Rational solitons of resonant wave interaction models

Antonio Degasperis
National Institute of Nuclear Physics, Rome, Italy
S. Lombardo
Northumbria University, Newcastle, UK

VDM60
Cagliari, September 02-05, 2013

BACKGROUND : RATIONAL SOLUTIONS

dispersive LINEAR equations cannot have rational solutions

++++++++++++++++++++++++++++++++++++

Korteweg-deVries equation $u_t + u_{xxx} - 6uu_x = 0$

$$u_n(x,t) = -2\partial_x^2 \log(P_n(x,t)) \ , \quad n \geq 0$$

Adler-Moser polynomials : $P_0 = 1$, $P_1 = x$, $P_2 = x^3 + 12t$, ...

$$u_0 = 0 \, , \; u_1 = \frac{2}{x^2} \, , \; u_2 = 6x\frac{x^3 - 24t}{(x^3 + 12t)^2} \, , \; \ldots$$

Boussinesq equation $u_{tt} \pm u_{xxxx} + (u^2)_{xx} = 0$
(1977–78) Airault, McKean, Moser, Ablowitz, Satsuma

defocusing Nonlinear Schroedinger equation $iu_t + u_{xx} - 2|u|^2 u = 0$
(1985) Nakamura, Hirota, (1996) Hone, (2006) Clarkson

$$u_n = \frac{g_n}{f_n} \ , \quad n \geq 0$$

$$g_1 = 1 \, , \, f_1 = x \, , \, g_2 = -2x^3 + 12it \, , \, f_2 = x^4 - 12t^2 \, , \, ...$$

++++++++++++++++++++++++++++++++++++

focusing Nonlinear Schroedinger equation $iu_t + u_{xx} + 2|u|^2 u = 0$
(1983) Peregrine, (2010) Clarkson

$$u_n = \frac{G_n}{F_n} \, e^{2it} \ , \quad n \geq 0$$

$$G_0 = 1 \, , \, F_0 = 1 \, , \, G_1 = 4x^2 + 16t^2 - 4it - 3 \, , \, F_1 = 4x^2 + 16t^2 + 1 \, , \, ...$$

++++++++++++++++++++++++++++++++++++

connection to Painleve' II and IV : (1959–1965) Yablonskii–Vorob'ev polynomials, (1999) Noumi, Yamada (generalized Hermite polynomials and generalized Okamoto polynomials)

APPLICATIONS TO WAVES

- wave-packet (continuum spectrum)
- shock wave
- plane wave
- periodic train
- solitons (discrete spectrum)
- lumps (rational solitons)   (alias *rogue waves*, *freakons*)

" A *rogue wave* is large, unexpected, and dangerous" (National Ocean Service)

Figure: ship damage (2004)

rational soliton as ratio of polynomials of degree 2

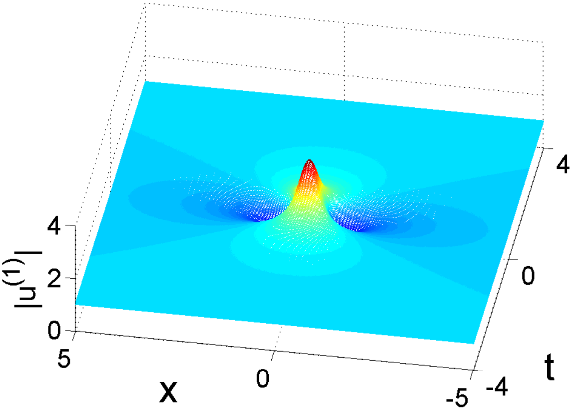

Figure: background amplitude=1 , peak amplitude = 3

EXTENSIONS

- extensions of solutions : higher order rational solitons
-
-
- extensions to other integrable models such as:
  vector nonlinear Schroedinger equations
  Hirota equation and coupled Hirota equations
  three wave resonant interaction model

