, R_k^{(\alpha,\beta)}(x),$$

where $Q_k(i;\beta,\alpha,n)$ are Hahn orthogonal polynomials,

$$Q_k(x;\alpha,\beta,N) := \sum_{i=0}^{k} \frac{(-k)_i(k+\sigma)_i}{(\alpha+1)_i(-N)_i} \frac{(-x)_i}{i!},$$

and $\sigma := \alpha + \beta + 1$, $K := \dfrac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}$.

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- Short representations

$$D_i^n(x;\alpha,\beta) = \frac{(-1)^{n-i}(\sigma+1)_n}{K\,(\alpha+1)_{n-i}(\beta+1)_i} \sum_{k=0}^{i} \frac{(-i)_k}{(-n)_k} R_{n-k}^{(\alpha,\beta+k+1)}(x),$$

$$D_{n-i}^n(x;\alpha,\beta) = \frac{(-1)^{i}(\sigma+1)_n}{K\,(\alpha+1)_{i}(\beta+1)_{n-i}} \sum_{k=0}^{i} (-1)^k \frac{(-i)_k}{(-n)_k} R_{n-k}^{(\alpha+k+1,\beta)}(x)$$

Dual Bernstein polynomials: Bézier form

- $D_i^n(x;\alpha,\beta) = \sum_{j=0}^{n} C_{ij}(n,\alpha,\beta) B_j^n(x)$

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$$C_{ij}(n,0,0) = \frac{(-1)^{i+j}}{\binom{n}{i}\binom{n}{j}} \sum_{h=0}^{\min(i,j)} (2h+1) \binom{n+h+1}{n-i}\binom{n-h}{n-i}\binom{n+h+1}{n-j}\binom{n-h}{n-j}.$$

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- L&W, 2011 ($\alpha, \beta > -1$):

$$C_{ij}(n,\alpha,\beta) = \frac{1}{B(\alpha+1,\beta+1)} \sum_{m=0}^{n} \frac{(2m/\sigma+1)(\beta+1)_m(\sigma)_m}{m!(\alpha+1)_m} \times Q_m(i;\beta,\alpha,n) Q_m(j;\beta,\alpha,n),$$

where $Q_k(x;\alpha,\beta,N)$ are Hahn orthogonal polynomials, and $\sigma := \alpha+\beta+1$

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$$\Downarrow$$

- $\mathbf{L}_x Q_m(x;\beta,\alpha,n) = m(m+\sigma)Q_m(x;\beta,\alpha,n)$,
- where

$$\mathbf{L}_x y(x) = a(x)y(x+1) - c(x)y(x) + b(x)y(x-1),$$

$$\begin{cases} a(x) := (x-n)(x+\beta+1), \\ b(x) := x(x-a-n-1), \\ c(x) := a(x)+b(x) \end{cases}$$

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Recurrence relation for $C_{ij}(n,\alpha,\beta)$

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- $$C_{i+1,j} = \frac{1}{A(i)}\Big\{(i-j)(2i+2j-2n-\alpha+\beta)C_{ij}+ B(j)C_{i,j-1}+A(j)C_{i,j+1}-B(i)C_{i-1,j}\Big\},$$

  where $C_{ij} \equiv C_{ij}(n,\alpha,\beta)$, and

$$\begin{cases} A(u) := (u-n)(u+1)(u+\beta+1)/(u+1), \\ B(u) := u(u-n-\alpha-1)(u-n-1)/(u-n-1) \end{cases}$$

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- Cross rule: $$\begin{array}{ccc} & C_{i-1,j} & \\ C_{i,j-1} & C_{ij} & C_{i,j+1} \\ & \boxed{C_{i+1,j}} & \end{array}$$

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- Jüttler, Rababah and Al-Natour $\Longrightarrow$ complexity $O(n^3)$

Part II. Applications

Generalizations of Bernstein polynomials

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Now, we focus on the discrete dual Bernstein polynomials and show that

- these polynomials have a very nice application in CAGD
- and are closely related to the (classical) dual Bernstein polynomials

