Multiple-pole solutions (MPS) of the Nonlinear Schrödinger equation

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Contents

1. Topics on MPS

(1) Explicit formulas for MPS

[Schiebold, Habilitation Thesis 2004]
[Aktosun/Demontis/van der Mee, Inv. Probl. 2007]

(2) Asymptotic description

[Zakharov/Shabat, Sov. Phys. JETP 1972]

- coalescence of two simple poles

[Olmedilla, Physica D 1986]

- conjecture for one pole of order L
- proof for $L = 2, 3$

(3) Understand transition to higher order degeneracy phenomena

2. Operator formulas

Proposition

Let A, C be $n \times n$-matrices, and denote by $\overline{A}$ the matrix obtained from A by taking the complex conjugate entries. Then

$$Q = (I + L\overline{L})^{-1}(AL + L\overline{A}) \quad \text{where } L(x,t) = \exp(Ax - \mathrm{i}A^2 t)C,$$

is a solution of the matrix NLS equation

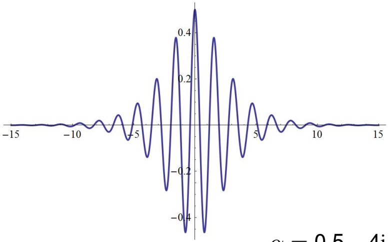
$$\mathrm{i}Q_t = Q_{xx} + 2Q\overline{Q}Q$$

provided that $I + L\overline{L}$ is invertible.

For $n = 1$, $A = \alpha$, and $C = \exp(\varphi)$,

$$
\begin{aligned}
q(x,t) &= 2\mathrm{Re}(\alpha)\frac{L(x,t)}{1+|L(x,t)|^2} \\
&= \mathrm{Re}(\alpha)\, e^{\mathrm{i}\,\mathrm{Im}(\alpha x - \mathrm{i}\alpha^2 t + \varphi)} \cosh^{-1}\left(\mathrm{Re}(\alpha x - \mathrm{i}\alpha^2 t + \varphi)\right),
\end{aligned}
$$

the 1-soliton solution.



$\alpha = 0.5 - 4\mathrm{i}$

- α characterizes the soliton
  - amplitude: $\mathrm{Re}(\alpha)$
  - velocity: $-2\mathrm{Im}(\alpha)$
- φ gives its initial position shifts

Towards a solution formula

First ingredient:

Proposition

Let $Q = Q(x,t)$ be a solution of the matrix NLS equation which, for some $a, c \in \mathbb{C}^n$, can be written in the form

$$Q(x,t) = Q_0(x,t)c\overline{a}^t.$$

Then $q(x,t) = a^t Q_0(x,t)c$ solves the NLS equation.

$$\begin{aligned} Q\overline{Q}Q &= Q_0 c\overline{a}^t \ \ \overline{Q}_0\overline{c}a^t \ \ Q_0 c\overline{a}^t \\ &= Q_0 c \ \ \overline{a}^t\overline{Q}_0\overline{c} \ \ a^t Q_0 c \ \overline{a}^t = Q_0 c \ \overline{q}q \ \overline{a}^t \end{aligned}$$

Towards a solution formula

Second ingredient:

- $Q = (I + L\overline{L})^{-1} e^{Ax - iA^2 t} (AC + C\overline{A})$
- For any right-hand side Y, the Sylvester equation $AX + XB = Y$ has a unique solution if and only if $0 \notin \operatorname{spec}(A) + \operatorname{spec}(B)$.

Third ingredient:

- From traces to determinants

$$\begin{aligned} q &= a^t (I + L\overline{L})^{-1} e^{Ax - iA^2 t} c \\ &= \operatorname{tr}\Big((I + L\overline{L})^{-1} e^{Ax - iA^2 t} c a^t \Big) \\ &= \ldots \end{aligned}$$

Theorem

Let $A \in \mathcal{M}_{n,n}(\mathbb{C})$ *with* $0 \notin spec(A) + spec(\overline{A})$*, and* $a, c \in \mathbb{C}^n$*.*

Denote by C *the unique solution of the Sylvester equation* $AC + C\overline{A} = c\overline{a}^t$*, and define*

$$L(x,t) = e^{Ax - \mathrm{i}A^2 t} C, \qquad L_0(x,t) = e^{Ax - \mathrm{i}A^2 t} c a^t.$$

Then a solution of the NLS equation, which is smooth on $\mathbb{R}^2$*, is given by*

$$q = 1 - \frac{\det \begin{pmatrix} I - L_0 & -L \\ \overline{L} & I \end{pmatrix}}{\det \begin{pmatrix} I & -L \\ \overline{L} & I \end{pmatrix}}.$$