BACKGROUND: resonant waves and integrability

generic wave equation in 1+1 dimensions

$$u_t + iF(-i\partial_x)u = N(u) \ , \ \ u = u(x,t)$$

linear dispersion relation $\omega(k) = F(k)$
multi-scale perturbation of monochromatic waves

$$u(x,t) = \epsilon \sum_{\alpha_1,\alpha_2,-\infty}^{+\infty} A^{(\alpha_1,\alpha_2)}(\xi,t_1,t_2) e^{i\alpha_1[k_1 x - \omega(k_1)t]} e^{i\alpha_2[k_2 x - \omega(k_2)t]}$$

$$\xi = \epsilon x \ , \ t_1 = \epsilon t \ , \ t_2 = \epsilon^2 t$$

MONOCHROMATIC WAVES

$$\omega(\alpha_1 k_1 + \alpha_2 k_2) = \alpha_1 \omega(k_1) + \alpha_2 \omega(k_2)$$

BACKGROUND: resonant waves and integrability, cont'd

$$\omega(\alpha_1 k_1 + \alpha_2 k_2) = \alpha_1 \omega(k_1) + \alpha_2 \omega(k_2)$$

trivial solutions:

$$\alpha_1 = 1 \ , \ \alpha_2 = 0 \ \text{ and } \ \alpha_1 = 0 \ , \ \alpha_2 = 1$$

if no other solutions:

$$A^{(1,0)}_{t_1} + v_1 A^{(1,0)}_{\xi} = 0 \ , \ A^{(0,1)}_{t_1} + v_2 A^{(0,1)}_{\xi} = 0 \ , \quad v_j = \omega'(k_j)$$

$$A^{(1,0)}_{t_2} = i(\gamma_1 A^{(1,0)}_{\xi\xi} + g_1 |A^{(1,0)}|^2 A^{(1,0)}) \ , \ A^{(0,1)}_{t_2} = i(\gamma_2 A^{(0,1)}_{\xi\xi} + g_2 |A^{(0,1)}|^2 A^{(0,1)})$$

$$\gamma_j = \frac{1}{2} \omega''(k_j)$$

BACKGROUND: resonant waves and integrability, cont'd

weak resonant condition

$$\omega'(k_1) \;=\; \omega'(k_2)$$

Vector Nonlinear Schrödinger (VNLS) equation

$$\begin{cases} A^{(1,0)}_{t_2} &= & i[\gamma_1 A^{(1,0)}_{\xi\xi} + (g_1|A^{(1,0)}|^2 + g_{12}|A^{(0,1)}|^2)A^{(1,0)})] \\ A^{(0,1)}_{t_2} &= & i[\gamma_2 A^{(0,1)}_{\xi\xi} + (g_2|A^{(0,1)}|^2 + g_{21}|A^{(1,0)}|^2)A^{(0,1)}] \end{cases}$$

BACKGROUND: resonant waves and integrability, cont'd

$$\omega(\alpha_1 k_1 + \alpha_2 k_2) = \alpha_1 \omega(k_1) + \alpha_2 \omega(k_2)$$

strong resonant condition

$$\alpha_1 = \alpha_2 = 1 \ , \ \ \omega(k_1 + k_2) = \omega(k_1) + \omega(k_2)$$

3 Wave Resonant Interaction (3WRI) equation

$$\begin{cases} A^{(1,0)}_{t_1} + v_1 A^{(1,0)}_{\xi} & = & g_1 A^{(0,1)*} A^{(1,1)} \\ A^{(0,1)}_{t_1} + v_2 A^{(0,1)}_{\xi} & = & g_2 A^{(1,0)*} A^{(1,1)} \\ A^{(1,1)}_{t_1} + v_3 A^{(1,1)}_{\xi} & = & g_3 A^{(1,0)} A^{(0,1)} \end{cases}$$

$$v_3 = \omega'(k_1 + k_2)$$

Vector Nonlinear Schrödinger + 3Wave Resonant Interaction
+ Nonlocal 2Wave Interaction

