Spectral problems associated with the matrix-valued AKNS system

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The AKNS system

The AKNS (Ablowitz-Kaup-Newell-Segur) system is

$$v' = \begin{pmatrix} -i\xi I_n & Q \\ R & i\xi I_m \end{pmatrix} v, \qquad x \in \mathbb{R},$$

where $m \geq n \geq 1$ and

- Q and R are $n \times m$ and $m \times n$ complex-valued matrix functions.
- I_n, I_m, are the $n \times n$ and $m \times m$ identity matrices, respectively.
- ξ is a complex-valued eigenvalue parameter.

$n = m = 1$: Introduced in 1970s by AKNS to solve certain nonlinear evolution equations by the inverse scattering transform technique.

The AKNS system (cont.)

- Nonlinear matrix PDE associated with AKNS:

$$
\begin{aligned}
iQ_x =& Q_{tt} - 2QRQ \\
iR_x =& - R_{tt} + 2RQR
\end{aligned}
$$

- $R = \pm Q^*$: $\quad iQ_x = Q_{tt} \mp 2QQ^*Q$

- $n = m = 1$: Nonlinear Schrödinger equation (NLS), where the $+$ $(-)$ sign corresponds to the focusing (defocusing) case, respectively.

Motivation

The AKNS system has an interesting spectral theory. Some prior work on the subject:

- $n = m = 1$: N. Asano and Y. Kato, JMP, 22 (1981) and JMP, 25 (1984).
- $n = 1, m = 2$: B. Prinari, M. Ablowitz, G. Biondini, JMP, 47 (2006).
- $n = 1, m \geq 1$: B. Prinari, G. Biondini, A. D. Trubatch, Studies in Appl. Math. 126 (2011).
- Any n, m: F. Demontis (thesis).
- $n = 1, m \geq 1$: F. Demontis and C. van der Mee: Serdica Math. J. 36 (2010).

Goal: Study the inverse scattering theory of new, more general matrix evolution equations.

The AKNS with constant coefficients

Consider the system

$$v' = \underbrace{\begin{pmatrix} -i\xi I_n & Q \\ R & i\xi I_m \end{pmatrix}}_{A(\xi)} v, \qquad x \in \mathbb{R},$$

where Q and R are constant matrices.

Set

$$H_0(Q,R) = iJ\,\frac{d}{dx} + \begin{pmatrix} 0 & -iQ \\ iR & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}.$$

Then

$$\boxed{v' = A(\xi)v \iff H_0(Q,R)v = \xi v}$$

Constant potentials

A Fourier transform $v(x) \to \widehat{v}(p)$, $\quad d/dx \to -ip$, $\quad H_0 \to \widehat{H}_0(p)$ gives

$$\widehat{H}_0(p) = \begin{pmatrix} pI_n & -iQ \\ iR & -pI_m \end{pmatrix}.$$

Suppose QR has

- κ distinct nonzero eigenvalues with algebraic multiplicities ν_k.
- (possibly) an eigenvalue zero with algebraic multiplicity ν_0.

  Note that $\dim \ker[RQ] \geq m - n$ and the algebraic multiplicity of the eigenvalue zero of RQ is $m - n + \nu_0$.

Constant potentials (cont.)

Spectrum of $\widehat{H}_0(p)$: Set

$$\det(\widehat{H}_0(p) - \xi I_{n+m}) = (-1)^m (p+\xi)^{m-n} \det[(p^2 - \xi^2) I_n + QR]$$
$$= (-1)^m (p+\xi)^{m-n+\nu_0} (p-\xi)^{\nu_0} \prod_{k=1}^{\kappa} (p - \mu(\xi))^{\nu_k} (p + \mu(\xi))^{\nu_k} = 0,$$

where

$$\mu(\xi) = \sqrt{\xi^2 - \omega_k}, \quad \text{with} \quad \mathrm{Im}[\sqrt{\xi^2 - \omega_k}] \geq 0.$$

Define curves (branches of hyperbolas):

$$\Gamma^{(r)}_{\omega_k} = \{\sqrt{\omega_k + t} : \ t \geq 0\}, \qquad \Gamma^{(\ell)}_{\omega_k} = -\Gamma^{(r)}_{\omega_k}.$$

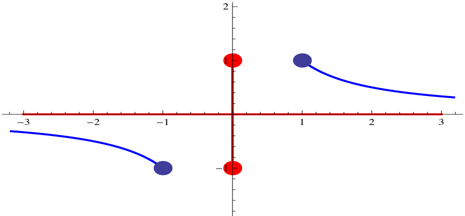
Constant potentials (cont.)

$$\sigma(H_0) = \bigcup_{j=1}^{\kappa} \left(\Gamma^{(r)}_{\omega_j} \cup \Gamma^{(\ell)}_{\omega_j} \right) \quad [\cup \mathbb{R}]$$

$$m = n \text{ and } \nu_0 = 0 \text{ [if } m > n \text{ or } \nu_0 > 0].$$

Branches for square roots:

$$\operatorname{Im} \sqrt{\xi^2 - \omega_k} > 0 \quad \text{for } \xi \in \mathbb{C} \setminus (\Gamma^{(r)}_{\omega_k} \cup \Gamma^{(\ell)}_{\omega_k})$$



Constant potentials: Eigenvalues of $A(\xi)$

Eigenvalues of $A(\xi)$:

$$\det(A(\xi) - \lambda I_{n+m}) =$$

$$(-1)^{\nu_0}(i\xi - \lambda)^{m-n+\nu_0}(i\xi + \lambda)^{\nu_0} \prod_{k=1}^{\kappa} (\lambda - \lambda_k(\xi))^{\nu_k}(\lambda + \lambda_k(\xi))^{\nu_k}$$

where

$$\lambda_k(\xi) = i\sqrt{\xi^2 - \omega_k} \quad (\omega_k \neq 0).$$

Eigenvalues:

- $\pm\lambda_k(\xi)$, $k = 1, \ldots, \kappa$, if $QR \neq 0$.
- $i\xi$: If $m - n > 0$ or $\nu_0 > 0$.
- $-i\xi$: If $\nu_0 > 0$.

Constant potentials: Eigenvalues of $A(\xi)$ (cont.)

We are interested in the

- Jordan structure of $A(\xi)$ at an eigenvalue λ: Number and sizes of Jordan blocks of $A(\xi)$ at its eigenvalues, denoted by $\mathcal{J}[A(\xi), \lambda]$.

THEOREM 1

The Jordan structures of $A(\xi)$ and QR (resp., RQ) are related as follows:

(i) $\mathcal{J}[A(\xi), \pm\lambda_k(\xi)] = \mathcal{J}[QR, \omega_k]$, provided $\xi^2 \neq \omega_k$ and $k = 1, \ldots, \kappa$.