Further aspects:

- The theorem is a particular case of a more general result in [Schiebold, J. Phys. A 2010] where a solution formula is given
  - for the whole AKNS system
  - depending on *operator* parameters
- Advantage: access to wider solution classes
  - solutions to the initial value problem
  - matrix solitons
  - countable superpositions
- Systematic explanation of the choices via functional analysis

3. MPS via ISM

To principal parts

$$r_j(k) = \frac{r_{jn_j}}{(k-k_j)^{n_j}} + \frac{r_{jn_j-1}}{(k-k_j)^{n_j-1}} + \ldots + \frac{r_{j1}}{k-k_j}$$

given at N points $k_1, \ldots, k_N$ in the upper half plane $\mathbb{H}$, we associate the GLM-kernel

$$\Omega(z;t) = -2\mathrm{i} \sum_{j=1}^{N} \mathrm{res}_{k=k_j} \left(r_j(k) e^{2\mathrm{i}kz - 4\mathrm{i}k^2 t} \right).$$

Lemma

There are vectors d, f such that

$$\Omega(z;t) = -2\mathrm{i} f^t \, e^{\Lambda x + \mathrm{i}\Lambda^2 t} \, d,$$

where $\Lambda = \mathrm{diag}\{\Lambda_j \mid j = 1, \ldots, N\}$ and Λ_j is $2\mathrm{i}$ times the $n_j \times n_j$-Jordan block w.r.t. the eigenvalue k_j.

Lemma

$$K(x,y) := 2\mathrm{i}f^t\left(I + \overline{G(x,t)}\ G(x,t)\right)^{-1} e^{\overline{\Lambda}(x+y)-\mathrm{i}\overline{\Lambda}^2 t}\ \overline{d}$$

solves the GLM-equation

$$K(x,y) + \int_0^\infty \int_0^\infty K(x,s)\Omega(x+s+z)\overline{\Omega(x+y+z)}\, ds\, dz = \overline{\Omega(x+y)}.$$

The associated solution $q(x) = -K(x,0)$ of the NLS equation can be obtained via:

$$A = \overline{\Lambda},$$
$$a = -2\mathrm{i}f, \quad c = \overline{d}.$$

4. Asymptotic description of MPS

Definition

Two functions $f = f(x,t)$, $g = g(x,t)$ have the same asymptotic behavior for $t \to \infty$ ($t \to -\infty$),

$$f(x,t) \approx g(x,t) \text{ for } t \approx \infty \ (t \approx -\infty).$$

if for every $\epsilon > 0$ there is a t_ϵ such that for $t > t_\epsilon$ ($t < t_\epsilon$) we have $|f(x,t) - g(x,t)| < \epsilon$ uniformly in x.

The following technical assumptions can be made without loss of generality:

Assumption 1: The matrix $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is in Jordan form with N Jordan blocks of sizes $n_j \times n_j$ and eigenvalues α_j.

Adapt notation to this Jordan form: For $v \in \mathbb{C}^n$, write $v = (v_j)_{j=1}^N$ with $v_j = (v_j^{(\mu)})_{\mu=1}^{n_j} \in \mathbb{C}^{n_j}$.

Assumption 2: The vectors $a, c \in \mathbb{C}^n$ satisfy $a_j^{(1)} c_j^{(n_j)} \neq 0$.

Assumption 3: $\mathrm{Re}(\alpha_j) > 0$.

We also need:

Assumption 4: The $\mathrm{Im}(\alpha_j)$ are pairwise different.

Theorem

For the associated solution it holds:

$$q(x,t) \approx \sum_{j=1}^{N} \sum_{j'=0}^{n_j-1} q^{\pm}_{jj'}(x,t) \text{ for } t \approx \pm\infty,$$

where

$$q^{\pm}_{jj'}(x,t) = (-1)^{j'+1}\,\mathrm{Re}(\alpha_j) e^{-\mathrm{i}\,\mathrm{Im}\left(\Gamma^{\pm}_{jj'}(x,t)\right)} \cosh^{-1}\left(\mathrm{Re}\left(\Gamma^{\pm}_{jj'}(x,t)\right)\right)$$

and $$\Gamma^{\pm}_{jj'}(x,t) = \alpha_j x - \mathrm{i}\alpha_j^2 t \mp J' \log|t| + \varphi_j + \varphi_j^{\pm} + \varphi^{\pm}_{jj'},$$

where we have set $J' = -(n_j - 1) + 2j'$*, and the quantities* φ_j, $\varphi_j^{\pm}$, $\varphi^{\pm}_{jj'}$ *are given by*