$$\begin{cases} u_t^{(1)} = i\alpha[u_{xx}^{(1)} - 2(s_1 \left|u^{(1)}\right|^2 + s_2 \left|u^{(2)}\right|^2)u^{(1)}] + \beta(-c_1 u_x^{(1)} - s_1 w^* u^{(2)}) \\ u_t^{(2)} = i\alpha[u_{xx}^{(2)} - 2(s_1 \left|u^{(1)}\right|^2 + s_2 \left|u^{(2)}\right|^2)u^{(2)}] + \beta(-c_2 u_x^{(2)} + s_2 w u^{(1)}) \\ 0 = \beta\,(w_x + s_1 s_2 (c_1 - c_2)\, u^{(1)*} u^{(2)}) \end{cases}$$

$$\psi_x = X\psi \quad , \quad \psi_t = T\psi$$

$$X(x,t,k) = ik\sigma + Q(x,t) \quad , \quad T(x,t,k) = \alpha\, T_{nls}(x,t,k) + \beta\, T_{3w}(x,t,k)$$

$$T_{nls} = 2ik^2\sigma + 2kQ + i\sigma(Q^2 - Q_x) \;, \quad T_{3w} = 2ikC - \sigma W + \sigma[C, Q(x,t)]$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \; C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & c_2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & s_1 u^{(1)*} & s_2 u^{(2)*} \\ u^{(1)} & 0 & 0 \\ u^{(2)} & 0 & 0 \end{pmatrix} \quad , \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -s_1 w^* \\ 0 & s_2 w & 0 \end{pmatrix}$$

Darboux construction

$$\Psi(x,t,k) = \left[\mathbf{1} + \left(\frac{\chi-\chi^*}{k-\chi}\right)P(x,t)\right]\Psi_0(x,t,k)$$

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x,t) \\ u_0^{(2)}(x,t) \end{pmatrix} + \frac{2i(\chi-\chi^*)\zeta^*}{|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$w(x,t) = w_0(x,t) - \frac{2is_1s_2(c_1-c_2)(\chi-\chi^*)z_1^*z_2}{|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2}$$

$$Z(x,t) = \begin{pmatrix} \zeta(x,t) \\ z_1(x,t) \\ z_2(x,t) \end{pmatrix} = \Psi_0(x,t,\chi^*)Z_0$$

$$\begin{pmatrix} u_0^{(1)}(x,t) \\ u_0^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} a_1 e^{i(qx-\nu_1 t)} \\ a_2 e^{-i(qx+\nu_2 t)} \end{pmatrix}$$

$$w_0(x,t) = is_1 s_2 (c_2 - c_1) \frac{a_1 a_2}{2q} e^{-i[2qx+(\nu_2-\nu_1)t]}$$

$$\begin{cases} \nu_1 &= \alpha[q^2 + 2(s_1 a_1^2 + s_2 a_2^2)] + \beta[c_1 q + s_2 \frac{a_2^2}{2q}(c_1 - c_2)] \;, \\ \nu_2 &= \alpha[q^2 + 2(s_1 a_1^2 + s_2 a_2^2)] + \beta[-c_2 q + s_1 \frac{a_1^2}{2q}(c_1 - c_2)] \;. \end{cases}$$

$$\Psi_0(x,t,k) = G(x,t) e^{i(\Lambda(k)x - \Omega(k)t)}$$

$$[\Lambda(k)\,,\,\Omega(k)] = 0$$

$$Z(x,t) = G(x,t) e^{i(\Lambda(\chi^*)x - \Omega(\chi^*)t)} Z_0$$

ELEMENTARY OBSERVATION : if N is nilpotent, $N^{m+1} = 0$, $N^m \neq 0$, then e^{zN} is a matrix-valued polynomial of z of degree m

PROPOSITION : necessary condition for $\Lambda(k)$ to be similar to a Jordan form is that at least two eigenvalues of $\Lambda(k)$ be equal to each other.