(ii) $\mathcal{J}[A(\xi), -i\xi] = \mathcal{J}[QR, 0]$ and $\mathcal{J}[A(\xi), i\xi] = \mathcal{J}[RQ, 0]$ provided $\xi \neq 0$.

(iii) $A(\pm\sqrt{\omega_k})$ at 0 and QR at ω_k have the same number of Jordan blocks but the block sizes for $A(\pm\sqrt{\omega_k})$ are twice the block sizes for QR.

Constant potentials: Jordan chains of $A(\xi)$

The case $\xi = 0$ and $\lambda = 0$ is special:

- The Jordan structure of $A(0)$ at $\lambda = 0$ depends on the kernels of $(QR)^s$, $R(QR)^s$, $(RQ)^s$, and $Q(RQ)^s$ for $s = 1, 2 \ldots$.
- Jordan chains correspond to paths on a certain graph associated with these kernels.

  Example:

$$Q = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

- QR: Eigenvalue $\omega_1 = 1$, $\nu_1 = 1$ and eigenvalue 0, $\nu_0 = 2$ (semi-simple: no Jordan blocks of size greater than 1).
- RQ eigenvalue $\omega_1 = 1$, $\nu_1 = 1$ and eigenvalue 0, algebraic mult. 3, geometric mult. 2.

Constant potentials: Jordan chains of $A(\xi)$ (cont.)

$$(k,\alpha) \longleftrightarrow \alpha_k = \dim \mathcal{A}_k$$

$$(k,\beta) \longleftrightarrow \beta_k = \dim \mathcal{B}_k$$

$$\mathcal{A}_{2k} = \ker[(RQ)^k], \qquad \mathcal{A}_{2k-1} = \ker[Q(RQ)^{k-1}]$$

$$\mathcal{B}_{2k} = \ker[(QR)^k], \qquad \mathcal{B}_{2k-1} = \ker[R(QR)^{k-1}]$$

In the example: $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$ and $\beta_1 = 1, \beta_2 = 2, \beta_3 = 2$

$$\begin{tikzcd}
(1,\alpha)[1] & (2,\alpha)[2] \arrow[dl] & (3,\alpha)[3] \arrow[dl] \\
(1,\beta)[1] & (2,\beta)[2] \arrow[ul] & (3,\beta)[2]
\end{tikzcd}$$

Constant potentials: Jordan chains of $A(\xi)$ (cont.)

Some more results:

- If $m > n$ ($m = n$) then for any $\xi \in \mathbb{C}$, the maximum length of a Jordan chain for $A(\xi)$ at any eigenvalue is $2n + 1$ ($2n$).

- $A(\xi)$ is diagonalizable for every $\xi \neq 0$ if and only if $RQ = QR = 0$.

- $A(\xi)$ is diagonalizable for every $\xi \in \mathbb{C}$ if and only if $Q = R = 0$.

  Embedded in the graphical picture is a proof of H. Flander's theorem on elementary divisors of AB and BA: Proc. Amer. Math. Soc. (1951).

Constant potentials: Reduction of $A(\xi)$

Let U $(n \times n)$ and V $(m \times m)$ be unitary matrices so that

$$U^* Q V = \Sigma,$$

where

$$\Sigma = \Sigma_p \oplus 0_{(n-p)\times(m-p)}, \qquad \Sigma_p = \operatorname{diag}(\sigma_1, \ldots, \sigma_p),$$

and $\sigma_1 \leq \cdots \leq \sigma_p$ are the singular values of Q. Then

$$\begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} -i\xi I_n & Q \\ R & i\xi I_m \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} -i\xi I_n & \Sigma \\ R^\sharp & i\xi I_m \end{pmatrix},$$

where $R^\sharp = V^* R U$. Introducing the partitions

$$\Sigma = \begin{pmatrix} \widehat{\Sigma} & 0_{n\times(m-p)} \end{pmatrix}, \quad \widehat{\Sigma} = \begin{pmatrix} \Sigma_p \\ 0_{(n-p)\times p} \end{pmatrix}, \quad R^\sharp = \begin{pmatrix} R_1^\sharp \\ R_2^\sharp \end{pmatrix},$$

Constant potentials: Reduction of $A(\xi)$ (cont.)

we obtain

$$\begin{pmatrix} -i\xi I_n & \Sigma \\ R^\sharp & i\xi I_m \end{pmatrix} = \left(\begin{array}{cc|c} -i\xi I_n & \widehat{\Sigma} & \\ R_1^\sharp & i\xi I_p & 0 \\ \hline R_2^\sharp & 0 & i\xi I_{m-p} \end{array}\right).$$

If $r_1 := \mathrm{rank}[R_1^\sharp] < \mathrm{rank}[R] = \mathrm{rank}[R^\sharp] =: r$, rearrange rows and columns $n+p+1$ through $n+m$ so that

$$R^\sharp \longrightarrow \widehat{R}^\sharp = \begin{pmatrix} \widehat{R}_1^\sharp \\ \widehat{R}_2^\sharp \end{pmatrix},$$

where $\widehat{R}_1^\sharp$ has size $(p+r-r_1) \times n$ and is of rank r. This turns $A(\xi)$ into a similar matrix:

$$\left(\begin{array}{cc|c} -i\xi I_n & \Sigma & \\ \widehat{R}_1^\sharp & i\xi I_{p+r-r_1} & 0 \\ \hline \widehat{R}_2^\sharp & 0 & i\xi I_{m-p-r+r_1} \end{array}\right).$$

Constant potentials: Reduction of $A(\xi)$ (cont.)

