Advanced School on Integral Equations and Applications

Lisbon, May 18-20, 2017

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1 ABSTRACTS OF THE COURSES

Integral equations: from basic tools to advanced methods

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Abstract

Integral equations are an important subject within applied mathematics. They are used as mathematical models for many and varied physical situations, and also occur as reformulations of other mathematical problems. The aim of this course is to provide basic notions for a complete knowledge of the classical and advanced methods for the numerical solution of integral equations.

Numerical Analysis of Fractional Differential Equations of Caputo type

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Abstract

In this course we focus on numerical methods for differential equations of noninteger order, in the Caputo sense. The course is divided in two main parts. In the first one, after a brief review of the basic definitions of fractional derivatives we start with the theory of initial value problems of Caputo type. We emphasize the equivalence of such problems with singular Volterra integral equations, and we present and discuss several numerical schemes to solve them. Then terminal boundary value problems are considered and the equivalence between them and singular Fredholm integral equations is established. Once again some numerical methods are presented. In both cases, we consider ordinary single-term equations, and we highlight the main issues arising in the numerical approximation of this kind of problems, which are mainly related to the non-locality of the fractional differential operator, and to the existence of singular solutions. Later, we discuss how these ideas can be extended to solve multi-term or partial differential problems. The second part is devoted to the numerical analysis of distributed-order ordinary and partial differential equations. The distributed-order derivative is a linear operator, defined as a weighted integral of different differentiation orders over a certain range. Thus, distributed-order differential equations can be regarded as natural generalizations of differential equations of integer and fractional order. The available literature on this type of derivatives is still scarce and only very few studies were devoted to the numerical approximation of such models. We will present the ones for Caputo type derivatives.

A numerical library for the solution of integro-differential problems

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Abstract

The tau method, introduced by Cornelius Lanczos, is a spectral method originally developed to compute a polynomial that approximates the solution of a linear ordinary differential problem with polynomial coefficients. The method has been used since then and extended to problems with non polynomial coefficients, to functional and to nonlinear integro-differential equations, among others. This widespread was only possible from the pioneering work of Ortiz and Samara with the introduction of an algebraic formulation of the method. The Tau Toolbox is a project to aggregate all these contributions, to enhance the use of the method by developing more stable algorithms and to offer efficient implementations of its algebraic formulation. It is able to solve various integro-differential problems, linear and nonlinear, with initial and/or boundary or others conditions, working with the most common polynomial orthogonal basis. Non expert can now profit from this spectral method and its solutions properties. On the other hand, experts can easily analyze new problems by exploring the large set of building blocks functions provided.

Drug diffusion in viscoelastic media: analysis, numerics and applications

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Abstract

It is well known that the use of the classical diffusion equation to describe drug diffusion in viscoelastic materials leads to inaccurate results. This fact motivates the inclusion of the viscoelastic properties of the material in the diffusion equation. Consequently integro-differential equations of Volterra type arise in the description of such phenomena. In this course we present an overview on mathematical models of non-fickian type, their analysis and numerical simulation in different contexts.

2 ABSTRACTS OF THE SEMINARS

Building test examples for projection methods in the solution of nonlinear integral equations

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Abstract

We will consider Fredholm integral equations with nonlinear integral operators and describe the classical projection methods applied to these equations, as well as more recent projection methods. Their numerical solution is based on matrix problems and the formulation of the representation matrices will be addressed and fully built for an example. The function solution is then obtained from the matrix problem solution.

Modeling electromagnetic wave's propagation in human eye's Structure

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Abstract

In this talk we will discuss the a mathematical model that describes the electromagnetic wave's propagation through the eye's structures in order to create a virtual OCT scan. Our model is based on time-dependent Maxwel's equations, that we deduce in the integral form. Maxwell's equations can also be cast in the differential form which we consider for the numerical discretization. We focus on deriving stability and convergent estimates a leap-frog type discontinuous Galerkin method. In the model we consider anisotropic permittivity tensors which arise naturally in our application of interest. We illustrate the performance of the method with some numerical experiments.

On some classes of Volterra integral equations

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Abstract

Many physical, chemical and biological problems are modelled as nonlinear singular Volterra integral equations (VIEs). One can find in the literature numerous papers which deal with the analytical properties and/or numerical approximations of specific applied problems, namely: superfluidity and travelling wave analysis in nonlinear reaction-convection-diffusion problems, nonlinear waves, heat flow problems with radiation cooling at the boundary, absorption of gas through a liquid surface, industrial problems related with the ageeing of stainless steel; we also refer to the problem of a competitive chemical reaction between an antigen and a labeled antigen for antibody sites on a cell wall, with applications to some medical devices. In this talk we will give a review of some of these works.

