Integrable Flows for Starlike Curves in Centroaffine Space

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Outline

Introduction

For integrable PDE describing geometric evolutions of curves, the differential invariants (*curvatures*) play a fundamental role in revealing integrability.

Example: Vortex Filament Flow

$\gamma(x,t) \in \mathbb{R}^3$: position vector of an evolving space curve.
x: arclength parameter.
κ, τ: differential invariants (curvature and torsion).

$$\underset{\text{binormal evolution}}{\gamma_t = \gamma_x \times \gamma_{xx} = \kappa B} \quad \xrightarrow[q = \frac{1}{2}\kappa e^{\mathrm{i}\int \tau\, dx}]{\text{Hasimoto Map}} \quad \underset{\text{cubic focusing NLS}}{\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0}$$

This work: investigates the relation between a *non-stretching* curve evolution in centro-affine space and the completely integrable PDE system for the differential invariants, by seeking a natural Hamiltonian formulation of the curve flow.

Inspiration: Hamiltonian setting for the Vortex Filament Flow

Pre-symplectic structure on the space of curves:

$$\omega_\gamma(X,Y) = \int_\gamma |X, \gamma', Y| \, \mathrm{d}x. \quad \text{(Marsden \& Weinstein, 1983)}$$

A geometric recursion operator $X_{j+1} = -T \times X_j'$ plus the non-stretching condition generate an infinite hierarchy (Langer & Perline, 1991):

Commuting flows: $\gamma_t = X_j$

$$\begin{aligned} X_1 &= \kappa B \\ X_2 &= \tfrac{1}{2}\kappa^2 T + \kappa' N + \kappa\tau B, \\ X_3 &= \kappa^2\tau T + (2\kappa'\tau + \kappa\tau')N \\ &\quad + (\kappa\tau^2 - \kappa'' - \tfrac{1}{2}\kappa^3)B \\ &\vdots \end{aligned}$$

Conserved integrals: $\int \rho_j \, \mathrm{d}x$

$$\begin{aligned} \rho_0 &= \|T\|, \\ \rho_1 &= -\tau, \\ \rho_2 &= \tfrac{1}{2}\kappa^2, \\ \rho_3 &= \tfrac{1}{2}\kappa^2\tau, \\ \rho_4 &= \tfrac{1}{2}((\kappa')^2 + \kappa^2\tau^2) - \tfrac{1}{8}\kappa^4, \\ &\vdots \end{aligned}$$

where X_{j+1} is the Hamiltonian vector field for $\int \rho_j \, \mathrm{d}x$

Starlike curves in centroaffine $\mathbb{R}^3$

– The "isometry group" is $SL(3,\mathbb{R})$ without translations.

– A smooth curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ is *starlike* if $|\gamma, \gamma', \gamma''| \neq 0$. This condition is invariant under the linear action of $SL(3,\mathbb{R})$.

– Define centroaffine arclength $\displaystyle\int |\gamma, \gamma', \gamma''|^{1/3}\,\mathrm{d}x$. A curve γ is arclength parametrized if

$$|\gamma, \gamma', \gamma''| = 1. \tag{1}$$

From now on, let $\gamma(x)$ be a starlike curve with arclength parameter x.

Differentiating (1) with respect to x gives $|\gamma, \gamma', \gamma'''| = 0$, implying

$$\gamma''' = p_0\gamma + p_1\gamma'.$$

p_0, p_1 are the differential invariants (*Wilczynski invariants*) of γ, and compare with Euclidean torsion and curvature (resp.):

- If $p_0 = 0$ then γ is planar.
- If $p_0 = \frac{1}{2}p_1'$ then γ lies on a conic in $\mathbb{RP}^2$.

Hamiltonian structure on starlike loops

Let $\widehat{M} \subset \mathrm{Map}(S^1, \mathbb{R}^3)$ be the space of starlike loops $\gamma : S^1 \to \mathbb{R}^3$ parametrized by centroaffine arclength.

The vector field $X = \delta_X \gamma = a\gamma + b\gamma' + c\gamma''$ (a, b, c smooth, periodic) is in $T_\gamma \widehat{M}$ (i.e. is *non-stretching*) if $\delta_X |\gamma, \gamma', \gamma''| = 0$:

$$a + b' + \frac{1}{3}(c'' + 2p_1 c) = 0.$$

For $\gamma \in \widehat{M}$, the closed skew-symmetric 2-form

$$\omega_\gamma(X, Y) = \oint_\gamma |X, \gamma', Y| \mathrm{d}x, \qquad X, Y \in T_\gamma \widehat{M}$$

gives *pre-symplectic structure* on $\widehat{M}$ with 2-dimensional kernel spanned by

$$Z_0 = \gamma', \quad Z_1 = \gamma'' - \frac{2}{3} p_1 \gamma.$$

Hamiltonian vector fields: examples

Using the correspondence between Hamiltonians $H \in C^\infty(\widehat{M})$ and Hamiltonian vector fields X_H:

$$\mathrm{d}H[X] = \omega_\gamma(X, X_H), \quad \forall X \in T_\gamma \widehat{M} \tag{2}$$

find that:

1. $Z_1 = \gamma'' - \frac{2}{3} p_1 \gamma$ is the Hamiltonian vector field for $\oint_\gamma |\gamma, \gamma', \gamma''|^{1/3} \mathrm{d}x$ (total arclength);
2. $Z_2 = p_1 \gamma'' + (p_0 - p_1')\gamma' + (\frac{2}{3}(p_1'' - p_1^2) - p_0')\gamma$ is the Hamiltonian vector field for $\oint_\gamma (-p_1)\mathrm{d}x$ (minus the total curvature).