Since $\widehat{R}_1^\sharp$ and $\widehat{R}^\sharp$ have the same rank, there exists an $(m-p-r+r_1)\times(p+r-r_1)$ matrix W such that

$$W\widehat{R}_1^\sharp = \widehat{R}_2^\sharp.$$

With the help of the similarity

$$S = \left[\begin{array}{c|c} \begin{array}{cc} \multicolumn{2}{c}{I_{n+p+r-r_1}} \\ 0 & W \end{array} & \begin{array}{c} 0 \\ I_{m-p-r+r_1} \end{array} \end{array}\right], \quad S^{-1} = \left[\begin{array}{c|c} \begin{array}{cc} \multicolumn{2}{c}{I_{n+p+r-r_1}} \\ 0 & -W \end{array} & \begin{array}{c} 0 \\ I_{m-p-r+r_1} \end{array} \end{array}\right].$$

we can finally transform $A(\xi)$ into block-diagonal form and state:

THEOREM 2

$A(\xi)$ is similar (the similarity does not depend on ξ) to

$$\left(\begin{array}{cc|c} -i\xi I_n & \Sigma & \\ \widehat{R}_1^\sharp & i\xi I_{p+r-r_1} & 0 \\ \hline 0 & 0 & i\xi I_{m-p-r+r_1} \end{array}\right).$$

Reduction of $A(\xi)$: Special cases

Special cases:

Corollary 3

Suppose $n = 1$ and $m \geq 2$. Let $\sigma = \|Q\|$ and $\omega = QR$ ($\in \mathbb{C}$). Then $A(\xi)$ is similar to one of the following block-diagonal matrices:

$$\left(\begin{array}{cc|c} -i\xi & \sigma & \\ \omega\sigma^{-1} & i\xi & \smash{0} \\ \hline \multicolumn{2}{c|}{0} & i\xi I_{m-1} \end{array}\right), \qquad \omega \neq 0,$$

$$\left(\begin{array}{ccc|c} -i\xi & \sigma & 0 & \\ 0 & i\xi & 0 & 0 \\ \|R\| & 0 & i\xi & \\ \hline \multicolumn{3}{c|}{0} & i\xi I_{m-2} \end{array}\right), \qquad \omega = 0.$$

Reduction of $A(\xi)$: Special cases (cont.)

Corollary 4

Suppose that $R = \pm Q^*$. Then $A(\xi)$ is unitarily similar to the direct sum

$$\begin{pmatrix} -i\xi & \sigma_1 \\ \pm\sigma_1 & i\xi \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} -i\xi & \sigma_p \\ \pm\sigma_p & i\xi \end{pmatrix}$$

$$\oplus \underbrace{-i\xi I_1 \oplus \cdots \oplus -i\xi I_1}_{n-p} \oplus \underbrace{i\xi I_1 \oplus \cdots \oplus i\xi I_1}_{m-p}.$$

Potentials with different limits as $\pm\infty$

Suppose

$$Q(x) = Q_\pm \qquad R(x) = R_\pm \quad \text{for} \quad x \in \mathbb{R}^\pm,$$

where

- $Q_\pm$ and $R_\pm$ are constant matrices on $\mathbb{R}^\pm$ and such that $Q_+ \neq Q_-$ and $R_+ \neq R_-$ in general.

Define

$$V(x) = \begin{pmatrix} 0 & -iQ_+ \\ iR_+ & 0 \end{pmatrix}, \; x > 0, \qquad V(x) = \begin{pmatrix} 0 & -iQ_- \\ iR_- & 0 \end{pmatrix}, \; x < 0,$$

$$V_\pm(x) = V_\pm = \begin{pmatrix} 0 & -i\,Q_\pm \\ i\,R_\pm & 0 \end{pmatrix}, \qquad -\infty < x < \infty.$$

Let

$$W(x) = V(-x).$$

Different limits (cont.)

Define

$$H_V = iJD + V, \quad H_{V_\pm} = iJD + V_\pm, \quad H_W = iJD + W.$$

Theorem 5

- $\sigma_{\mathrm{ess}}(H_V) = \sigma(H_+) \cup \sigma(H_-)$
- $\sigma_{\mathrm{ess}}(H_V) = -\sigma_{\mathrm{ess}}(H_V)$ (only true for σ_{ess}!)

Proof with the help of the "twisting trick": E.B. Davies and B. Simon, Commun. Math. Phys. 63 (1978).

Sketch: Define a unitary $2(n+m) \times 2(n+m)$ matrix $U(x)$ such that

$$U(x) = \begin{pmatrix} I_{n+m} & 0 \\ 0 & I_{n+m} \end{pmatrix} \quad x > 1, \quad U(x) = \begin{pmatrix} 0 & -I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} \quad x < 0,$$

so that $U(x)$ is absolutely continuous on $\mathbb{R}$.

Different limits (cont.)

For example, choose

$$U(x) = \begin{pmatrix} u_1(x) I_{n+m} & -u_2(x) I_{n+m} \\ u_2(x) I_{n+m} & u_1(x) I_{n+m} \end{pmatrix}, \qquad 0 \le x \le 1,$$

where $u_1(x) = \sin(\pi x/2)$, $u_2(x) = \cos(\pi x/2)$.
Then

$$U^*(H_V \oplus H_W)U = \begin{pmatrix} H_{V_+} & 0 \\ 0 & H_{V_-} \end{pmatrix} + Z(x),$$

where $Z(x)$ has compact support in $[0,1]$ and thus is a relatively compact perturbation. Therefore

$$\sigma_{\mathrm{ess}}(H_V) \cup \sigma_{\mathrm{ess}}(H_W) = \sigma_{\mathrm{ess}}(H_{V_+}) \cup \sigma_{\mathrm{ess}}(H_{V_-}) = \sigma(H_{V_+}) \cup \sigma(H_{V_-}).$$

Difficulty: H_V and H_W are in general not unitarily equivalent! H_V and $-H_W$ are (via JP where $P: x \to -x$, $J = \mathrm{diag}(I_n, -I_m)$)!

Eigenvalues/spectral singularities of H_V

H_V may have discrete or embedded eigenvalues, or spectral singularities (defined below).

To determine these points we need the Jost solutions:

Suppose $\xi \in \rho(H_V) \cap \mathbb{C}^+$ (other cases ar similar).Then there exist m (resp. n) linearly independent solutions of the AKNS system that are in $L^2(\mathbb{R}^+)^{n+m}$ (resp. $L^2(\mathbb{R}^-)^{n+m}$). Choose these vectors as the column of two matrices $F_{0,\pm}(x,\xi)$.

Let $W_0(\xi) = \det \begin{pmatrix} F_{0,-}(0,\xi) & F_{0,+}(0,\xi) \end{pmatrix}$.

Definition

A point ξ is called a spectral singularity if $\xi \in \sigma_{\mathrm{ess}}(H_V) \setminus \sigma_p(H_V)$ and $W_0(\xi) = 0$.

Eigenvalues/spectral singularities of H_V (cont.)