Due to the singularities in the kernels of the integral equations, their exact solutions will in general not be differentiable at an end point of the integration interval; this causes a loss of the optimal (global) convergence orders of the classical product integration and collocation methods based on piecewise polynomial approximations. If one wants a high order method, the nonsmooth behaviour of the solutions has to be taken into account. We illustrate some techniques that allow us to recover the optimal orders.

For comprehensive studies and many references we refer to [1], [2].

References

[1] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, Cambridge, 2004.

[2] H. Brunner, *Volterra Integral Equations. An Introduction to Theory and Applications*, Cambridge University Press, Cambridge, 2017.

Numerical solution of an integro-differential equation modelling the neuronal activity

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Abstract

Modelling the neuronal activity in the cerebral cortex is a very challenging task, which has nowadays multiple applications not only in Medicine (interpretation of data, such as EEG, fMRI and optical imaging) but also in Robotics. One of the most promising approaches in this domain are the so called Neural Field Equations (NFE), where a certain region of the cortex is considered as a *continuous field of neurons* with certain electrical properties. In this work we consider a two-dimensional NFE in the form

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \int_{\Omega} K(|\bar{x}-\bar{y}|)S(V(y,t-\tau(\bar{x},\bar{y}))d\bar{y}, \tag{1}$$

$\bar{x} \in \Omega \subset R^2, t \in [0,T]$, where the unknown $V(\bar{x},t)$ is a continuous function $V : \Omega \times [0,T] \to R$, I, K and S are given functions; c is a constant. We search for a solution V of this equation which satisfies the initial condition $V(\bar{x},t) = V_0(\bar{x},t), \quad \bar{x} \in \Omega, \quad t \in [-\tau_{max}, 0]$, where $\tau_{max} = \max_{\bar{x},\bar{y} \in \Omega} \tau(\bar{x},\bar{y})$. Here τ is a delay depending on $\bar{x}$ and $\bar{y}$ (as a particular case, we also consider the case $\tau \equiv 0$).

Equation (**??**) without delay was introduced first by Wilson and Cowan [3], and then by Amari [1], to describe excitatory and inhibitory interactions in populations of neurons.

We describe a numerical method recently introduced [2] to approximate the solution of equation (**??**). The accuracy and efficiency of the method are discussed and some numerical examples are presented which illustrate its performance.

This talk is based in a joint work with E. Buckwar, from the University of Linz.

References

[1] S. L. Amari, *Dynamics of pattern formation in lateral-inhibition type neural fields* Biol. Cybernet. **27** (2) (1977), pp. 77–87.

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[3] H. R. Wilson and J.D. Cowan, *Excitatory and inhibitory interactions in localized populations of model neurons*, Bipophys. J. **12** (1972), pp.1–24.

Fractional diffusion problems

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Abstract

Fractional derivatives have been used to model anomalous diffusion and therefore the fractional diffusion equation has been presented as a suitable model for many problems that appear in different fields, such as biology, finance and hydrology. This equation can be obtained from the standard diffusion equation by replacing the second order spatial derivative by a fractional operator of an order between one and two. This operator is defined by a combination of the left and right Riemann-Liouville derivatives. The most simple problems involving Riemann-Liouville derivatives lead to challenging questions, from different point of views, such as, mathematical or physical and in this talk we will discuss some of them. We start with the development of a numerical method for a diffusion equation defined in an open domain and show some numerical solutions to observe the anomalous diffusive process. Then, we present some interesting properties of the Riemann-Liouville derivative to discuss a few of the challenges we are faced with when we want to define a well-posed problem in a bounded domain, subject to Dirichlet boundary conditions. We also analyse the additional difficulties encountered related to the numerical discretisation of this operator.

A numerical library for the solution of integro-differential problems

Paulo Vasconcelos
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Abstract

The tau method, introduced by Cornelius Lanczos, is a spectral method originally developed to compute a polynomial that approximates the solution of a linear ordinary differential problem with polynomial coefficients. The method has been used since then and extended to problems with non polynomial coefficients, to functional and to nonlinear integro-differential equations, among others. This widespread was only possible from the pioneering work of Ortiz and Samara with the introduction of an algebraic formulation of the method. The Tau Toolbox is a project to aggregate all these contributions, to enhance the use of the method by developing more stable algorithms and to offer efficient implementations of its algebraic formulation. It is able to solve various integro-differential problems, linear and nonlinear, with initial and/or boundary or others conditions, working with the most common polynomial orthogonal basis. Non expert can now profit from this spectral method and its solutions properties. On the other hand, experts can easily analyze new problems by exploring the large set of building blocks functions provided.

