Breaking mechanism from a vacuum point in the defocusing NLS equation

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joint work with Stefano Trillo

The nonlinear Schrödinger (NLS) equation

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + \sigma|\psi|^2\psi = 0$$

where

$$\epsilon > 0, \qquad \sigma = 1 \ \rightarrow \text{focusing}, \qquad \sigma = -1 \ \rightarrow \text{defocusing}$$

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- Ubiquitous $\leftrightarrow$ Universal
- Integrable $\leftrightarrow$ Exactly solvable (Zakharov-Shabat, '72)

Weak dispersive regime

Study asymptotics of fast oscillating solutions $\psi(x, t; \epsilon)$

Madelung transform

$$\psi(x, t; \epsilon) = \sqrt{u(x, t)} \exp\left(-\frac{i}{\epsilon} \int^{x} v(x', t)\, dx'\right)$$

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Hydrodynamic form

$$\begin{aligned} &u_t + (uv)_x = 0 \\ &v_t + vv_x + u_x - \frac{\epsilon^2}{4}\left(\frac{u_{xx}}{u} - \frac{u_x^2}{2u^2}\right)_x = 0 \qquad \epsilon << 1 \end{aligned}$$

Standard Approach

Look for solutions of the form

$$u_\epsilon = u + \epsilon u_1 + \epsilon^2 u_2 + \dots \qquad v_\epsilon = v + \epsilon v_1 + \epsilon^2 v_2 + \dots .$$

Leading (**dispersionless**) order $\rightarrow$ shallow water equations (SWE)

$$\begin{aligned} u_t + (uv)_x &= 0 \\ v_t + vv_x + u_x &= 0 \end{aligned}$$

Note

$$u_j \text{ and } v_j, \text{ with } j = 1, 2, \dots \quad \longleftrightarrow \quad \text{transport equations}$$

Riemann Invariants

Introduce the variables

$$\xi = v + 2\sqrt{u}, \qquad \eta = v - 2\sqrt{u}$$

such that the SWE takes the diagonal form

$$\xi_t + \lambda \xi_x = 0 \qquad \eta_t + \mu \eta_x = 0,$$

where $\lambda(\xi, \eta)$ and $\mu(\xi, \eta)$ are the characteristic speeds

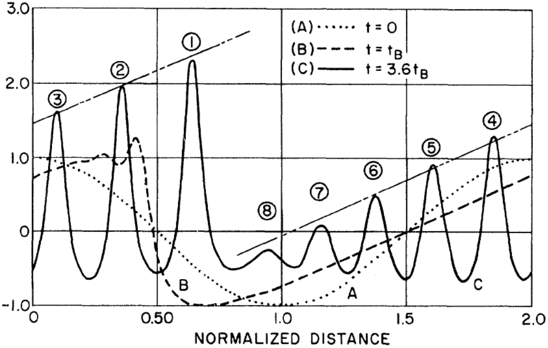
$$\lambda = \frac{3\xi + \eta}{4} \qquad \mu = \frac{\xi + 3\eta}{4}.$$

Note In general, Riemann invariants break in finite time. If they break at the same point (x, t) the corresponding initial datum is said to be **non-generic.**

Weak dispersive phenomenology

Consider KdV equation

$$u_t + uu_x + \epsilon^2 u_{xxx} = 0$$



Zabusky and Kruskal, '65

Universality (Dubrovin, '06)

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Defocusing NLS equation near the critical point of gradient catastrophe (*generic case*)

$$(v - v_c) + (u - u_c) \simeq \epsilon^{4/7}\left[\frac{x_+}{\alpha} + \sigma U''\left(\frac{\nu_+ x_+}{\epsilon^{6/7}}; \frac{\nu_+ x_-}{\epsilon^{4/7}}\right)\right]$$

$$(v - v_c) - (u - u_c) \simeq \epsilon^{2/7}\beta U\left(\frac{\nu_+ x_+}{\epsilon^{6/7}}; \frac{\nu_+ x_-}{\epsilon^{4/7}}\right)$$

where

$$x_\pm = (x - x_c) + (v_c \pm 2\sqrt{u_c})(t - t_c)$$

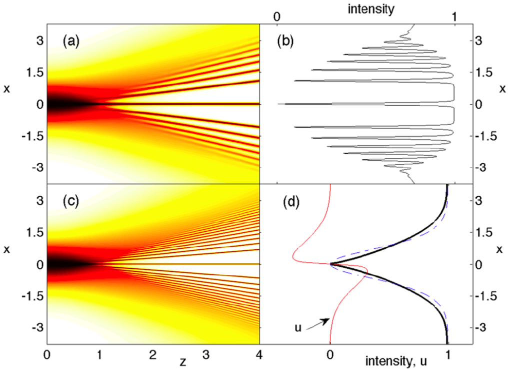
and $U(X, T)$ satisfies P_I^2 equation

$$X = UT - \left[\frac{1}{6}U^3 + \frac{1}{24}(U'^2 + 2UU'') + \frac{1}{240}U^{(IV)}\right]$$