A special case:

- Suppose $Q_\pm R_\pm$ and $R_\pm Q_\pm$ are diagonalizable.
- Let $n^\pm$ denote the number of eigenvalues (counting multiplicities) of $Q_\pm R_\pm$ different from 0.
- Let $\omega_1^\pm, \omega_2^\pm, \ldots, \omega_{n^\pm}^\pm$ be the nonzero eigenvalues of $Q_\pm R_\pm$.
- If $Q_\pm R_\pm$ has eigenvalue zero, let $\nu_0^\pm$ denote its multiplicity.
- Let $\Phi_\pm$ be matrices whose columns are an eigenbasis for $Q_\pm R_\pm$ so that $Q_\pm R_\pm \Phi_\pm = \Phi_\pm \mathrm{diag}(\omega_1^\pm, \ldots, \omega_{n^\pm}^\pm)$.
- Set $\mu_k^\pm(\xi) = \sqrt{\xi^2 - \omega_k^\pm}, \qquad M_\pm = \mathrm{diag}(\mu_1^\pm, \ldots, \mu_{n^\pm}^\pm)$.

Hence $n^\pm = n - \nu_0^\pm$. Since $\sigma(Q_+R_+) \setminus \{0\} = \sigma(R_+Q_+) \setminus \{0\}$, we have that $\dim \ker[R_+Q_+] = m - n^+ = m - n + \nu_0^+$.

Let N_+ be an $m \times (m - n^+)$ matrix whose columns form a basis for $\ker[R_+Q_+]$ and let $\widehat{N}_-$ be an $n \times \nu_0^-$ matrix whose columns form a basis for $\ker[Q_-R_-]$.

Eigenvalues/spectral singularities of H_V (cont.)

Then, if $\operatorname{Im}\xi > 0$, we have

$$F_{0,+}(x,\xi) = \begin{pmatrix} 0 & i\Phi_+(M_+ - \xi I_{n^+}) \\ N_+ & R_+\Phi_+ \end{pmatrix} \begin{pmatrix} e^{i\xi x} I_{m-n^+} & 0 \\ 0 & e^{iM_+ x} \end{pmatrix}, \ x > 0,$$

$$F_{0,-}(x,\xi) = \begin{pmatrix} \widehat{N}_- & -i\Phi_-(M_- + \xi I_{n^-}) \\ 0 & R_-\Phi_- \end{pmatrix} \begin{pmatrix} e^{-i\xi x} I_{\nu_0^-} & 0 \\ 0 & e^{-iM_- x} \end{pmatrix}, \ x < 0.$$

Hence

$$W_0(\xi) = \begin{vmatrix} \widehat{N}_- & -i\Phi_-(M_- + \xi I_{n^-}) & 0 & i\Phi_+(M_+ - \xi I_{n^+}) \\ 0 & R_-\Phi_- & N_+ & R_+\Phi_+ \end{vmatrix}.$$

Eigenvalues/spectral singularities of H_V (cont.)

If $n = m = 1$, $Q_\pm = q_\pm$, $R_\pm = r_\pm$ ($q_\pm \neq 0, r_\pm \neq 0$) the decaying solutions are

$$F_{0,\pm}(x,\xi) = \begin{pmatrix} \pm i(\mu_\pm \mp \xi) \\ r_\pm \end{pmatrix} e^{\pm i\mu_+ x}, \quad x \in \mathbb{R}^\pm.$$

Then

$$W_0(\xi) = -ir_+(\mu_-(\xi) + \xi) - ir_-(\mu_+(\xi) - \xi).$$

Zeros of $W_0(\xi)$: $$\xi = \pm \frac{(q_- r_+ - q_+ r_-)}{2(q_- - q_+)^{1/2}(r_+ - r_-)^{1/2}}$$

Only one root can be an eigenvalue but there may be two spectral singularities.

Question: How many zeros does $W_0(\xi)$ have? Can we find a bound for the number of zeros?

Eigenvalues/spectral singularities of H_V (cont.)

Theorem 6

Suppose $m = n$ and $Q_\pm R_\pm$ are diagonalizable and have only nonzero eigenvalues. Then the number of eigenvalues (counted according to their multiplicities) is at most $n2^{2n} - 2n$. Moreover, the number of eigenvalues ξ such that $-\xi$ is not an eigenvalue is bounded by $n2^{2n-1} - n$.

- Exact for $n = 1$ and the nonzero eigenvalues. Not exact for $\xi = 0$, since 0 is always simple. For $n = 2$ the bound is 28.
- The number 2^{2n} is the sum of the binomial coefficients of the form $\binom{2n}{s}$, $s = 0, \ldots, 2n$, which represents the number of ways in which we can pick s factors $\mu_k^\pm(\xi)$ from a total of $2n$ such factors. If the ω_k^+ and ω_s^- are all distinct these products are also distinct, and no two factors can "annihilate" each other.
- The factor n is technical. The term $-2n$ comes from exploiting certain symmetries.

Eigenvalues/spectral singularities of H_V (cont.)

Idea behind proof: Construct a function $h(\xi)$ that is analytic on $\rho(H_V)$ such that $W_0(\xi)h(\xi)$ is equal to a polynomial of degree N. Then the number of eigenvalues will be bounded above by N.

$n = m = 1, q_+ = 1, q_- = i, r_+ = i, r_- = 1$:
$h(\xi) = -i\xi + \sqrt{\xi^2 - i}$.
$W_0(\xi) = (1+i)\xi + (1-i)\sqrt{\xi^2 - i}$.
Thus

$$W_0(\xi)h(\xi) = -1 - i + 2(1-i)\xi^2.$$

$W_0(\xi) = 0$ for $\xi = \pm\sqrt{i/2}$, $-\sqrt{i/2}$ is the eigenvalue. Then, in this case, $h(-\sqrt{i/2}) \neq 0$, but $h(\sqrt{i/2}) = 0$. In general we cannot rule out common zeros of $h(\xi)$ and $W_0(\xi)$.

Eigenvalues/spectral singularities of H_V (cont.)

Examples with eigenvalues/spectral singularities: $n = m = 1$.

(1) $q_+ = 1, r_+ = 1, q_- = 1-\rho, r_- = 1-\rho$, with $\rho \geq 0$.
We have

$$\sigma_{\text{ess}}(H_V) = \begin{cases} \mathbb{R} \setminus (-|1-\rho|, |1-\rho|), & \rho \in [0,2) \setminus \{1\}, \\ \mathbb{R}, & \rho = 1, \\ \mathbb{R} \setminus (-1,1), & \rho \geq 2. \end{cases}$$

From the formula for ξ we get $\xi = 0$ for all $\rho > 0$. But $\xi = 0$ is an eigenvalue only when $\rho > 1$. For $\rho = 0$, $\xi = \pm 1$ are both spectral singularities.

(2) $q_\pm = q_0 e^{i\varphi_\pm}$, where $q_0 > 0$, $\varphi_\pm \in \mathbb{R}$, $\varphi_- - \varphi_+ := \widehat{\varphi}$. If $\widehat{\varphi} \in (2\pi m, 2\pi(m+1))$. Then $\xi_0 = (-1)^{m+1} q_0 \cos(\widehat{\varphi}/2)$ is an eigenvalue in the spectral gap $(-q_0, q_0)$.