Remark: Correspondence (2) is not an isomorphism. For those H's for which X_H exists, X_H is defined up to addition of elements in the kernel of ω_γ.

General curvature evolutions

Switch to $k_1 = p_1$, $k_2 = p_0 - p_1'$. (Then $Z_2 = k_1\gamma'' + k_2\gamma' + \ldots$).

Let $\gamma_t = r_0\gamma + r_1\gamma' + r_2\gamma''$ be a general non-stretching flow (i.e. with $r_0 = -r_1' - \frac{1}{3}(r_2'' + 2k_1 r_2)$.)

Then, the differential invariants evolve by

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

where $\mathcal{P}$ is a skew-adjoint 5th order matrix differential operator:

$$\begin{pmatrix} -2D^3 + Dk_1 + k_1 D & -D^4 + D^2 k_1 + 2Dk_2 + k_2 D \\ D^4 - k_1 D^2 + 2k_2 D + Dk_2 & \frac{2}{3}(D^5 + k_1 D k_1 - k_1 D^3 - D^3 k_1) + [k_2, D^2] \end{pmatrix},$$

with $D = D_x$.

Remark: $\mathcal{P}$ plays a key role in the integrability of $\gamma_t = Z_1$.

Bi-hamiltonian formulation

The curvature evolution induced by $\gamma_t = Z_1 = \gamma'' - \frac{2}{3}k_1\gamma$ can be written in Hamiltonian form in two distinct ways:

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\mathsf{E}\rho_1, \qquad \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{Q}\mathsf{E}\rho_3, \tag{$\ddagger$}$$

where

$$\rho_1 = k_2, \qquad \rho_3 = \frac{1}{3}(k_1')^2 + k_2k_1' + (k_2)^2 + \frac{1}{9}k_1^3$$

are conserved densities, E is the Euler operator

$$\mathsf{E}f = \left(\sum_{j\geq 0} (-D)^j \frac{\partial f}{\partial k_1^{(j)}}, \sum_{j\geq 0} (-D)^j \frac{\partial f}{\partial k_2^{(j)}} \right)^T,$$

and $\mathcal{Q} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$.

Since $\mathcal{P}$, $\mathcal{Q}$ form a compatible pair of Hamiltonian operators (related to the Adler-Gel'fand-Dikii bracket for $\mathfrak{sl}(3)$ and its companion), ($\ddagger$) is a bi-Hamiltonian system.

A double hierarchy: Recursion Operators

The curvature evolution induced by $\gamma_t = Z_2 = k_1\gamma'' + \kappa_2\gamma' + \ldots$ is also bi-Hamiltonian for $\mathcal{P}$ and $\mathcal{Q}$, with respect to the densities:

$$\rho_2 = k_1 k_2, \quad \rho_4 = \tfrac{1}{3}(k_1'')^2 + k_1''(k_2' - k_1^2) - k_1(k_1')^2 + (k_2')^2 - k_1^2 k_2' + \tfrac{1}{9}k_1^4 + 2k_1 k_2^2.$$

Define a sequence of evolution equations for k_1, k_2

$$\frac{\partial}{\partial t_j}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = F_j[k_1, k_2],$$

via the recursion

$$F_{j+2} = \mathcal{P}\mathcal{Q}^{-1}F_j,$$

with initial data given by

$$F_0 = \begin{pmatrix} k_1' \\ k_2' \end{pmatrix}, \qquad F_1 = \begin{pmatrix} k_1'' + 2k_2' \\ \frac{2}{3}(k_1 k_1' - k_1''') - k_2'' \end{pmatrix},$$

and a sequence of conserved densities given by

$$\mathsf{E}\rho_{j+2} = \mathcal{Q}^{-1}\mathcal{P}\mathsf{E}\rho_j, \qquad \text{with} \qquad \rho_0 = k_1, \rho_1 = k_2.$$

Connection with the Boussinesq hierarchy

The curvature evolution induced by $\gamma_t = Z_1$:

$$\frac{\partial}{\partial t}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1'' + 2k_2' \\ \frac{2}{3}(k_1 k_1' - k_1''') - k_2'' \end{pmatrix},$$

is equivalent to the Boussinesq equation

$$\frac{\partial}{\partial t}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6}q_1''' - \frac{2}{3}q_1 q_1'' \\ 2q_0' \end{pmatrix}$$

under the change of variables $k_1 = -q_1$, $k_2 = \frac{1}{2}q_1' - q_0$. (See also Chou & Qu, 2002.)

