# A symbol approach in IgA matrix analysis (and in the design of efficient multigrid methods)

Stefano Serra-Capizzano, Dept. of Science and high Technology, U. Insubria, Como

Joint work with C. Garoni, C. Manni, F. Pelosi, H. Speleers

# Model problem

## Model problem

$$\begin{cases} -\Delta u + \beta \cdot \nabla u + \gamma u = f & \text{on } (0,1)^d \\ u = 0 & \text{on } \partial((0,1)^d) \end{cases} \tag{1}$$

$(d \geq 1, \quad f \in L^2((0,1)^d), \quad \beta \in \mathbb{R}^d, \quad \gamma \geq 0)$

## Weak form

Find $u \in H_0^1((0,1)^d)$ such that

$$a(u,v) = F(v) \qquad \forall v \in H_0^1((0,1)^d) \tag{2}$$

where

$$a(u,v) := \int_{(0,1)^d} (\nabla u \cdot \nabla v + \beta \cdot \nabla u \; v + \gamma u \, v) \qquad\qquad F(v) := \int_{(0,1)^d} f \, v$$

$\exists!$ solution $u \in H_0^1((0,1)^d)$ to (2), called the weak solution of (1).

## Galerkin method and IgA

To approximate $u$ we consider the Galerkin method.

### Galerkin method

1. Choose a subspace $W \subset H_0^1((0,1)^d)$ with $\dim W =: N < \infty$
2. Find $u_W \in W$ such that

$$a(u_W, v) = F(v) \qquad \forall v \in W \tag{3}$$

$$\Downarrow$$

$\exists!$ solution $u_W \in W$ to (3) (whatever $W$).

*Chosen a basis* $\{\varphi_1, \ldots, \varphi_N\}$ *for* $W$, $u_W$ *has the representation* $u_W = \sum_{i=1}^N u_i \varphi_i$, *with* $[u_1 \ u_2 \ \cdots \ u_N]^T =: \mathbf{u} \in \mathbb{R}^N$, *and problem* (3) *is equivalent to the following: find* $\mathbf{u} \in \mathbb{R}^N$ *such that*

$$A\mathbf{u} = \mathbf{f},$$

*where* $A = [a(\varphi_j, \varphi_i)]_{i,j=1}^N$ *is the* **stiffness matrix** *and* $\mathbf{f} = [F(\varphi_i)]_{i=1}^N$ .
*In the IgA setting* $W$ *is chosen as a space of splines.*

# Spectral analysis (1D case)... the 2D case is similar

To simplify both the notation and the presentation, we focus on the model problem (1) in the case $d = 1$. In this case, we made the following choices in the Galerkin method.

- $W = W_n^{[p]}$, where $p \geq 1$, $n \geq 2$ and

$$W_n^{[p]} := \left\{ s \in C^{p-1}[0,1] : \; s_{|\left[\frac{i}{n}, \frac{i+1}{n}\right)} \in \mathbb{P}_p \;\; \forall i = 0, \dots, n-1, \; s(0) = s(1) = 0 \right\} \subset H_0^1(0,1)$$

  is the space of polynomial splines of degree $p$ defined over the unifor grid $\frac{i}{n}$, $i = 0, \dots, n$, and vanishing at $x = 0, 1$ ($\dim W_n^{[p]} = n + p - 2$).
- $\{\varphi_1, \dots, \varphi_{n+p-2}\}$ = basis formed by the polynomial B-splines vanishing at $0, 1$.

$A_n^{[p]}$ := *stiffness matrix for the Galerkin problem resulting from these choices*

① Construction of the matrix $A_n^{[p]}$ and computation of a proper symbol $f_p$.

② Taking the symbol in mind, analysis of the spectral properties of $A_n^{[p]}$ with particular attention to the following:

- (Asymptotic) spectral distribution in the Weyl sense of the sequence of matrices $\left\{ \frac{1}{n} A_n^{[p]} : n = 2, 3, 4, \ldots \right\}$, for fixed $p \geq 1$.
- Estimates for the extremal eigenvalues.
- Spectral conditioning $\kappa_2(A_n^{[p]})$.

③ Design of fast (iterative) solvers and constructive use of the symbol.

# Spectral distribution of a sequence of matrices $\{X_n\}$

We denote by $\mu_m$ the Lebesgue measure in $\mathbb{R}^m$.