DEFINITION : k_c is a *critical value* of k if $\Lambda(k_c)$ is similar to a Jordan form Λ_J :

$$\Lambda(k_c) = T\,\Lambda_J\,T^{-1}$$

REMARK :

$$\Omega(k_c) = T\,\widehat{\Omega}\,T^{-1}\ ,\ [\Lambda_J\,,\,\widehat{\Omega}] = 0$$

RATIONAL SOLUTIONS - 2

$\Delta(k) =$ discriminant of the characteristic polynomial of $\Lambda(k)$
$=$ fourth degree polynomial

$$\Delta(k_c) = 0 \quad , \quad k_c \neq k_c^*$$

compute:

1. the critical value k_c
2. the similarity matrix T, the Jordan form Λ_J and the matrix $\widehat{\Omega}$
3. the vector

$$\begin{pmatrix} v(x,t) \\ v_1(x,t) \\ v_2(x,t) \end{pmatrix} = T e^{i(\Lambda_J x - \widehat{\Omega} t)} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

the solution

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{i(qx-\nu_1 t)} & 0 \\ 0 & e^{-i(qx+\nu_2 t)} \end{pmatrix} \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \right.$$

$$\left. \frac{2i(k_c^* - k_c)v^*}{|v|^2 - s_1|v_1|^2 - s_2|v_2|^2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right]$$

$$w(x,t) = is_1 s_2(c_2 - c_1)e^{-i[2qx+(\nu_2-\nu_1)t]} \left[\frac{a_1 a_2}{2q} + \frac{2(k_c^* - k_c)v_1^* v_2}{|v|^2 - s_1|v_1|^2 - s_2|v_2|^2} \right]$$

CLASSIFICATION - 1

$$\text{Case}[\lambda_1 = \lambda_2 = \lambda_3] \quad :$$

$$\Lambda_J = \begin{pmatrix} \lambda_1 & \mu_1 & 0 \\ 0 & \lambda_1 & \mu_1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \ , \ \ \mu_1 \neq 0 \ , \ \ \widehat{\Omega} = \begin{pmatrix} \omega_1 & \rho_1 & \rho_2 \\ 0 & \omega_1 & \rho_1 \\ 0 & 0 & \omega_1 \end{pmatrix}$$

$$q \neq 0\,, \quad k_c = \pm i\frac{\sqrt{27}}{2}q\,, \quad s_1 = s_2 = -1\,, \quad a_1 = a_2 = 2q$$

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{i(qx-\nu_1 t)} & 0 \\ 0 & e^{-i(qx+\nu_2 t)} \end{pmatrix} \frac{1}{M_4} \begin{pmatrix} P_4^{(1)} \\ P_4^{(2)} \end{pmatrix}$$

$$w(x,t) = is_1 s_2 (c_2 - c_1) e^{-i[2qx+(\nu_2-\nu_1)t]} \frac{P_4}{M_4}$$

CLASSIFICATION - 2

$$\text{Case}[\lambda_1 \; = \; \lambda_2 \neq \lambda_3] \quad :$$

$$\Lambda_J = \begin{pmatrix} \lambda_1 & \mu & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \; , \;\; \mu \neq 0 \;\; , \;\; \widehat{\Omega} = \begin{pmatrix} \omega_1 & \rho & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}$$

Subcases

- $q = 0 \; , \; s_1 = s_2 = -1 \;\; ,$ vector Peregrine solution

$$\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} = e^{2i\omega t} \left[\frac{(P_2 + |f|^2 e^{2px})}{(M_2 + |f|^2 e^{2px})} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{f P_1 \, e^{(px + i\omega t)}}{(M_2 + |f|^2 e^{2px})} \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} \right]$$

- $q \neq 0 \; , \; a_1 = a_2 \; , \; s_1 = \pm 1 \; , \; s_2 = \pm 1$
- $q \neq 0 \; , \; a_1 \neq a_2 \; , \; s_1 = \pm 1 \; , \; s_2 = \pm 1$

Vector Nonlinear Schrödinger Equations (VNLS)

$$\begin{cases} u_t^{(1)} = i[u_{xx}^{(1)} - 2(s_1 \left|u^{(1)}\right|^2 + s_2 \left|u^{(2)}\right|^2)u^{(1)}] \\ u_t^{(2)} = i[u_{xx}^{(2)} - 2(s_1 \left|u^{(1)}\right|^2 + s_2 \left|u^{(2)}\right|^2)u^{(2)}] \end{cases}$$