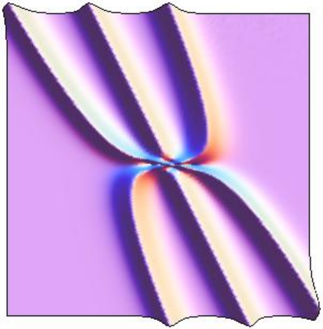
$$\begin{aligned}
\exp(\varphi_j) &= a_j^{(1)} c_j^{(n_j)} / (\alpha_j + \overline{\alpha}_j)^{n_j}, \\
\exp(\varphi_j^{\pm}) &= \prod_{k \in \Lambda_j^{\pm}} \left[\frac{\alpha_j - \alpha_k}{\alpha_j + \overline{\alpha}_k}\right]^{2n_k}, \\
\exp\left(\pm\varphi^{\pm}_{jj'}\right) &= \left(-\mathrm{i}(\alpha_j + \overline{\alpha}_j)^2\right)^{-J'} \frac{j'!}{(j' - J')!},
\end{aligned}$$

for the index sets $\Lambda_j^{\pm} = \{k \mid \mathrm{Im}(\alpha_j) \lessgtr \mathrm{Im}(\alpha_k)\}$.

Geometric interpretation and examples

One single $n \times n$-Jordan block:

- wave packet with n solitons
- the packet itself moves with constant velocity $v = -2\mathrm{Im}(\alpha)$
- the solitons approach/drift away from its center on logarithmic curves.

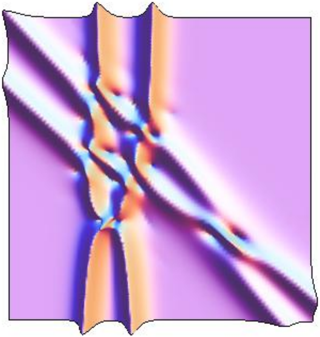


$n = 3$, $\alpha = 1 + 0.1\mathrm{i}$

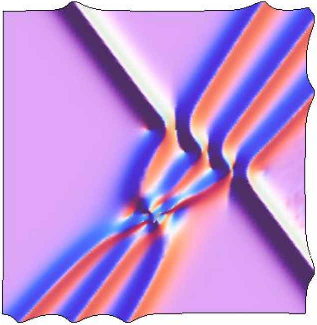
Note that the internal collisions of the solitons do not affect the path of the wave packets center.

N Jordan blocks:

- superposition of N wave packets as just described
- collisions between the wave packets are elastic and result only in a phase-shift



$n_1 = n_2 = 2$,
$\alpha_1 = 1$, $\alpha_2 = \frac{1}{2} + \frac{1}{2}\mathrm{i}$,



$n_1 = 3$, $n_2 = 1$,
$\alpha_1 = 1 - \frac{1}{2}\mathrm{i}$, $\alpha_2 = 1 + \frac{1}{2}\mathrm{i}$

N-soliton solutions versus MPS

- Asymptotic curves

$$\mathrm{Re}\Big(\alpha_j x - \mathrm{i}\alpha_j^2 t \mp J' \log|t| \ + \varphi_j + \varphi_j^{\pm} + \varphi_{jj'}^{\pm}\Big) = 0$$

are no longer straight lines.

- Phase-shifts $\varphi_{jj'}^{\pm}$ due to collisions between solitons within the same wave packet,
- phase-shifts $\varphi_j^{\pm}$ due to collisions between wave packets

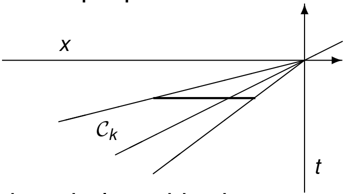
$$\exp(\varphi_j^{\pm}) \ = \ \prod_{k\in\Lambda_j^{\pm}} \left[\frac{\alpha_j - \alpha_k}{\alpha_j + \overline{\alpha}_k}\right]^{2n_k} .$$

Each soliton in the *j*th wave packet experiences shifts from all n_k solitons within a colliding *k*th packet.

Main ingredients of the proof:

1. Regularity of MPS via factorization of solutions of the Sylveter equation over $L_2(-\infty, 0]$.
2. MPS asymptotically behave as the superposition of N single wave packets

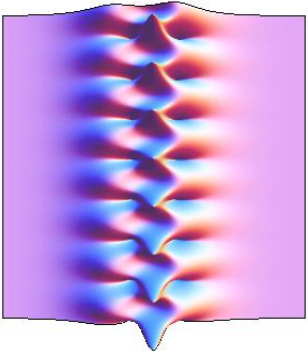
Contribution from cones $\mathcal{C}_k$ around the straight lines
$x = -2\mathrm{Im}(\alpha_k)t$



3. Asymptotics for single wave packets in logarithmic sectors
4. Evaluation of Cauchy-type determinants [Schiebold, LAA 2012]

5. Solutions of higher degeneracy

Wave packets having the same velocity cannot be distinguished by asymptotic analysis.