## Definition: (asymptotic) spectral distribution of a sequence of matrices

Let $\{X_n\}$ be a sequence of matrices with increasing dimension ($X_n \in \mathbb{C}^{d_n \times d_n}$ with $d_n < d_{n+1} \quad \forall n$) and let $f : D \subset \mathbb{R}^m \to \mathbb{C}$ be a measurable function defined on the measurable set $D$ with $0 < \mu_m(D) < \infty$.
We say that $\{X_n\}$ is distributed like $f$ in the sense of the eigenvalues, and we write $\{X_n\} \overset{\lambda}{\sim} f$, if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\mu_m(D)} \int_D F(f(x_1, ..., x_m))dx_1...dx_m \qquad \forall F \in C_c(\mathbb{C}, \mathbb{C})$$

## Spectral distribution of $\left\{\frac{1}{n}A_n^{[p]}: \ n = 2, 3, 4, \dots\right\}$ in the sense of Weyl: Szegö type results (Garoni, Manni, Pelosi, S., Speleers, 2012, under revision for NUMER.MATH.)

$\forall p \geq 0$ denote by $\phi_{[p]} : \mathbb{R} \to \mathbb{R}$ the cardinal B-spline of degree $p$ over the uniform knot sequence $\{0, 1, \dots, p+1\}$:

$$\phi_{[p]}(x) = \begin{cases} \chi_{[0,1)}(x) & \text{se } p = 0 \\ \dfrac{x}{p}\phi_{[p-1]}(x) + \dfrac{p+1-x}{p}\phi_{[p-1]}(x-1) & \text{se } p \geq 1 \end{cases}$$

$\forall p \geq 1$ let $f_p : [-\pi, \pi] \to \mathbb{R}$,

$$f_p(\theta) = (2 - 2\cos\theta)\left(\phi_{[2p-1]}(p) + 2\sum_{k=1}^{p-1}\phi_{[2p-1]}(p-k)\cos(k\theta)\right)$$

### Theorem

$\forall p \geq 1$, $\left\{\dfrac{1}{n}A_n^{[p]}: \ n = 2, 3, 4, \dots\right\} \overset{\lambda}{\sim} f_p$ i.e.

$$\lim_{n\to\infty} \frac{1}{n+p-2}\sum_{j=1}^{n+p-2} F\left(\lambda_j\left(\frac{1}{n}A_n^{[p]}\right)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f_p(\theta))d\theta \qquad \forall F \in C_c(\mathbb{C}, \mathbb{C})$$

# Properties of the symbol $f_p$

$f_p$ is called the **symbol** of the sequence of matrices $\left\{\frac{1}{n}A_n^{[p]} : n = 2, 3, 4, \ldots\right\}$; $\forall p \geq 1$,

- $f_p(0) = 0$ and $\theta = 0$ is the only zero of $f_p$ over $[-\pi, \pi]$;
- $\lim_{\theta \to 0} \dfrac{f_p(\theta)}{\theta^2} = 1 \;\Rightarrow\; \theta = 0$ is a zero of order 2;
- $f_p(\theta) > 0 \quad \forall \theta \in [-\pi, \pi] \backslash \{0\}$.

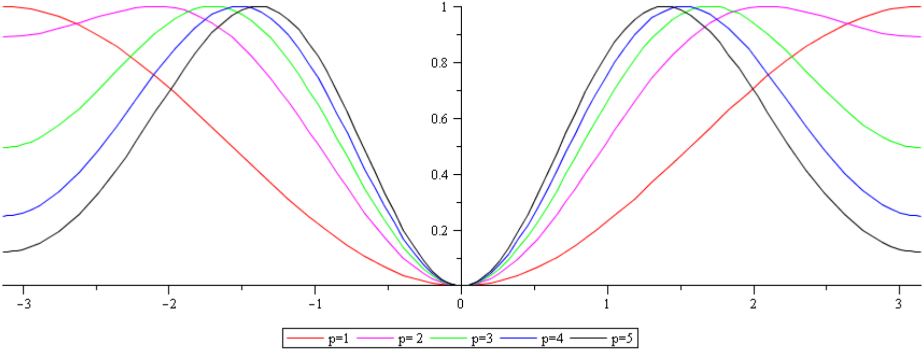


Figure: graph of the normalized symbol $f_p/M_{f_p}$ for $p = 1, 2, 3, 4, 5$, where $M_{f_p} = \max_{\theta \in [-\pi,\pi]} f_p(\theta)$

## Estimates for the extremal eigenvalues and for the spectral conditioning (Garoni, Manni, Pelosi, S., Speleers, 2012, under revision for NUMER. MATH.)