Dual discrete Bernstein polynomials

- Discrete Bernstein basis polynomials of degree n (Sablonnière, 1992)

$$b_i^n(x;N) = \frac{1}{(-N)_n}\binom{n}{i}(-x)_i(x-N)_{n-i} \qquad (0 \le i \le n \le N;\ N \in \mathbb{N})$$

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$$d_0^n(x;\alpha,\beta,N),\ d_1^n(x;\alpha,\beta,N),\ldots,d_n^n(x;\alpha,\beta,N) \in \Pi_n,$$

are defined so that

$$\left\langle d_i^n,\ b_j^n \right\rangle_H = \delta_{ij} \qquad (i,j=0,1,\ldots,n).$$

- Here

$$\langle f,\ g\rangle_H := \sum_{x=0}^{N}\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}f(x)g(x) \qquad (\alpha,\ \beta > -1)$$

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- Recall that Hahn polynomials are orthogonal with respect to this inner product

Dual discrete Bernstein polynomials: difference-recurrence relation

Theorem. Dual discrete Bernstein polynomials $d_i^n(x) \equiv d_i^n(x;\alpha,\beta,N)$ satisfy the following difference-recurrence relation:

$$a_N(x)d_i^n(x+1) + [c_n(i) - c_N(x)]\, d_i^n(x) + b_N(x)d_i^n(x-1) - a_n(i)d_{i+1}^n(x) - b_n(i)d_{i-1}^n(x) = 0,$$

where $0 \le i \le n \le N$, $d_{-1}^n(x) = d_{n+1}^n(x) := 0$, and

$$a_n(x) := (x-n)(x+\alpha+1), \quad b_n(x) := x(x-\beta-n-1), \quad c_n(x) := a_n(x)+b_n(x).$$

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Remark. Thanks to the above result, we can propose an efficient algorithm of solving the so-called problem of the degree reduction of Bézire curves, which is important in CAGD.

Multi-degree reduction of Bézier curves with constraints

- Given a Bézier curve of degree n, with control points $p_i \in \mathbb{R}^d$,

$$P_n(t) = \sum_{i=0}^{n} p_i \, B_i^n(t) \qquad (0 \le t \le 1),$$

where

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (0 \le i \le n)$$

are Bernstein basis polynomials.

Multi-degree reduction of Bézier curves with constraints

- Given a Bézier curve of degree n, with control points $p_i \in \mathbb{R}^d$,

$$P_n(t) = \sum_{i=0}^{n} p_i \, B_i^n(t) \qquad (0 \le t \le 1),$$

where B_i^n $(0 \le i \le n)$ are Bernstein basis polynomials.

- **Problem.** Find a degree m $(m < n)$ Bézier curve,

$$Q_m(t) = \sum_{i=0}^{m} q_i \, B_i^m(t) \qquad (0 \le t \le 1),$$

such that the value of the error

$$\int_0^1 (1-t)^\alpha t^\beta \, \|P_n(t) - Q_m(t)\|^2 \, dt \qquad (\alpha, \ \beta > -1)$$

is minimized in the space Π_m^d under the additional conditions that

$$P_n^{(i)}(0) = Q_m^{(i)}(0) \qquad (0 \le i < k), \qquad P_n^{(j)}(1) = Q_m^{(j)}(1) \qquad (0 \le j < l),$$

where $k + l \le m$, and $\| \cdot \|$ denote the Euclidean vector norm

Degree reduction: previous results

- Many papers relevant to this problem: Eck, 1995; Brunett et al., 1996; Farouki, 2000; Chen and Wang, 2002; Lee et al., 2002, Ahn, 2003; Ahn et al., 2004; Sunwoo and Lee, 2004; Sunwoo, 2005; Zang and Wang, 2005; Lu and Wang, 2006, 2007; Rababah et al., 2006.

- Part of them deal with the unconstrained case, i.e., $k = l = 0$.

- In most cases, $k = l$, and the Legendre parameters, i.e., $\alpha = \beta = 0$, are chosen.