Example 1:

$$n_1 = n_2 = 1$$
$$\alpha_1 = 4,\ \alpha_2 = 2$$

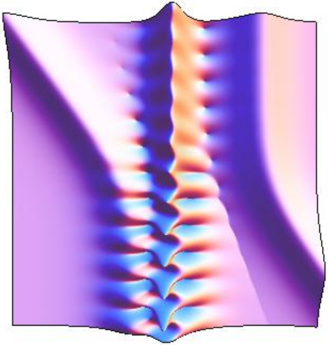
(two stationary solitons)

[Aktosun/Demontis/van der Mee, Inverse Problems 2007]

Example 2:

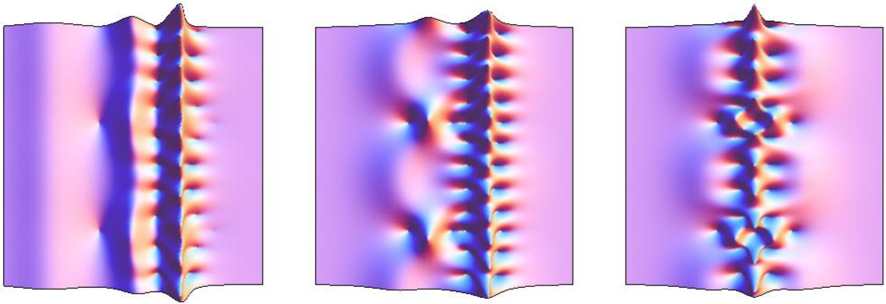
$n_1 = n_2 = n_3 = 1$
$\alpha_1 = 4,\ \alpha_2 = 2,\ \alpha_3 = 2 + \frac{1}{2}\mathrm{i}$

(the solution from Example 1 meets a non-stationary soliton)



Asymptotic analysis confirms that each of the two stationary solitons experiences a different phase-shift from the collision.

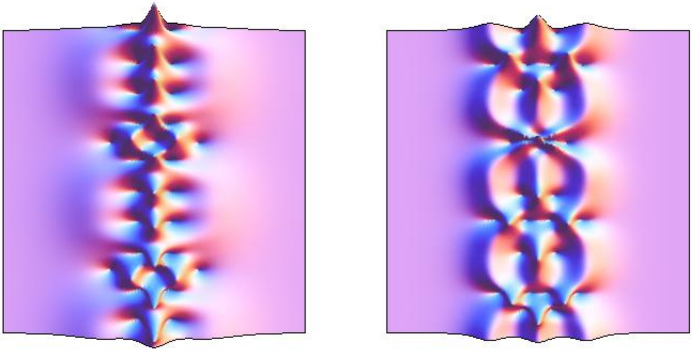
Example 3: Three stationary solitons which only differ in their initial position shifts.



$$n_1 = n_2 = n_3 = 1$$

$$\alpha_1 = 4,\ \alpha_2 = 2,\ \alpha_1 = 1$$

Example 4: On the left the solution from the last example. On the right one of the eigenvalues has been changed.

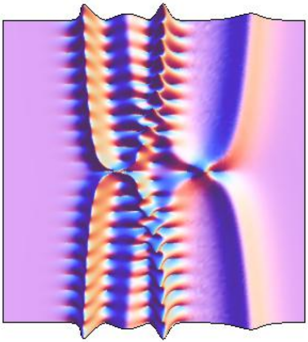


$n_1 = n_2 = n_3 = 1$

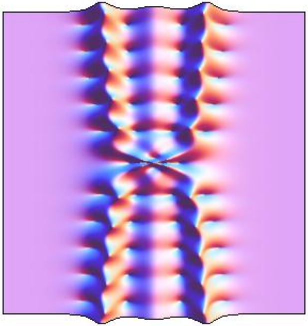
$\alpha_1 = 4,\ \alpha_2 = 2,\ \alpha_3 = 1$

$\alpha_1 = 4,\ \alpha_2 = 3,\ \alpha_3 = 2$

Example 5: Two stationary constellations of higher complexity.



$n_1 = n_2 = 2$
$\alpha_1 = 4, \alpha_2 = 2$



$n_1 = 2, n_2 = 1$
$\alpha_1 = 4, \alpha_2 = 2$

Selected references

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In preparation.