### Theorem

$\forall p \geq 1$ there exists a constant $C_p > 0$ such that

$$\left|\lambda_{\min}(A_n^{[p]})\right| \geq \lambda_{\min}(\mathrm{Re}\, A_n^{[p]}) \geq \frac{C_p(\pi^2+\gamma)}{n} \qquad \forall n \geq 2$$

where

- $\lambda_{\min}(A_n^{[p]})$ is an eigenvalue of $A_n^{[p]}$ with minimum modulus;
- $\mathrm{Re}\, A_n^{[p]} = \dfrac{A_n^{[p]} + {A_n^{[p]}}^T}{2}$;
- $\gamma$ is the parameter appearing in (1).

### Theorem

$\forall p \geq 1$ there exists a constant $\alpha_p$ such that

$$\kappa_2(A_n^{[p]}) \leq \alpha_p n^2 \qquad \forall n \geq 2$$

# Properties of the symbol $f_p$

When $p$ increases, the value $f_p(\pi)/M_{f_p}$ decreases and, apparently, converges exponentially to 0 as $p \to \infty$.

| $p$ | $\varepsilon_p$ |
|---|---|
| 1 | 1.0000 |
| 2 | 0.8889 |
| 3 | 0.4941 |
| 4 | 0.2494 |
| 5 | 0.1289 |
| 6 | 0.0570 |
| 7 | 0.0264 |
| 8 | 0.0120 |
| 9 | 0.0054 |
| 10 | 0.0024 |

Table: computation of $\varepsilon_p := f_p(\pi)/M_{f_p}$ for increasing values of $p$. Notice that, for $p = 3, ..., 10$, we have (roughly) $\varepsilon_p \approx \frac{1}{2} \cdot \varepsilon_{p-1}$.

When $p$ increases, the symbol shows some non-canonical behavior:

- Non-canonical behavior at $\theta = \pi$ and for large $p$ of $f_p$, when compared with the symbols occurring in the FD/FE approximating matrices.
- Ill-conditioning at the low frequencies ($\theta = 0$: canonical) and, for large $p$, at the high frequencies ($\theta = \pi$: non-canonical).

$$\Downarrow$$

Difficulty in the design of efficient multigrid methods.

- One purpose of the spectral analysis: design efficient preconditioners and multigrid methods for the fast solution of linear systems with coefficient matrix $A_n^{[p]}$ (or $\frac{1}{n}A_n^{[p]}$).
- There exists a 'canonical procedure' for creating, on the base of the symbol $f_p$, a two-grid $TG_n^{[p]}$ (for $\frac{1}{n}A_n^{[p]}\mathbf{u} = \mathbf{g}$) from which we expect an optimal convergence rate $\rho(TG_n^{[p]})$ (optimal = bounded by a constant $c_p < 1$ independent of the matrix size).
- Underlying idea: consider $\frac{1}{n}A_n^{[p]}$ as if it were $\tau_{n+p-2}(f_p)$ or $T_{n+p-2}(f_p)$ and design a two-grid method that has been already proved to have an optimal convergence rate both for sequences of $\tau$ and Toeplitz matrices (S., NUMER. MATH. 2002).

### Definition ($\tau$ and Toeplitz matrices associated with a symbol $f$)

Given $m \geq 1$ and a real even trigonometric polynomial $f(\theta) = \sum_{j=0}^{\ell} a_j \cos(j\theta)$,

$$\tau_m(f) = S_m \cdot \operatorname*{diag}_{j=1,\ldots,m}\left[f\left(\frac{j\pi}{m+1}\right)\right] \cdot S_m, \qquad S_m = \sqrt{\frac{2}{m+1}}\left[\sin\frac{ij\pi}{m+1}\right]_{i,j=1}^{m}$$

$$T_m(f) = [f_{j-k}]_{j,k=1}^{m}, \qquad f_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\mathrm{i}j\theta}d\theta, \qquad j \in \mathbb{Z}$$