Self-adjoint H_V

Suppose that $R_\pm = Q_\pm^*$. Suppose $Q_\pm R_\pm = Q_\pm Q_\pm^*$ have nonzero distinct eigenvalues $0 < \omega_1^\pm < \cdots < \omega_{\kappa^\pm}^\pm$ and possibly eigenvalue 0 with multiplicities $\nu_0^\pm$. Let $\mathcal{S}_\pm = \{|\omega_1^\pm|^{1/2}, \ldots, |\omega_{\kappa^\pm}^\pm|^{1/2}\}$. Set

$$\alpha_0 = \min\left\{|\omega_1^-|^{1/2}, |\omega_1^+|^{1/2}\right\}, \qquad \beta_0 = \min\left\{|\omega_{\kappa^-}^-|^{1/2}, |\omega_{\kappa^+}^+|^{1/2}\right\}.$$

THEOREM 7

(i) If $\nu_0^\pm$ are both zero, then all embedded zeros of $W_0(\xi)$ (i.e., zeros in $\sigma_{\rm ess}(H_V)$) satisfy $\alpha_0 \le |\xi| \le \beta_0$. All spectral singularities are contained in $\pm(\mathcal{S}_- \cup \mathcal{S}_+)$. Every embedded zero of $W_0(\xi)$ not in $\pm(\mathcal{S}_- \cup \mathcal{S}_+)$ corresponds to an embedded eigenvalue.

(ii) If one or both of $\nu_0^\pm$ are nonzero, then all zeros of $W_0(\xi)$ lie in $|\xi| \le \beta_0$. The other conclusions are as in (i). In particular, if $W_0(0) = 0$, then 0 is an eigenvalue of H_V.

Self-adjoint H_V (cont.)

In the self-adjoint case ($R_\pm = Q_\pm^*$) we have:

- $n^\pm = n - \nu_0^\pm =$ number of nonzero eigenvalues (counting multiplicities) of $Q_\pm Q_\pm^*$. This is the same as the number of nonzero eigenvalues (counting multiplicities) of $Q_\pm^* Q_\pm$. Then $m - n^+ = m - (n - \nu_0^+) = \dim \ker[Q_+]$.
- N_+ is an $m \times (m - n^+)$ matrix whose columns form an orthonormal basis for $\ker[Q_+] = \ker[Q_+^* Q_+]$.
- $\widehat{N}_-$ is an $n \times \nu_0^-$ matrix whose columns form an orthonormal basis for $\ker[Q_-^*] = \ker[Q_- Q_-^*]$.

Self-adjoint H_V(cont.)

The proof employs the identity

$$F_{0,+}(x,\xi)^* J F_{0,+}(x,\xi) = \begin{pmatrix} -N_+^* N_+ & 0 \\ 0 & C_+(\xi) \end{pmatrix}, \qquad \xi \in \mathbb{R},$$

for all $x \in \mathbb{R}$, where

$$C_+(\xi) = M_+(\xi)^* M_+(\xi) - \xi(M_+(\xi) + M_+(\xi)^*) + \xi^2 - \Omega_+$$

and $Q_\pm Q_\pm^* \Phi_\pm = \Phi_\pm \Omega_\pm$. There are similar relations for $F_{0,-}(x,\xi)^* J F_{0,-}(x,\xi)$. Moreover, it uses the fact that there are vectors $\beta_- \in \mathbb{C}^n$, $\beta_+ \in \mathbb{C}^m$ such that

$$F_{0,+}(x,\xi)\beta_+ + F_{0,-}(x,\xi)\beta_- = 0.$$

Skew-selfadjoint potential V

Suppose $R_{\pm} = -Q_{\pm}^*$. Then the Wronskian is:

$$W_0(\xi) = \begin{vmatrix} \widehat{N}_- & -i\Phi_-(M_- + \xi I_{n^-}) & 0 & i\Phi_+(M_+ - \xi I_{n^+}) \\ 0 & -Q_-^*\Phi_- & N_+ & -Q_+^*\Phi_+ \end{vmatrix}.$$

Let

$$A = \begin{pmatrix} \widehat{N}_- & -i\Phi_-(M_- + \xi I_{n^-}) \end{pmatrix} \qquad B = \begin{pmatrix} 0 & i\Phi_+(M_+ - \xi I_{n^+}) \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & -Q_-^*\Phi_- \end{pmatrix} \qquad D = \begin{pmatrix} N_+ & -Q_+^*\Phi_+ \end{pmatrix}$$

Then

$$W_0(\xi) = (\det A)(\det D)\det(I_n - BD^{-1}CA^{-1}).$$

Define

$$\mathcal{V}(\xi) = -BD^{-1}CA^{-1} = \mathcal{V}_+(\xi)\mathcal{V}_-(\xi)$$
$$\mathcal{V}_+(\xi) = \Phi_+(M_+(\xi) + \xi I_{n^+})^{-1}\Phi_+^*Q_+,$$
$$\mathcal{V}_-(\xi) = Q_-^*\Phi_-(M_-(\xi) + \xi I_{n^-})^{-1}\Phi_-^*.$$

Skew-selfadjoint potential V (cont.)

Suppose $Q_{\pm}$ are both nonzero. Let

$$\gamma_0 = \min\left\{|\omega^+_{\kappa^+}|^{1/2}, |\omega^-_{\kappa^-}|^{1/2}\right\}, \quad \gamma_1 = \max\left\{|\omega^+_{\kappa^+}|^{1/2}, |\omega^-_{\kappa^-}|^{1/2}\right\}.$$

Then $\sigma_{\mathrm{ess}}(H_V) = i[-\gamma_1, \gamma_1] \cup \mathbb{R}$. For every $s \in (0, \gamma_1]$, let

$$\rho_{\pm}(s) = \#\{k : |\omega^{\pm}_k|^{1/2} < s\}.$$

We partition $\Phi_{\pm}$ in the form

$$\Phi_{\pm} = \mathrm{row}(\Phi^{(1)}_{\pm}, \ldots, \Phi^{(\mathrm{s})}_{\pm}, \ldots, \Phi^{(\kappa^{\pm})}_{\pm}),$$

where $\Phi^{(s)}_{\pm}$ is the $n \times \nu^{\pm}_s$ matrix whose columns are an orthonormal basis for the eigenspace associated with the eigenvalue $\omega^{\pm}_s$ of $-Q_{\pm}Q^*_{\pm}$.