- Chebyshev parameters, i.e., $\alpha = \beta = \pm 1/2$, are also considered: Rababah et al., 2006; Lu and Wang, 2007.

- The main tool used was transformation between the Bernstein and orthogonal polynomial bases.

- The total complexity of known algorithms for optimal multi-degree reduction of Bézier curves with constraints is $O(n^3)$

Degree reduction: motivation

- Data transfer and exchange between design systems.
- Data compression.

Multi-degree reduction of Bézier curves with constraints

- Given the polynomial $P_n \in \Pi_n$,

$$P_n(t) := \sum_{i=0}^{n} p_i \, B_i^n(t).$$

- We look for a polynomial $Q_m \in \Pi_m$ $(m < n)$,

$$Q_m(t) := \sum_{i=0}^{m} q_i \, B_i^m(t),$$

which gives minimum value of the squared norm

$$\|P_n - Q_m\|_{L_2}^2 := \langle P_n - Q_m, \, P_n - Q_m \rangle_J$$

with the constraints

$$P_n^{(i)}(0) = Q_m^{(i)}(0) \qquad (i = 0, 1, \ldots, k-1),$$

$$P_n^{(j)}(1) = Q_m^{(j)}(1) \qquad (j = 0, 1, \ldots, l-1),$$

where $k + l \leq m$

Solution

- Constraints

$$q_0, q_1, \ldots, q_{k-1}, \qquad q_{m-l+1}, q_{m-l+2}, \ldots, q_m$$

Solution

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$$q_0, q_1, \ldots, q_{k-1}, \qquad q_{m-l+1}, q_{m-l+2}, \ldots, q_m.$$

- Other coefficients

$$q_i = \sum_{j=k}^{n-l} w_j \, \Phi_{ij} \qquad (k \le i \le m-l),$$

where

$$\Phi_{ij} := \langle B_j^n, D_i^{(m,k,l)} \rangle_J$$

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$$\underline{\Phi_{ij} := \langle B_j^n, D_i^{(m,k,l)} \rangle_J}$$

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Problem

For $k \le i \le m - l$ and $k \le j \le n - l$, propose an efficient algorithm of computing the quantities Φ_{ij}, where

$$\Phi_{ij} \equiv \Phi_{ij}^{(n,m,k,l)}(\alpha, \beta) := \langle B_j^n, D_i^{(m,k,l)} \rangle_J$$

Relation between Φ_{ij} and dual discrete Bernstein polynomials

Theorem. For $i = k, k+1, \ldots, m-l$ $(0 \le k+l \le m)$, and $j = 0, 1, \ldots, n$ the following formula holds:

$$\begin{aligned}\Phi_{ij} &:= \langle B_j^n, D_i^{(m,k,l)} \rangle_J \\ &= \binom{m-k-l}{i-k}\binom{n}{j}\binom{m}{i}^{-1} \frac{(\alpha+2l+1)_{n-l-j}(\beta+2k+1)_{j-k}}{(n-k-l)!}\Psi_{ij}\end{aligned}$$

with

$$\Psi_{ij} \equiv \Psi_{ij}^{(n,m,k,l)}(\alpha,\beta) := d_{i-k}^{m-k-l}(j-k; \beta+2k, \alpha+2l, n-k-l).$$

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Remark. All we need is a fast method for evaluation of

$$d_{i-k}^{m-k-l}(j-k; \beta+2k, \alpha+2l, n-k-l)$$

for $k \le i \le m-l,\ k \le j \le n-l$

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Difference-recurrence relation

Theorem. Dual discrete Bernstein polynomials $d_i^n(x) \equiv d_i^n(x;\alpha,\beta,N)$ satisfy the following difference-recurrence relation:

$$a_N(x)d_i^n(x+1) + [c_n(i) - c_N(x)]\, d_i^n(x) \\ + b_N(x)d_i^n(x-1) - a_n(i)d_{i+1}^n(x) - b_n(i)d_{i-1}^n(x) = 0,$$

where $0 \le i \le n \le N$, $d_{-1}^n(x) = d_{n+1}^n(x) := 0$, and

$$a_n(x) := (x-n)(x+\alpha+1), \quad b_n(x) := x(x-\beta-n-1), \quad c_n(x) := a_n(x)+b_n(x).$$