For simplicity we assume $\beta = \gamma = 0$ and set $\frac{1}{n}A_n^{[p]} =: K_n^{[p]}$.
Fix $p \geq 1$ and an infinite set of indices $\mathcal{I}_p \subseteq \{n \geq 2 : \; n + p - 2 \geq 3 \text{ odd}\}$.
We want to solve $K_n^{[p]}\mathbf{u} = \mathbf{g}$, with $n \in \mathcal{I}_p$ and $\mathbf{g} \in \mathbb{R}^{n+p-2}$.

- Smoother: $S_n^{[p]} = I - \omega_p K_n^{[p]}$, with $\omega_p$ a relaxation parameter.
- $\theta = 0$ is the only zero of $f_p$ over $[-\pi, \pi]$ and $f_p(\theta) > 0, \; \forall\theta \in [-\pi, \pi]\backslash\{0\} \Rightarrow$ we choose a real even trigonometric polynomial $q_p$ vanishing at the 'mirror point' $\pi - 0 = \pi$ of $0$ ($=$ the zero of $f_p$) and satisfying

$$q_p^2(\theta) + q_p^2(\pi - \theta) > 0, \quad \forall\theta \in [0, \pi], \qquad \limsup_{\theta\to 0} \frac{q_p^2(\pi - \theta)}{f_p(\theta)} < +\infty.$$

  $\theta = 0$ is a zero of order 2 for $f_p \Rightarrow$ the simple choice $q_p(\theta) = 1 + \cos\theta$ works.

- Projector: $P_n^{[p]} = T_{n+p-2} \cdot \tau_{n+p-2}(1 + \cos\theta)$, where $T_{n+p-2}$ is the 'cutting matrix'

$$T_{n+p-2} = \begin{bmatrix} 0 & 1 & & & & & 0 \\ & & 0 & 1 & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{\frac{(n+p-2)-1}{2} \times (n+p-2)}.$$

  $P_n^{[p]}$ is full-rank: $\text{rank}(P_n^{[p]}) = \frac{(n+p-2)-1}{2}, \; \forall n \in \mathcal{I}_p$.

Given an approximation $\mathbf{u}_0 \in \mathbb{R}^{n+p-2}$ to the solution $\mathbf{u}$ of $K_n^{[p]}\mathbf{u} = \mathbf{g}$, we construct a new approximation $\mathbf{u}_1$ according to the following scheme.

$\mathbf{u}_0 \to \mathbf{u}_1$

1. Compute residual: $\mathbf{r} = \mathbf{f} - K_n^{[p]}\mathbf{u}_0$
2. Project residual: $\mathbf{r} = P_n^{[p]}\mathbf{r}$
3. Compute coarse-grid correction: $\mathbf{e} = \left(P_n^{[p]} K_n^{[p]} {P_n^{[p]}}^T\right)^{-1} \mathbf{r}$
4. Prolongate coarse-grid correction: $\mathbf{e} = {P_n^{[p]}}^T \mathbf{e}$
5. Correct initial approximation: $\mathbf{u}_1 = \mathbf{u}_0 + \mathbf{e}$
6. Relax one time: $\mathbf{u}_1 = S_n^{[p]}\mathbf{u}_1 + \omega_p \mathbf{f}$

We have

$$\mathbf{u}_1 = TG_n^{[p]}\mathbf{u}_0 + (I - TG_n^{[p]})\mathbf{u}$$

where

$$TG_n^{[p]} := S_n^{[p]} \cdot CGC_n^{[p]} \qquad CGC_n^{[p]} := \left(I - {P_n^{[p]}}^T \left(P_n^{[p]} K_n^{[p]} {P_n^{[p]}}^T\right)^{-1} P_n^{[p]} K_n^{[p]}\right)$$