Theorem 8

(i) $\|\mathcal{V}(\xi)\|_{\mathbb{C}^{n\times n}} \le 1$ for all $\xi \in \overline{\mathbb{C}^+}$ and $\|\mathcal{V}(\xi)\|_{\mathbb{C}^{n\times n}} < 1$ on $\overline{\mathbb{C}^+} \setminus [0, i\gamma_0]$.

(ii) $W_0(\xi) = 0$ if and only if $\mathcal{V}(\xi)$ has eigenvalue -1 and this can happen only if $\xi = is_0$ for some $0 \le s_0 \le \gamma_0$. Moreover, $\dim(\ker[W_0(is_0)]) = \dim(\ker[I_n + \mathcal{V}(is_0)])$.

(iii) If -1 is an eigenvalue of $\mathcal{V}(is_0)$ for some $0 \le s_0 \le \gamma_0$ with associated normalized eigenvector $f \in \mathbb{C}^n$, then

$(\mathrm{iii})_1$ $\|\mathcal{V}_+(is_0)^* f\|_{\mathbb{C}^m} = \|\mathcal{V}_-(is_0) f\|_{\mathbb{C}^m} = 1$ and $\mathcal{V}_+(is_0)^* f = -\mathcal{V}_-(is_0) f$.

$(\mathrm{iii})_2$ $\|\Phi_+^* f\|_{\mathbb{C}^{n^+}} = \|\Phi_-^* f\|_{\mathbb{C}^{n^-}} = 1$ and $(\Phi_\pm^{(k)})^* f = 0$ for $k = 1, \ldots, \rho_\pm(s_0)$.

(iv) If -1 is an eigenvalue of $\mathcal{V}(is_0)$, then it is semi-simple. Moreover, $\mathcal{V}(is_0) f = -f$ implies $\mathcal{V}(is_0)^* f = -f$.

Skew-selfadjoint potential V (cont.)

Proof:

(i) The function $\xi \to \sqrt{\xi^2 + a^2} + \xi$ with $a > 0$ maps $\overline{\mathbb{C}^+} \setminus [0, ia]$ to the outside of the semi-disk of radius a centered at 0 in the closed upper half-plane.

(ii) $f \in \ker[I_n + \mathcal{V}(is_0)] \longleftrightarrow \begin{pmatrix} A^{-1}f \\ -D^{-1}CA^{-1}f \end{pmatrix} \in \ker[W_0(\xi)].$

$(\mathrm{iii})_1$ Suppose that $\mathcal{V}(is_0)f = -f$ where $\|f\|_{\mathbb{C}^n} = 1$. Then ((, $)_{\mathbb{C}^n}$ denotes the inner product)

$$(f, \mathcal{V}(is_0)f)_{\mathbb{C}^n} = (\mathcal{V}_+(is_0)^* f, \mathcal{V}_-(is_0)f)_{\mathbb{C}^n} = -1.$$

Since $\|\mathcal{V}_+(is_0)\|_{\mathbb{C}^{n\times m}} \leq 1$ and $\|\mathcal{V}_-(is_0)\|_{\mathbb{C}^{m\times n}} \leq 1$, an application of the Schwarz inequality, which here is an equality, proves $(\mathrm{iii})_1$.

Skew-selfadjoint potential V (cont.)

$(\mathrm{iii})_2$ This says that the eigenvalues of $Q_\pm Q_\pm^*$ below s_0^2 do not matter. (Details are omitted).

(iv) The first assertion in (iv) follows from the fact that -1 is a "peripheral" eigenvalue of the spectrum because the spectral radius of $\mathcal{V}(is_0)$ is 1. Also, $\mathcal{V}(is_0)$ is a contraction and this alone allows the conclusion that

$$(\mathrm{ran}(\mathcal{V}(is_0) + I_n))^\perp = \ker(\mathcal{V}(is_0) + I_n).$$

Then, if -1 were not semisimple, then there would exist nonzero vectors $f, g \in \mathbb{C}^n$ such that $(\mathcal{V}(is_0) + I_n)g = f$ and $(\mathcal{V}(is_0) + I_n)f = 0$. But then $\|f\|^2 = f^*(\mathcal{V}(is_0) + I_n)g = 0$, a contradiction.

Skew-selfadjoint potential V (cont.)

Corollary 9

In the skew-adjoint case with constant one-step potentials, embedded eigenvalues cannot occur.

Proof: This follows from $(\mathrm{iii})_2$, which says that $(\Phi_{\pm}^{(k)})^* f = 0 \quad$ for $k = 1, \ldots, \rho_{\pm}(s_0)$. This eliminates the components of f that would be needed for an L^2- eigenfunction.

Skew-selfadjoint potential V (cont.)

Theorem 10

(i) If $Q_\pm$ are self-adjoint and positive semidefinite, then spectral singularities can occur only at the points $i|\omega_k^+|^{1/2}$ and $i|\omega_s^-|^{1/2}$ and this happens if and only if $|\omega_k^+|^{1/2} = |\omega_s^-|^{1/2}$ for some k and s and $\mathrm{ran}[\Phi_+^{(k)}] \cap \mathrm{ran}[\Phi_-^{(s)}] \neq \{0\}$. In particular, if $(\sigma(Q_+) \setminus \{0\}) \cap (\sigma(Q_-) \setminus \{0\}) = \emptyset$, then there are no nonzero spectral singularities.

(ii) If one of $Q_\pm$ is self-adjoint and positive semidefinite and the other, $Q_\mp$, is merely self-adjoint, then the conclusions of (i) hold true.

(iii) If one of $Q_\pm$ is self-adjoint and positive semidefinite and the other, $Q_\mp$, is negative semidefinite, then there are no nonzero spectral singularities.

Skew-selfadjoint potential V (cont.)

Proof (sketch):
Since $Q_\pm$ is self-adjoint (so $m = n$), we may choose $\Phi_\pm$ so that its columns are an orthonormal eigenbasis for the nonzero eigenvalues of $Q_\pm$, that is, $Q_\pm \Phi_\pm = \Phi_\pm \Lambda^\pm$, where $\Lambda^\pm$ is a diagonal $n^\pm \times n^\pm$ matrix whose entries are the nonzero eigenvalues of $Q_\pm$.
The columns of $\Phi_\pm$, from left to right, correspond to the eigenvalues in order of increasing absolute values, so $|\Lambda_1^\pm| < |\Lambda_2^\pm| < \cdots < |\Lambda_{\kappa^\pm}|$ and $(\Lambda_k^\pm)^2 = |\omega_k^\pm|$.
This allows us to write

$$\mathcal{V}_-(is_0) = \Phi_- \Lambda^- (M_-(is_0) + is_0 I_{n^-})^{-1} \Phi_-^*,$$

$$\mathcal{V}_+(is_0) = \Phi_+ \Lambda^+ (M_+(is_0) + is_0 I_{n^+})^{-1} \Phi_+^*,$$

where is_0 is a spectral singularity.