3Wave Resonant Interaction Equations (3WRI)

$$\begin{cases} u_t^{(1)} = (-c_1 u_x^{(1)} - s_1 w^* u^{(2)}) \\ u_t^{(2)} = (-c_2 u_x^{(2)} + s_2 w u^{(1)}) \\ 0 = (w_x + s_1 s_2 (c_1 - c_2) u^{(1)*} u^{(2)}) \end{cases}$$

parameter space 1

structural parameters: $s_1\,,\, s_2\,,\, c_1\,,\, c_2 \quad , \quad [c_1 c_2 (c_1 - c_2) \neq 0]$

background solution parameters : $q\,,\, a_1\,,\, a_2$

$$\begin{pmatrix} u_0^{(1)}(x,t) \\ u_0^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} a_1 e^{i(qx-\nu_1 t)} \\ a_2 e^{-i(qx+\nu_2 t)} \end{pmatrix}$$

$$w_0(x,t) = i s_1 s_2 (c_2 - c_1) \frac{a_1 a_2}{2q} e^{-i[2qx+(\nu_2-\nu_1)t]}$$

VNLS : $\nu_1 = \nu_2 = q^2 + 2(s_1 a_1^2 + s_2 a_2^2)$

3WRI : $\nu_1 = c_1 q + s_2 \frac{a_2^2}{2q}(c_1 - c_2)\,,\ \nu_2 = -c_2 q + s_1 \frac{a_1^2}{2q}(c_1 - c_2)$

Darboux construction

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{i(qx-\nu_1 t)} & 0 \\ 0 & e^{-i(qx+\nu_2 t)} \end{pmatrix} \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \right.$$

$$\left. \frac{2i(k_c^* - k_c)v^*}{|v|^2 - s_1|v_1|^2 - s_2|v_2|^2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right]$$

$$w(x,t) = i s_1 s_2 (c_2 - c_1) e^{-i[2qx + (\nu_2 - \nu_1)t]} \left[\frac{a_1 a_2}{2q} + \frac{2(k_c^* - k_c) v_1^* v_2}{|v|^2 - s_1|v_1|^2 - s_2|v_2|^2} \right]$$

$$[v(x,t)\,,\ v_1(x,t)\,,\ v_2(x,t)\,] = P_1(x,t) + P_2(x,t) \times \text{exponential}(x,t)$$

P_1 and P_2 have one complex parameter and have degree 2 or 4

parameter space 3

systematic search:

- $q = 0 \rightarrow$ VNLS, $s_1 = s_2 = -1$
- $q \neq 0\,,\; a_1 = a_2 = 2q \rightarrow s_1 = s_2 = -1$
- $q \neq 0\,,\; a_1 = a_2 \neq 2q\,,\; q^2 \geq 2a_1^2\,,\; s_1 = s_2 = 1$ no solutions
- $q \neq 0\,,\; a_1 = a_2 \neq 2q\,,\; q^2 < 2a_1^2\,,\; s_1 = s_2 = 1$
- $q \neq 0\,,\; a_1 = a_2 \neq 2q\,,\; s_1 = s_2 = -1$
- $q \neq 0\,,\; a_1 \neq a_2\,,\; s_1$ and s_2 arbitrary

VNLS solutions 1

Baronio F, Degasperis A, Conforti M, Wabnitz S (2012). Phys. Rev. Lett., vol. 109; p. 044102-044106

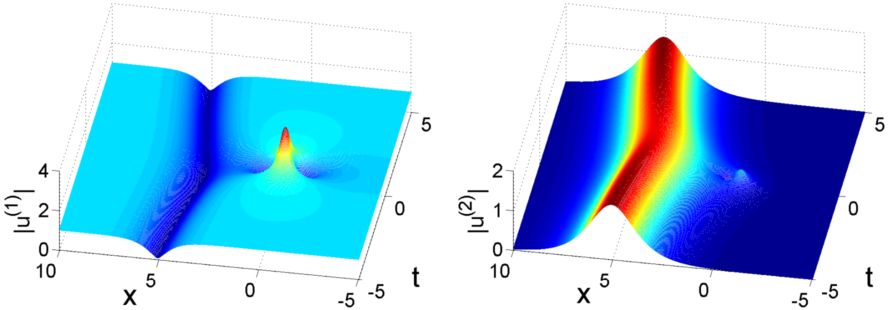

Figure: $f = 0.1, a_1 = 1, a_2 = 0$.