One-dimensional model for blood flow based on Cosserat Theory

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Abstract

In this talk, we study the unsteady motion of a generalized viscoelastic fluid of third-grade where specific normal stress coefficient depends on the shear rate by using a power-law model. For this issue, we use the Cosserat theory approach which reduces the exact three-dimensional equations to a system depending only on time and on a single spatial variable. This one-dimensional system is obtained by integrating the linear momentum equation over the cross-section of the tube, taking a velocity field approximation provided by the Cosserat theory. The velocity field approximation satisfies exactly both the incompressibility condition and the kinematic boundary condition. From this reduced system, we obtain unsteady equations for the wall shear stress and mean pressure gradient depending on the volume flow rate, Womersley number, viscoelastic coefficients and flow index over a finite section of the tube geometry with constant circular cross-section. The attention is focused on some numerical simulations.

References

[1] C. Truesdell, W. Noll, *The non-linear field theories of mechanics*, 2nd edition , Springer, New-York, 1992.

[2] R.L. Fosdick, K.R. Rajagopal, *Thermodynamics and stability of fluids of third grade*, Proc. R. Soc. Lond. A., **339** (1980), pp. 351–377.

[3] D. A. Caulk, P.M. Naghdi, *Axisymmetric motion of a viscous fluid inside a slender surface of revolution*, Journal of Applied Mechanics, **54**(1) (1987), pp. 190–196.

A Quadrature-Difference Method to solve a System of coupled second order Fredholm Integro-Differential Equations with constant Coefficients

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Abstract

The use of integro-differential equations plays an important role in modeling many mathematical and physical problems nowadays. One example is Option Pricing in the field of Financial Mathematics. When the underlying asset follows a jump-diffusion process, the resulting equation after applying appropriate stochastic tools, is a partial integro-differential equation. If a more complex underlying process for the asset is used like regime-switching or jump-telegraph diffusion process, then we obtain a system of partial integro-differential equations. In most of the cases, close solution does not exist and the system can only be solved numerically.

In this paper we present a Quadrature-Difference Method (QDM) to solve a system of second order coupled Fredholm Integro-Differential Equation with constant coefficients. The basic idea is to fully discretize the coupled equations of the system using a finite difference technique on the differential part and combine it with a quadrature formula on the integral parts of the equation. After the overall discretization we arrive to two systems of linear algebraic equations that can be solve simultaneously as a matrix block system. Some numerical problems are presented to test the performance and accuracy of the proposed method. Moreover, we compute the numerical order of convergence of the proposed method.

Boundary Domain Integral Equations for a Mixed Elliptic BVP with Variable Coefficient in Bounded Domains

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Abstract

A mixed boundary value problem for the diffusion partial differential equation with variable coefficient is reduced to a system of direct segregated parametrix-based Boundary-Domain Integral Equations (BDIEs). We use a parametrix different from the one employed by Mikhailov in [2, 3] and Chkadua, Mikhailov, Natroshvili in [1]. Mapping properties of the potential type integral operators appearing in these equations are presented in appropriate Sobolev spaces. We prove the equivalence between the original BVP and the corresponding BDIE system. The invertibility and Fredholm properties of the boundary-domain integral operators are also analysed.

Based on joint work with S. Mikhailov

References

[1] O. Chkadua, S.E. Mikhailov, D. Natroshvili, *Analysis of direct boundary-domain integral equations for variable-coefficient for a mixed BVP with variable coefficient, I: equivalence and invertibility*, J. Integral Equations and Applications **21(4)** (2009), pp. 499-543.