A previous study: phase jump dark initial datum

Solution for a jump phase-dark initial datum

$$\psi(x,0) = \tanh(x)\ \exp^{i\theta_0}, \qquad u(x,t) = |\psi(x,t)|^2$$



Conti, Fratalocchi, Peccianti, Ruocco, Trillo, PRL 2009

Our study: constant phase dark initial datum

We consider the dark constant phase initial condition

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Remark1

This case turns out to be **non-generic** $\rightarrow$ Universality conjecture does not apply.

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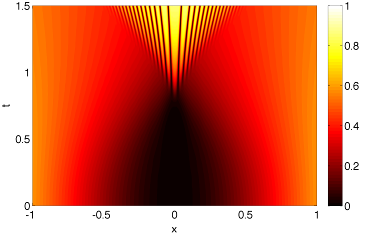
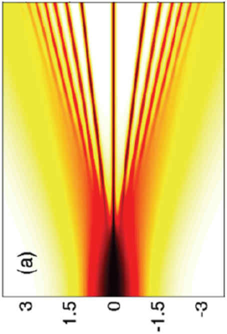
Remark2

Vacuum points in NLS dynamics appeared in El et al, ’95, Hoefer et al, ’08

Weak dispersive regime: comparison



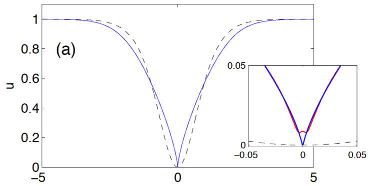
continuous phase

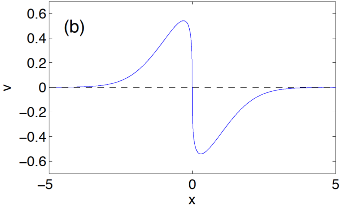


jump phase

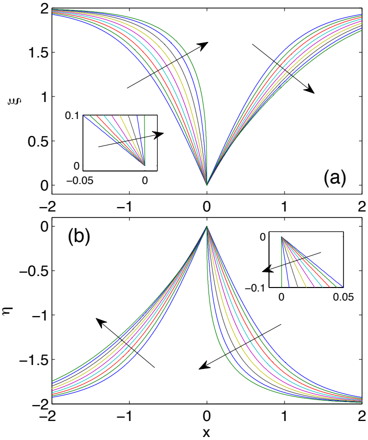
A dispersive shock that opens up in a characteristic fan turns out to resolve the singularity that occurs in the range $t = 0.75 - 0.78$ around the origin ($x = 0$)

Dispersionless regime





- Wave-breaking at $t = t_c \simeq 0.78$, $x \simeq 0$
- Preserved vacuum $u(0, t) = 0$, for $t < t_c$
- Gradient catastrophe scenario for $v_x(0, t_c) \sim \infty$
- Jump in the derivative for $u(x, t_c)$



Snapshots of RIs obtained by means of numerical integration SWE at time $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.784$

Main observations

Dispersionless regime:

- Dispersionless level: no qualitative difference
- Riemann Invariants (RIs) simultaneously break at the same point $\leftrightarrow$ The critical point is *non-generic*

Weak dispersion regime:

- Preserved vs non preserved vacuum
- Critical behaviour occurs around RIs breaking time

Analytic solution of SWE

Introduce the hodograph transform

$$(x, t) \leftrightarrow (u, v)$$

defined by

$$vt + f_u = x \qquad\qquad ut + f_v = 0,$$

where the function $f(u, v)$ is a solution to the Tricomi-type equation

$$f_{vv} - uf_{uu} = 0$$

Solution to IVP

$$\text{map:} \qquad (u(x, 0), v(x, 0)) \longleftrightarrow f(u, v)$$

Euler-Poisson-Darboux equation

In Riemann invariants, the hodograph equations in reads as

$$x - \lambda t = w_\xi \qquad\qquad x - \mu t = w_\eta$$

and the Tricomi type equation is replaced by the EPD equation for the function $w(\xi, \eta)$

$$w_{\xi\eta} = \frac{1}{2(\xi - \eta)}\left(w_\xi - w_\eta\right)$$

Differentiating the hodograph equations and solving w.r.t. ξ_x, ξ_t, η_x, η_t, we get

$$\begin{aligned} \xi_x &= \frac{2(\eta-\xi)}{2(\eta-\xi)w_{\xi\xi}+3(w_\xi-w_\eta)} \\ \eta_x &= \frac{2(\eta-\xi)}{2(\eta-\xi)w_{\eta\eta}+3(w_\xi-w_\eta)}. \end{aligned} \tag{4.1}$$

The solution $(\xi(x,t),\eta(x.t))$ is said to break, i.e. it develops a gradient catastrophe singularity, if there exists a *critical point* (x_c,t_c) such that

$$\xi_x \quad \text{and} \quad \eta_x \qquad \text{are bounded}$$

for any $t\in[0,t_c)$ and $|\xi_x(x_c,t_c)|=\infty$ or $|\eta_x(x_c,t_c)|=\infty$.