VNLS solutions 2

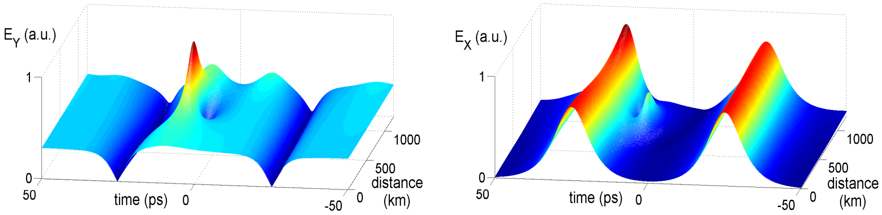

Figure: Numerical transmission of two 50 ps spaced dark-bright solitons in optical fibers, y-polarized dark waves (E_Y), and x-polarized bright envelopes (E_X).

VNLS solutions 3

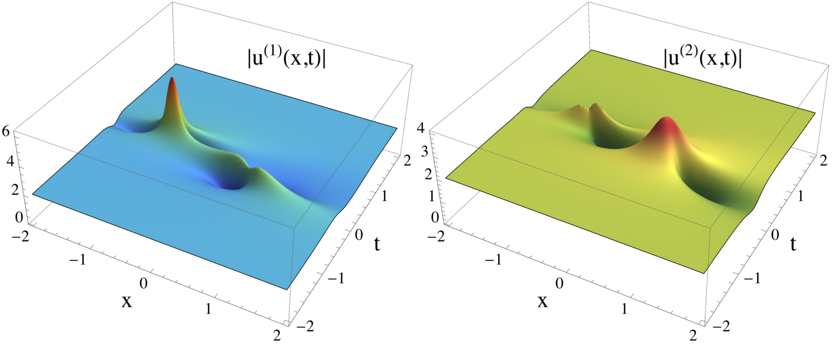

Figure: VNLS: $k_c = i\frac{\sqrt{27}}{2}$, $s_1 = s_2 = -1$, $a_1 = a_2 = 2$, $q = 1$, $\epsilon = 1$; $\gamma_1 = i$, $\gamma_2 = 0$, $\gamma_3 = 1$.

VNLS solutions 4

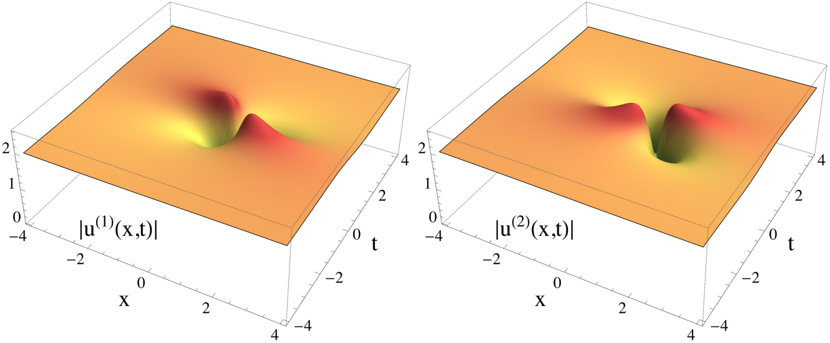

Figure: $k_c = \frac{i}{2}\sqrt{-13+16\sqrt{2}}$, $\lambda_1 = \lambda_2 \neq \lambda_3$, $s_1 = s_2 = 1$, $q = 1$, $a_1 = a_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$.