Skew-selfadjoint potential V (cont.)

We now exploit information about the numerical ranges of $\mathcal{V}_-(\xi)$ and $\mathcal{V}_+(\xi)$. The submatrices that are associated with the columns of $\Phi_\pm^{(k)}$ for $k > \rho_\pm(s_0)$ are unitary, since

$$|\mu_k^\pm(is_0) + is_0| = \left|(|\omega_k^\pm| - s_0^2)^{1/2} + is_0\right| = |\omega_k^\pm|^{1/2} = \Lambda_k^\pm.$$

We also know that

$$\boxed{(f, \mathcal{V}_-(is_0)f)_{\mathbb{C}^n} = -(f, \mathcal{V}_+(is_0)^* f)_{\mathbb{C}^n}}$$

where $\mathcal{V}(\xi)f = -f$ $(\|f\|_{\mathbb{C}^n} = 1)$.

- The numerical range of $\mathcal{V}_-(is_0)$ on the invariant subspace $\mathrm{ran}[\{\Phi_-^{(k)}\}_{k>\rho_-(s_0)}]$ is the polygon having $\Lambda_k^-(\mu_k^-(is_0) + is_0)^{-1}$ as vertices.

Skew-selfadjoint potential V (cont.)

- The numerical range of $-\mathcal{V}_+(is_0)^*$ on $\mathrm{ran}[\{\Phi_+^k\}_{k>\rho_+(s_0)}]$ is the polygon with vertices $-\Lambda_k^+(\mu_k^+(is_0) - is_0)^{-1}$.
- The two numerical ranges lie side by side on the unit circle in the third ($-\mathcal{V}_+(is_0)^*$) and fourth quadrant ($\mathcal{V}_-(is_0)$), respectively.
- The only way they can make contact is at $-i$ and this is possible only if $s_0 = |\omega_k^+|^{1/2} = |\omega_s^-|^{1/2}$ for some indices k and s.
- $f \in \mathrm{ran}[\Phi_+^{(k)}] \cap \mathrm{ran}[\Phi_-^{(s)}]$ follows.
- If $\Lambda^+ > 0$ but Λ^- has diagonal elements of arbitrary sign, then the numerical range of $\mathcal{V}_-(is_0)$ (the convex hull of the eigenvalues) lies inside the unit circle and above the line through $(-1, 0)$ and $(0, -1)$. The numerical range of $-\mathcal{V}_+(is_0)^*$ lies strictly below this line (except possibly for the point $(0, -1)$). The only possible point of contact is $-i$. Then there exist indices k and s such that $|\omega_k^+| = |\omega_s^-|$ and $s_0 = |\omega_k^+|^{1/2}$.

Skew-selfadjoint potential V (cont.)

- If $\Lambda^+ > 0$ and $\Lambda^- < 0$, then the numerical range of $\mathcal{V}_-(is_0)$ is in the second quadrant while that of $-\mathcal{V}(is_0)^*$ is in the third quadrant, and they are disjoint. No spectral singularities can occur.

Suppose the eigenvalue -1 of $\mathcal{V}(is_0)$ has (geometric) multiplicity $p \geq 1$. Let $f^{(1)}, \ldots, f^{(p)}$ be an associated orthonormal eigenbasis. Suppose that $s_0 = \sqrt{|\omega_k^+|} = \sqrt{|\omega_s^-|}$ for some k and s.

Theorem 11

$\mathcal{V}(\xi)$ is similar to a matrix $\widehat{\mathcal{V}}(\xi)$ having the following block structure:

$$\widehat{\mathcal{V}}(\xi) = \left(\begin{array}{c|c} \mathcal{A}(\xi) & \mathcal{B}(\xi) \\ \hline \mathcal{C}(\xi) & \mathcal{D}(\xi) \end{array} \right),$$

where

Skew-selfadjoint potential V (cont.)

$$\mathcal{A}(\xi) = \widehat{\mathcal{A}}(is_0)\, z + o(z), \quad z \to 0,$$

where $z = \sqrt{\xi^2 + |\omega_k^+|}$ and

$$[\widehat{\mathcal{A}}(is_0)]_{ij} = -is_0^{-1}(f^{(i)}, [P_+^{(k)} + P_-^{(s)}]f^{(j)})_{\mathbb{C}^n}.$$

Here $P_\pm^{(k)} = (\Phi_\pm^{(k)})(\Phi_\pm^{(k)})^*$.

$$\mathcal{B}(\xi) = O(z) \qquad \mathcal{C}(\xi) = O(z) \qquad \mathcal{D}(\xi) = \mathcal{D}(is_0) + O(z),$$

where $\mathcal{D}(is_0)$ is invertible.

The matrix $\widehat{\mathcal{A}}(is_0)$ may be zero. It is not zero if, for example, we are at the "highest" spectral singularity on the imaginary axis, that is, when $s_0 = |\omega_{\kappa^+}|^{1/2} = |\omega_{\kappa^-}|^{1/2}$.

Skew-selfadjoint potential V (cont.)

Using the formula

$$\begin{pmatrix} I_p & -\mathcal{B}(\xi)\mathcal{D}(\xi)^{-1} \\ 0 & I_{n-p} \end{pmatrix} \widehat{V}(\xi) \begin{pmatrix} I_p & 0 \\ -\mathcal{D}(\xi)^{-1}\mathcal{C}(\xi) & I_{n-p} \end{pmatrix} = \mathrm{diag}[\mathcal{U}(\xi), \mathcal{D}(k)],$$

where

$$\mathcal{U}(k) = \mathcal{A}(\xi) - \mathcal{B}(\xi)\mathcal{D}(\xi)^{-1}\mathcal{C}(\xi).$$

we find that

$$\widehat{\mathcal{V}}(\xi)^{-1} =$$

$$\begin{pmatrix} \mathcal{U}(\xi)^{-1} & -\mathcal{U}(\xi)^{-1}\mathcal{B}(\xi)\mathcal{D}(\xi)^{-1} \\ -\mathcal{D}(\xi)^{-1}\mathcal{C}(\xi)\mathcal{U}(\xi)^{-1} & \mathcal{D}(\xi)^{-1}\mathcal{C}(\xi)\mathcal{U}(\xi)^{-1}\mathcal{B}(\xi)\mathcal{D}(\xi)^{-1} + \mathcal{D}(\xi)^{-1} \end{pmatrix}$$

The divergence of $\widehat{\mathcal{V}}(\xi)^{-1}$ as ξ approaches a spectral singularity is determined by the inverse $\mathcal{U}(\xi)^{-1}$, which diverges like $z^{-1}\widehat{\mathcal{A}}(is_0)^{-1}$ (provided the inverse exists). (If the inverse does not exist, it diverges like $\sim z^{-2}$).