# Numerical experiments and optimality

| $n$ | $\rho(TG_n^{[1]})$ | $\rho(TG_n^{[3]})$ | $n$ | $\rho(TG_n^{[2]})$ | $\rho(TG_n^{[4]})$ |
|---|---|---|---|---|---|
| 20 | 0.3333 | 0.4507 | 21 | 0.0264 | 0.7391 |
| 40 | 0.3333 | 0.4486 | 41 | 0.0263 | 0.7378 |
| 80 | 0.3333 | 0.4476 | 81 | 0.0263 | 0.7373 |
| 160 | 0.3333 | 0.4470 | 161 | 0.0263 | 0.7371 |
| 320 | 0.3333 | 0.4472 | 321 | 0.0263 | 0.7371 |
| 640 | 0.3333 | 0.4472 | 641 | 0.0263 | 0.7371 |
| 1280 | 0.3333 | 0.4472 | 1281 | 0.0263 | 0.7371 |
| 2560 | 0.3333 | 0.4472 | 2561 | 0.0263 | 0.7371 |

Table: computation of $\rho(TG_n^{[p]})$, for $p = 1, 2, 3, 4$ and increasing values of $n$, with $\omega_1 = 1/3$, $\omega_2 = 0.7303$, $\omega_3 = 1.0365$, $\omega_4 = 1.2228$

From the table, whatever $p$, $\rho(TG_n^{[p]})$ is bounded by a constant $c_p < 1$ independent of $n$ .

**Theorem (Garoni, S., 2013)**

Let $p \in \{1, 2, 3\}$ and $\omega_p \in \left(0, \frac{2}{\rho_p}\right)$, where $\rho_p = \sup_{n \in \mathcal{I}_p} \rho(K_n^{[p]})$. Then, there exists a constant $c_p < 1$, independent of $n$, such that $\rho(TG_n^{[p]}) \leq c_p$, $\forall n \in \mathcal{I}_p$.

From the previous table: $\rho(TG_n^{[p]})$ converges to a limit $\rho(TG_\infty^{[p]})$ that worsens when $p$ increases from 2 to 4.
From the properties of $f_p$ at slide 10 and from the literature: the worsening is expected to became more and more evident as $p \to \infty$.

$$\lambda_p(\theta) = \frac{(1+\cos\theta)^2 f_p(\theta) s_p(\pi-\theta) + (1+\cos(\pi-\theta))^2 f_p(\pi-\theta) s_p(\theta)}{(1+\cos\theta)^2 f_p(\theta) + (1+\cos(\pi-\theta))^2 f_p(\pi-\theta)}$$

Possible solutions:

1. change the (relaxed Richardson) smoother $S_n^{[p]}$ with the (relaxed Gauss-Seidel) smoother $\widetilde{S}_n^{[p]}$, thus obtaining a new two-grid $\widetilde{TG}_n^{[p]}$;
2. change $S_n^{[p]}$ with $\widetilde{S}_n^{[p]}$ as before and add a pre-smoothing iteration with the (conjugate) gradient method, thus obtaining a multi-iterative method;
3. different size-reduction strategy: Donatelli, S., Sesana, BIT 2012.

In the table below, the system $K_n^{[4]}\mathbf{u} = \mathbf{g}$ resulting from the IgA approximation of

$$\begin{cases} -u'' = 1, & \text{on } (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (4)$$

is solved for different values of $n$, using first $TG_n^{[4]}$ (with $\omega_4 = 1.2228$), then $\widetilde{TG}_n^{[4]}$ (with relaxation parameter 1.0624) and finally a multi-iterative method with a pre-smoothing step by the gradient and a post-smoothing step by the relaxed Gauss-Seidel method (with relaxation parameter 0.9800): S., CMA 1993.
The methods start with $\mathbf{u}_0 = \mathbf{0}$ and stop at the first term $\mathbf{u}_{c(n)}$ satisfying

$$\|\mathbf{g} - K_n^{[4]}\mathbf{u}_{c(n)}\|_\infty \leq 10^{-14}.$$

| $n$ | $c(n)$ [ $TG_n^{[4]}$ ] | $c(n)$ [ $\widetilde{TG}_n^{[4]}$ ] | $c(n)$ [ multi-iterative method ] |
|---|---|---|---|
| 21 | 88 | 22 | 15 |
| 41 | 85 | 22 | 16 |
| 81 | 83 | 22 | 16 |
| 161 | 81 | 21 | 16 |
| 321 | 79 | 21 | 15 |
| 641 | 76 | 20 | 15 |
| 1281 | 74 | 20 | 14 |
| 2561 | 72 | 19 | 14 |
| 5121 | 69 | 18 | 13 |