VNLS solutions 5

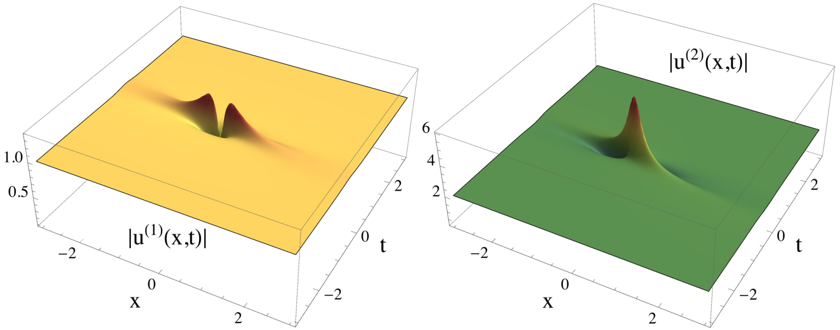

Figure: $k_c = 0.625 + 1.879i$, $\lambda_1 = \lambda_2 \neq \lambda_3$, $s_1 = 1$, $s_2 = -1$, $q = 1$, $a_1 = 1$, $a_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$.

3WRI solutions 1

Baronio F, Conforti M, Degasperis A, Lombardo S (2013).

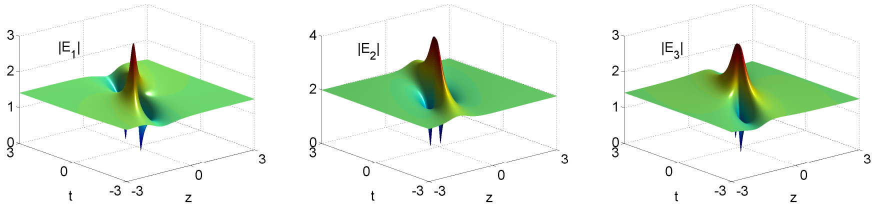

Figure: $V_1 = 1, V_2 = 0.5, q = 1, s_1 = s_2 = -1, \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 0.$

3WRI solutions 2

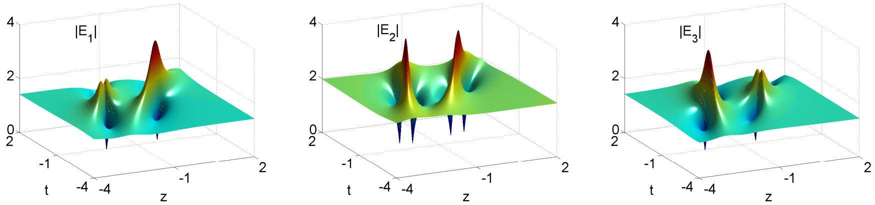

Figure: $V_1 = 1, V_2 = 0.5, q = 1, s_1 = s_2 = -1, \gamma_1 = 2, \gamma_2 = 7, \gamma_3 = 1.5 + i.$

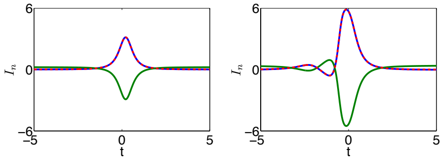

Figure: Effective energy evolution

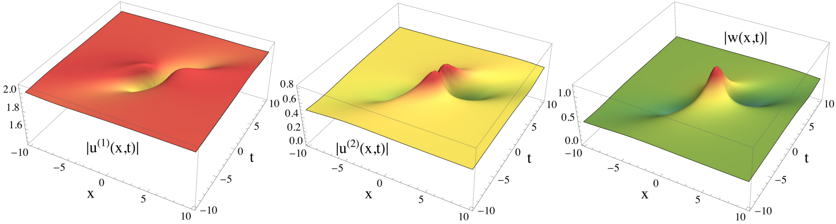

Figure: 3WRI: $k_c = 1.319 + 0.256i$, $\lambda_1 = \lambda_2 \neq \lambda_3$, $s_1 = s_2 = 1$, $q = 1$, $a_1 = 2$, $a_2 = 0.5$, $c_1 = 1$, $c_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$.

A Degasperis, S Lombardo, *Rational solitons of wave resonant interaction models*, arXiv:1305.6636v1 [nlin.SI], (2013)

BEST WISHES TO CORNELIS AND

CENTO DI QUESTI GIORNI !!!