Skew-selfadjoint potential V/transmission coefficient

The intended application of these small-z asymptotics concerns the singularities of the transmission coefficient, $T(\xi)$, and the reflection coefficients near spectral singularities.

Without going into detail we state the connection between $\mathcal{V}(\xi)$ and $T(\xi)$:

$$T(\xi) = \begin{pmatrix} \widehat{N}_+^* \\ (-2iM_+)^{-1}\Phi_+^* \end{pmatrix} \begin{pmatrix} \widehat{N}_- & -i\Phi_-(M_- + \xi I_{n^-}) \end{pmatrix} \begin{pmatrix} I_{\nu_0^-} & -i\widehat{N}_-^* U_1 \\ 0 & I_{n^-} + U_2 \end{pmatrix},$$

where

$$U_1 = \mathcal{V}(\xi)\Phi_-(M_- + \xi I_{n^-}), \quad U_2 = (M_- + \xi I_{n^-})^{-1}\Phi_-^*\mathcal{V}(\xi)\Phi_-(M_- + \xi I_{n^-})$$

Perturbed problem

Now we consider the perturbed problem.

Potentials:

$$Q_\pm(x) = Q_\pm + \widehat{Q}_\pm(x), \quad R_\pm(x) = R_\pm + \widehat{R}_\pm(x),$$

where $\widehat{Q}_\pm$ and $\widehat{R}_\pm$ have entries in $L^1(\mathbb{R}^\pm)$.
Let

$$\mathcal{W}_0(x,\xi) = \begin{pmatrix} F_{0,-}(x,\xi) & F_{0,+}(x,\xi) \end{pmatrix}.$$

The matrix resolvent kernel of H_V (for $\operatorname{Im}\xi > 0$) is given by

$$[(H_V - \xi I_{n+m})^{-1}](x,t) =$$

$$i\mathcal{W}_0(x,\xi)\begin{bmatrix} \theta(t-x)I_n & 0 \\ 0 & -\theta(x-t)I_m \end{bmatrix}\mathcal{W}_0(t,\xi)^{-1}J.$$

where $\widehat{Q}(t)^{1/2} = \widehat{Q}(t)\|\widehat{Q}(t)\|^{-1/2}$.

Resolvent/Birman-Schwinger kernel (cont.)

If v is a solution of the AKNS system, define

$$w = \begin{bmatrix} \|\widehat{R}(x)\|^{1/2} & 0 \\ 0 & \|\widehat{Q}(x)\|^{1/2} \end{bmatrix} v.$$

Then w satisfies

$$w(x,\xi) = \int_{-\infty}^{\infty} \mathcal{K}(x,t;\xi) w(t,\xi)\, dt,$$

where $\mathcal{K}(x,t;\xi)$ is the Birman-Schwinger kernel:

$$\mathcal{K}(x,t;\xi) = \begin{bmatrix} \|\widehat{R}(x)\|^{1/2} & 0 \\ 0 & \|\widehat{Q}(x)\|^{1/2} \end{bmatrix} \mathcal{W}_0(x,\xi)$$

$$\begin{bmatrix} \theta(t-x) I_n & 0 \\ 0 & -\theta(x-t) I_m \end{bmatrix} \mathcal{W}_0(t,\xi)^{-1} \begin{bmatrix} 0 & -\widehat{Q}(t)^{1/2} \\ -\widehat{R}(t)^{1/2} & 0 \end{bmatrix},$$

Resolvent/Birman-Schwinger kernel (cont.)

Let $\widehat{\mathcal{K}}(\xi)$ denote the Birman-Schwinger integral operator for the problem with $V = 0$.

Theorem 12

There is a positive number r_0 such that the region $|\xi| > r_0$ does not contain any eigenvalues or spectral singularities.

Proof: $\mathcal{K}(x, t; \xi)$ does not have a pointwise limit if $\xi \to \infty$ along lines where $\mathrm{Im}\, \xi$ is constant because it oscillates. But the following can be proved:

- $\|\mathcal{K}(\xi) - \widehat{\mathcal{K}}(\xi)\|_{H.S.} \to 0, \quad |\xi| \to \infty.$
- $\|\widehat{\mathcal{K}}(\xi)^2\|_{H.S.} \to 0 \quad |\xi| \to \infty.$
- Hence $\|\mathcal{K}(\xi)^2\|_{H.S.} \to 0$ follows, proving the assertion.

Theorem 8 generalized

Corollary 13

Suppose $\widehat{R}_{\pm}(x) = \widehat{Q}_{\pm}(x)^*$. Then Theorem 8 holds for the perturbed equation, provided we replace $W_0(\xi)$ by $\det\begin{pmatrix} F_-(0,\xi) & F_+(0,\xi) \end{pmatrix}$, where $F_{\pm}(x,\xi)$ denote the Jost solutions for the perturbed problem.

The reason is that the identities

$$F_+(x,\xi)^* J F_+(x,\xi) = \begin{pmatrix} -N_+^* N_+ & 0 \\ 0 & C_+(\xi) \end{pmatrix}, \qquad \xi \in \mathbb{R},$$

and that for $F_-(x,\xi)^* J F_-(x,\xi)$, also hold for the perturbed Jost solutions, since they only rely on asymptotic information as $x \to \pm\infty$.

Theorem 12 generalized

It is not clear to me at this point how far the results of Theorem 10 can be extended to the skew-selfadjoint perturbed problem.
We have some “extensions” under different conditions and with narrower conclusions. For example:

Let $n = 1$, $m \geq 1$, and suppose that $Q_\pm$, $\widehat{Q}_\pm$ have real components and that $R_\pm = -Q_\pm^T$, $\widehat{R}_\pm = -\widehat{Q}_\pm^T$ (here T denotes the transpose). Let $W(\xi)$ denote the Wronskian for the perturbed problem.

Set $\omega^\pm = Q_\pm Q_\pm^T > 0$. Then:

Theorem 14

Every zero of $W(\xi)$, where $\xi = is$ with $\sqrt{\omega^-} < s < \sqrt{\omega^+}$, corresponds to an embedded eigenvalue. In $0 \leq s \leq \sqrt{\omega^-}$ we can only have spectral singularities.

Thank you for your attention!