A class of initial value problems

General IVP solved by Geogdzhaev, '87 - Tian-Ye, '99

We focus on the family of initial data of the form

$$u(x,0) = u_0(x) \qquad\qquad v(x,0) = 0$$

For the sake of simplicity, u_0 is assumed to be a negative hump given by an even, continuous and differentiable function centred at $x = 0$.
Solution to the Tricomi type (EPD) equation

$$f(u,v) = u \int_{-1}^{1} g(v + 2\mu\sqrt{u})\sqrt{1-\mu^2}\, d\mu$$

where g is is fixed by the initial condition

For the initial datum

$$\psi(x,0) = |\tanh(x)| \; e^{i\theta_0}$$

in the hydrodynamic variables

$$u(x,0) = \tanh^2(x) \qquad v(x,0) = 0$$

The function $f(u,v)$ takes the form (in Riemann Invariants)

$$\begin{aligned} f(u(\xi,\eta), v(\xi,\eta)) = & \frac{1}{4}\int_0^{\xi} \frac{r\sqrt{(\xi - r)(r-\eta)}}{\sqrt{4-r^2}}\; dr \\ & + \frac{1}{4}\int_0^{\eta} \frac{r\sqrt{(\xi - r)(r-\eta)}}{\sqrt{4-r^2}}\; dr \end{aligned}$$

$$
\begin{aligned}
&\frac{1}{4}\int_0^{\xi}\frac{r\sqrt{(\xi-r)(r-\eta)}}{\sqrt{4-r^2}}\,dr = \\
&-2\left[\frac{\gamma}{2\alpha^2(k^2-\alpha^2)}(2-\xi)(\xi-\eta)\left(\alpha^2 E(\omega)+\right.\right. \\
&\left.\left.(\alpha^2-k^2)\omega+(\alpha^4-2\alpha^2+k^2)\Pi(\omega,\alpha^2)-\alpha^4\frac{\mathrm{sn}\omega\ \mathrm{cn}\omega\ \mathrm{dn}\omega}{1-\alpha^2\mathrm{sn}^2\omega}\right)\right] \\
&+\frac{\gamma}{\alpha^4}(2-\xi)(2+\xi)(\xi-\eta)\left[-k^2\omega+(3k^2-\alpha^2k^2-\alpha^2)\Pi(\omega,\alpha^2)\right. \\
&\left.+(2\alpha^2k^2+2\alpha^2-3k^2-\alpha^4)V_2+(\alpha^2-1)(\alpha^2-k^2)V_3\right]
\end{aligned}
$$

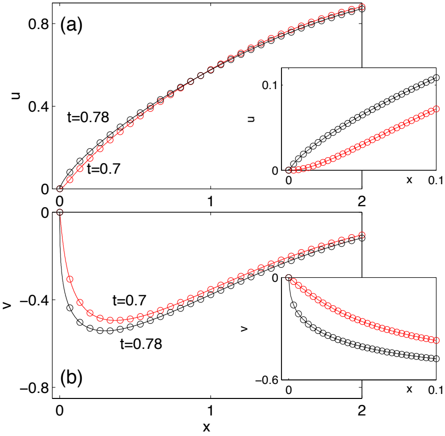
where $V_0 = \mathcal{F}(\phi, k) \qquad V_1 = \Pi(\phi, \alpha^2, k)$

$$V_2 = \frac{1}{2(\alpha^2 - 1)(k^2 - \alpha^2)} \left[\alpha^2 E(\omega) + (k^2 - \alpha^2)\omega \right.$$
$$+ (2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\Pi(\phi, \alpha^2, k)$$
$$\left. -\alpha^4 \frac{\mathrm{sn}\omega\ \mathrm{cn}\omega\ \mathrm{dn}\omega}{1 - \alpha^2 \mathrm{sn}^2\omega}\right] \qquad V_3 = \dots$$
$$\alpha^2 = \frac{\xi - \eta}{2 - \eta} \qquad k^2 = \frac{4(\xi - \eta)}{(2 - \eta)(2 + \xi)} \qquad \gamma = \frac{2}{\sqrt{(2 - \eta)(2 + \xi)}}$$
$$\phi = \sin^{-1}\left(\frac{\xi(2 - \eta)}{2(\xi - \eta)}\right) \qquad \omega = \mathrm{sn}^{-1}(\phi, k^2),$$

where $\mathcal{F}(\phi, k)$, $E(\omega)$ and $\Pi(\phi, \alpha^2, k)$ are the standard notations for the elliptic integral of first kind, the complete elliptic integral of second kind and the incomplete elliptic integral of the third kind respectively and sn, cn, dn stand for the Jacobian Elliptic functions.