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Remark. Thanks to the above result, we can propose an efficient algorithm of computing the quantities Ψ_{ij}

Ψ-table

- $\Psi_{ij} \equiv \Psi_{ij}^{(n,m,k,l)}(\alpha, \beta) := d_{i-k}^{m-k-l}(j-k; \beta+2k, \alpha+2l, n-k-l).$

Ψ-table

- $\Psi_{ij} \equiv \Psi_{ij}^{(n,m,k,l)}(\alpha, \beta) := d_{i-k}^{m-k-l}(j-k; \beta+2k, \alpha+2l, n-k-l)$.
- Quantities Ψ_{ij} $(k \le i \le m-l;\ k \le j \le n-l)$ can be put in a rectangular table,

$$
\begin{array}{cccc}
\Psi_{kk} & \Psi_{k,k+1} & \ldots & \Psi_{k,n-l} \\
\Psi_{k+1,k} & \Psi_{k+1,k+1} & \ldots & \Psi_{k+1,n-l} \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\Psi_{m-l,k} & \Psi_{m-l,k+1} & \ldots & \Psi_{m-l,n-l}
\end{array}
$$

Ψ-table

- $\Psi_{ij} \equiv \Psi_{ij}^{(n,m,k,l)}(\alpha, \beta) := d_{i-k}^{m-k-l}(j-k; \beta+2k, \alpha+2l, n-k-l)$.
- Quantities Ψ_{ij} $(k \le i \le m-l;\ k \le j \le n-l)$ can be put in a rectangular table.
- Using difference-recurrence relation for dual discrete Bernstein polynomials, one may obtain the element $\Psi_{i+1,j}$ in terms of four elements from the rows number i and $i-1$.

$$
\begin{array}{ccc}
 & \Psi_{i-1,j} & \\
\Psi_{i,j-1} & \Psi_{ij} & \Psi_{i,j+1} \\
 & \boxed{\Psi_{i+1,j}} &
\end{array}
$$

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 & \boxed{\Psi_{i+1,j}} &
\end{array}
$$

- More specifically,

$$
\begin{aligned}
\Psi_{i+1,j} \;=\; \{A(n,j)\,\Psi_{i,j-1} + [C(m,i)-C(n,j)]\,\Psi_{ij} + {}\\
B(n,j)\,\Psi_{i,j+1} - A(m,i)\,\Psi_{i-1,j}\}/B(m,i),
\end{aligned}
$$

$A(r,s) := (k-s)(r+l-s+\alpha+1)$, $B(r,s) := (s+l-r)(k+s+\beta+1)$, $C(r,s) := A(r,s)+B(r,s)$.

Ψ-table

- $\Psi_{ij} \equiv \Psi_{ij}^{(n,m,k,l)}(\alpha, \beta) := d_{i-k}^{m-k-l}(j-k; \beta+2k, \alpha+2l, n-k-l)$.
- Quantities Ψ_{ij} $(k \le i \le m-l;\ k \le j \le n-l)$ can be put in a rectangular table.
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$$\begin{array}{ccc} & \Psi_{i-1,j} & \\ \Psi_{i,j-1} & \Psi_{ij} & \Psi_{i,j+1} \\ & \boxed{\Psi_{i+1,j}} & \end{array}$$

- Cost: $O(nm)$

Degree reduction: conclusions

- We solved the problem of optimal multi-degree reduction of Bézier curves with constraints in the general case, i.e., for $\alpha, \beta > -1$, and arbitrary $k, l \in \mathbb{N}$.
- In our approach, we use the dual constrained Bernstein and dual discrete Bernstein polynomials.
- Our method does not use explicitly transformation between the Bernstein and orthogonal polynomial bases.
- The main tool is the difference–recurrence relation for dual discrete Bernstein polynomials.
- The complexity of the method is $O(nm)$, which seems to be significantly less than complexity of most known algorithms for multi-degree reduction of Bézier curves with constraints.