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Product Integration Rules on Plans Domains

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Abstract

This talk deals with the numerical computation of integrals of the type

$$I(f;\mathbf{t}) = \int_D f(\mathbf{x})K(\mathbf{x},\mathbf{t})w(\mathbf{x})d\mathbf{x}, \qquad \mathbf{x} = (x_1,x_2), \quad \mathbf{t} = (t_1,t_2), \tag{1}$$

where $D = [-1,1] \times [-1,1]$, f is a sufficiently smooth bivariate function defined on D, with possible algebraic singularities on the boundaries of D, w is the product of two Jacobi weight functions and the kernel K is defined in $D \times T$, where $T \subseteq D$. Kernels of this type, appear, for instance, in the numerical solution of integral for BEM 3D (see [1], [4]) and in the numerical solution of bivariate integral equations (see [2], [3]). We study the product cubature rule obtained by approximating the "regular"function f by suitable bivariate Lagrange polynomials interpolating the smooth function f at the zeros of univariate Jacobi polynomials. For some choices of the kernel K we determine suitable strategies to compute the coefficients of the cubature rule. For instance, we treat the cases of "nearly"singular kernels of the type

$$K(\mathbf{x},c) = \frac{1}{(x_1^2 + x_2^2 + c^2)^\lambda}, \quad 0 < |c|, \quad \lambda > 0,$$

for "small"values of the parameter c and of kernels like

$$K(\mathbf{x},\omega) = g(\omega x_1 x_1),$$

where g is a smooth high oscillatory function for "large"values of the parameter ω.

The stability and the convergence of the cubature rule are proved and some numerical tests, which confirm the theoretical estimates, are proposed.

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References

[1] B.M. Johnston, P.R. Johnston, D. Elliott, *A new method for the numerical evaluation of nearly singular integrals on triangular elements in the 3D boundary element method*, Journal of Computational and Applied Mathematics **Volume** (245) (2013), pp. 148–161.

[2] G. Mastroianni, G. Milovanović, D. Occorsio, *A Nyström method for two variables Fredholm integral equations on triangles*, Appl. Math. and Comput., **219** (2013), pp. 7653–7662.

[3] D. Occorsio, M. G. Russo, *Numerical methods for Fredholm integral equations on the square*, Appl. Math. and Comput., **218** (2011), pp. 2318–2333.

[4] L. Scuderi, *On the computation of nearly singular integrals in 3D BEM collocation*, Int. J. Numer. Meth. Engng., **74** (2008), pp. 1733–1770.

Stabilization of the spectral tau method for nonlinear integro-differential systems of equations

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Abstract

In this work an extension of the spectral Lanczos' tau method to systems of nonlinear integro-differential equations is proposed. This extension includes (i) linearization coefficients of orthogonal polynomials products issued from nonlinear terms and (ii) recursive relations to implement matrix inversion whenever a polynomial change of basis is required and (iii) orthogonal polynomial evaluations directly on the orthogonal basis. All these improvements ensure numerical stability and accuracy in the approximate solution. Exposed in detail, this novel approach is able to significantly outperform numerical approximations with other methods as well as different tau implementations. Numerical results on a set of problems illustrate the impact of the mathematical techniques introduced.

4 PROGRAM OF THE SCHOOL

SECOND ADVANCED SCHOOL ON INTEGRAL EQUATIONS AND APPLICATIONS
LISBON, 18-20 MAY, 2017
ROOM P3.10 of the DEPARTMENT OF MATHEMATICS

18 May – THURSDAY	19 May – FRIDAY	20 May – SATURDAY
9:00 **OPENING**		
9:15-10:00 10:15-11:00 **COURSE A** (*Fermo*)	9:15-10:00 10:15-11:00 **COURSE A** (*Fermo*)	9:15-10:00 10:15-11:00 **COURSE A** (*Fermo*)
11:00-11:30 COFFE BREAK	11:00-11:30 COFFE BREAK	11:00-11:30 COFFE BREAK
11:30-12:30 **COURSE B** (Ferreira)	11:30-12:30 **COURSE B** (Ferreira)	11:30-11:55 (*Serafini*) 11:55-12:20 (*Carapau*) **CONTRIBUTED TALKS**
12:30-14:30 LUNCH	12:30-14:30 LUNCH	12:30-14:30 LUNCH
14:30-16:00 **SEMINAR** (Vasconcelos)	14:30-15:30 15:45-16:45 **COURSE C** (Morgado, Rebelo)	14:30-15:30 15:45-16:45 **COURSE C** (Morgado, Rebelo)
16:00-16:25 (*Trindade*) 16:25-16:50 (Fresdena-Portillo) **CONTRIBUTED TALKS**		
16:50-17:15 COFFE BREAK	16:45-17:15 COFFE BREAK	16:45-17:15 COFFE BREAK
17:15-17:40 (Silva) **CONTRIBUTED TALKS**	17:15-18:00 **SEMINAR** (Barbeiro)	17:15-18:15 **SEMINAR** (Sousa)
17:40-18:25 **SEMINAR** (Almeida)	18:00-18:45 **SEMINAR** (Lima)	18:15-19:00 **SEMINAR** (Diogo)

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