Numerical vs analytic solution

$$vt + f_u = x \qquad\qquad ut + f_v = 0,$$



Near the vacuum...

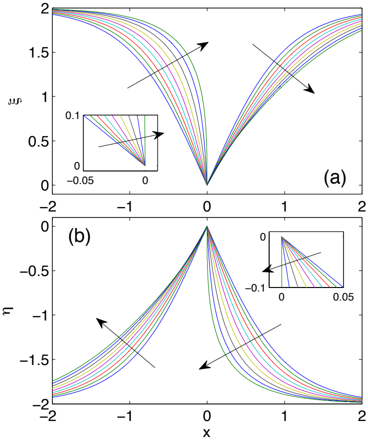
We compute

$$f(u(\xi,\eta),v(\xi,\eta)) \simeq \frac{1}{32}\left(\xi^2-\frac{2}{3}\eta\xi+\eta^2\right)(-\eta\xi)^{1/2}+ \\ +\frac{1}{64}(\xi-\eta)^2(\xi+\eta)\sin^{-1}\left(\frac{\xi+\eta}{\xi-\eta}\right)$$

and

$$x \simeq \frac{2(-\eta\xi)^{3/2}}{(\xi-\eta)^2}$$

$$t \simeq -\frac{\xi+\eta}{(\xi-\eta)^2}(-\eta\xi)^{1/2}+\frac{1}{2}\sin^{-1}\left(-\frac{\xi+\eta}{\xi-\eta}\right)$$



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On the positive half-line $x > 0$, the critical values such that

$$\eta_x(x_c, t_c) \to \infty$$

and

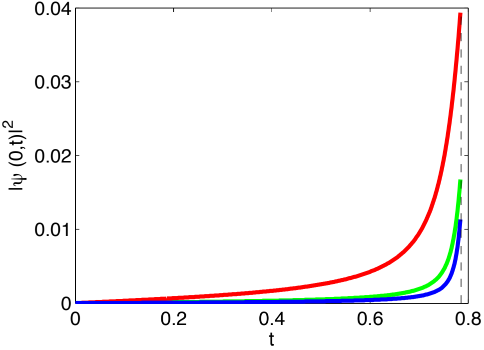
$$|\xi_x(x_c, t_c)| < \infty$$

are

$$x_c = \lim_{\eta \to 0} x(\xi, \eta)|_{\xi=0} = 0,$$

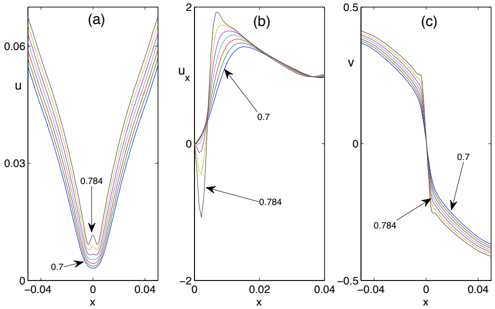
$$t_c = \lim_{\eta \to 0} t(\xi, \eta)|_{\xi=0} = \lim_{\eta \to 0} \frac{1}{2} \sin^{-1}(1) = \frac{\pi}{4} \simeq 0.785.$$

Dispersive effects: Initial vacuum point



Density $u(0, t) = |\psi(0, t)|^2$ vs. time t obtained from numerical integration of NLS equation with decreasing values of dispersion $\varepsilon = 5 \times 10^{-3}$ (red), $\varepsilon = 10^{-3}$ (green), $\varepsilon = 5 \times 10^{-4}$ (blue). The dashed vertical line stands for the critical time $t_c = \pi/4$.

Dispersive effects: critical phase



Hydrodynamic variables in the neighborhood of the origin, obtained from numerical integration of NLS equation with $\varepsilon = 5 \times 10^{-4}$. The snapshots are taken from time $t = 0.76$ to $t = 0.784$ with constant increment $\delta t = 0.004$.

Summary

- SWEs undergo the simultaneous breaking of RIs in the point of null density ($x = 0$)
- Breaking occurs in the opposite limits $x = 0^{\pm}$ for the two RIs
- Dispersion starts to play a role just approaching the breaking
- Slow adiabatic detachment of the min density points from zero that abruptly grows turns into a max near the breaking point for SWE

Outlook

- More general initial data
- Analytic asymptotic description of the weakly dispersive critical behaviour