We show:

- The curvature evolution induced by $\gamma_t = Z_2$ is equivalent to the second nontrivial flow in the Boussinesq hierarchy;
- The recursion operator $\mathcal{P}\mathcal{Q}^{-1}$ is equivalent to the Boussinesq recursion operator.

Relation with centroaffine curve flows

Theorem: *Each flow of the Boussinesq hierarchy is the curvature evolution induced by a geometric flow for centroaffine curves in* $\mathbb{R}^3$.

Proof: Define

$$X^{\rho_j} := Z_j = (\mathsf{E}\rho_j)_1\gamma' + (\mathsf{E}\rho_j)_2\gamma'' + r_0\gamma,$$

(r_0 given by the non-stretching condition), with ρ_j the j-th Boussinesq conserved density.

Then, $\gamma_t = Z_j$ induces the curvature evolution $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = F_j$.

Theorem: *Let* $H(\gamma) = \oint_\gamma(-\rho_j)\mathrm{d}x$ *and* $\mathsf{E}\rho_{j+2} = \mathcal{Q}^{-1}\mathcal{P}\mathsf{E}\rho_j$ *(the next density after* ρ_j *in the Boussinesq hierarchy). Then*

$$\mathrm{d}H[X] = \omega_\gamma(X, Z_{j+2}) \qquad \forall X \in T_\gamma\widehat{M}.$$

That is, $\gamma_t = Z_j, j \geq 2$ is a Hamiltonian evolution with Hamiltonian $\oint_\gamma(-\rho_{j-2})\mathrm{d}x$.

Summary

Non-stretching vector fields	Conserved densities
$Z_0 = \gamma'$	$\rho_0 = k_1$
$Z_1 = \gamma'' - \frac{2}{3}k_1\gamma$	$\rho_1 = k_2$
$Z_2 = k_1\gamma'' + k_2\gamma' + \dots$	$\rho_2 = k_1k_2$
$Z_3 = (k_1' + 2k_2)\gamma'' + \left(\frac{1}{3}k_1^2 - \frac{2}{3}k_1'' - k_2'\right)\gamma' + \dots$	$\rho_3 = \frac{1}{3}(k_1')^2 + k_1'k_2 + \frac{1}{9}k_1^3 + k_2^2$
$Z_4 = (-k_1''' - 2k_2'' + 2k_1k_1' + 4k_1k_2)\gamma'' + (\frac{2}{3}k_1^{(4)} + k_2''' - 2k_1k_1'' - (k_1')^2 - 2k_1k_2' + \frac{4}{9}k_1^3 + 2k_2^2)\gamma' + \dots$	$\rho_4 = \frac{1}{3}(k_1'')^2 + k_1''(k_2' - k_1^2) - k_1(k_1')^2 + (k_2')^2 - k_1^2k_2' + \frac{1}{9}k_1^4 + 2k_1k_2^2$

- The γ' and γ'' coefficients of Z_j match the components of $\mathsf{E}\rho_j$.
- Densities satisfy the recursion relation $\mathsf{E}\rho_{j+2} = \mathcal{Q}^{-1}\mathcal{P}\mathsf{E}\rho_j$
- $\gamma_t = Z_j$ induces curvature evolution $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\mathsf{E}\rho_j = \mathcal{Q}\mathsf{E}\rho_{j+2}$.
- For $j \geq 2$, Z_j is a Hamiltonian vector field for $-\int \rho_{j-2}\,\mathrm{d}x$.

An interesting sub-hierarchy

Example: The Vortex Filament Flow hierarchy has integrable sub-hierarchies preserving geometric invariants, e.g.:

- Under the even flows X_{2j}, planar curves remain planar.
- For each constant τ_0, there is a sequence of linear combinations of the X_j that preserves the constant torsion condition $\tau = \tau_0$. (These flows induce the mKdV hierarchy for κ.)

Fact: centroaffine curves with $p_0 = \frac{1}{2}p_1'$ (i.e. $k_2 = -k_1'$) lie on a quadric cone through the origin in $\mathbb{R}^3$.

Theorem: *If γ lies on a cone at time zero, and evolves under any of the following curve flows, then it stays on the same cone*

$$Z_0, \quad Z_3, \quad Z_4, \quad Z_7, \quad Z_8, \quad Z_{11}, \quad Z_{12}, \ \ldots (*)$$

Remark: $\gamma_t = Z_3$ restricted to a conical curve induces the Kaup-Kuperschmidt (KK) equation for k_1 (Chou & Qu, 2002):

$$(k_1)_t = k_1''''' - 5k_1k_1''' - \frac{25}{2}k_1'k_1'' + 5k_1^2k_1'.$$

In fact, we show that *the sequence* $(*)$ *realizes the KK hierarchy, when restricted to conical curves.*

References

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