Geometric means of matrices: analysis and algorithms

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Matrix geometric means provide a tool to average a set of positive definite matrices in such a way that the inverse of the matrix mean coincides with the mean of the inverse matrices. Different definitions of matrix mean have been given in the literature together with algorithms for their computations.

In some applications one needs to compute the geometric mean of a set of structured matrices. This happens, for instance, in radar detection problems where the matrices to average are Toeplitz. For physical reasons, one requires that the mean of structured matrices maintains the same structure of the input matrices. Unfortunately this requirement is not satisfied by the available definitions.

In this talk we give an overview on matrix geometric means, and recall their relationships with the Riemannian geometry of the cone of positive definite matrices. Then we treat with more attention the Karcher mean, relate it to a matrix equation, and provide numerical algorithms for its solution. Finally we present a modified version of the Karcher mean which preserves the structure of the input matrices and satisfies almost all the nice properties of the scalar geometric mean. An effective numerical algorithm for its computation is given in terms of solution of a vector equation.

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