| $n$ | $c(n)$ $[p=1]$ | $n$ | $c(n)$ $[p=2]$ | $n$ | $c(n)$ $[p=3]$ | $n$ | $c(n)$ $[p=4]$ |
|---|---|---|---|---|---|---|---|
| 16 | 11 | 15 | 7 | 14 | 9 | 13 | 13 |
| 32 | 11 | 31 | 8 | 30 | 8 | 29 | 12 |
| 64 | 10 | 63 | 7 | 62 | 7 | 61 | 11 |
| 128 | 9 | 127 | 7 | 126 | 7 | 125 | 10 |
| 256 | 8 | 255 | 7 | 254 | 6 | 253 | 9 |
| 512 | 7 | 511 | 6 | 510 | 6 | 509 | 9 |
| 1024 | 6 | 1023 | 6 | 1022 | 6 | 1021 | 8 |
| 2048 | 5 | 2047 | 5 | 2046 | 5 | 2045 | 7 |

Table: V-cycle for solving the system $K_n^{[p]}\mathbf{u} = \mathbf{g}$ resulting from the IgA approximation of (4). The method starts with $\mathbf{u}_0 = \mathbf{0}$ and stops at the first term $\mathbf{u}_{c(n)}$ such that $\|\mathbf{g} - K_n^{[p]}\mathbf{u}_{c(n)}\|_\infty \leq 10^{-8}$.

| $n$ | $c(n)$ $[p=1]$ | $n$ | $c(n)$ $[p=2]$ | $n$ | $c(n)$ $[p=3]$ | $n$ | $c(n)$ $[p=4]$ |
|---|---|---|---|---|---|---|---|
| 16 | 11 | 15 | 7 | 14 | 8 | 13 | 13 |
| 32 | 11 | 31 | 7 | 30 | 8 | 29 | 11 |
| 64 | 10 | 63 | 6 | 62 | 7 | 61 | 11 |
| 128 | 8 | 127 | 6 | 126 | 7 | 125 | 10 |
| 256 | 7 | 255 | 5 | 254 | 6 | 253 | 9 |
| 512 | 6 | 511 | 5 | 510 | 6 | 509 | 8 |
| 1024 | 5 | 1023 | 5 | 1022 | 6 | 1021 | 8 |
| 2048 | 4 | 2047 | 4 | 2046 | 5 | 2045 | 7 |

Table: W-cycle for solving the system $K_n^{[p]}\mathbf{u} = \mathbf{g}$ resulting from the IgA approximation of (4). The method starts with $\mathbf{u}_0 = \mathbf{0}$ and stops at the first term $\mathbf{u}_{c(n)}$ such that $\|\mathbf{g} - K_n^{[p]}\mathbf{u}_{c(n)}\|_\infty \leq 10^{-8}$.

# 2D numerical experiments and optimality

In the following, we consider our model problem (1) in the case $d = 2$ and (for simplicity) we assume $\beta = (0, 0)$, $\gamma = 0$. The IgA approximation of this problem leads to the system $K^{[p_1,p_2]}_{n,\nu n}\mathbf{u} = \mathbf{f}$ ($p_1, p_2 \geq 1$, $\nu \in \mathbb{Q}$, $\nu > 0$, $n, \nu n \geq 2$ integers), for which a two-grid $TG^{[p_1,p_2]}_{n,\nu n}$ (the 2D counterpart of $TG^{[p]}_n$) exists.

| $n$ | $\rho(TG^{[1,1]}_{n,n})$ | $\rho(TG^{[3,3]}_{n,n})$ | $n$ | $\rho(TG^{[2,2]}_{n,n})$ | $\rho(TG^{[4,4]}_{n,n})$ |
|---|---|---|---|---|---|
| 12 | 0.3258 | 0.9259 | 13 | 0.6081 | 0.9889 |
| 20 | 0.3306 | 0.9242 | 21 | 0.6081 | 0.9883 |
| 28 | 0.3319 | 0.9239 | 29 | 0.6096 | 0.9881 |
| 36 | 0.3325 | 0.9233 | 37 | 0.6098 | 0.9880 |
| 44 | 0.3328 | 0.9231 | 45 | 0.6103 | 0.9880 |
| 52 | 0.3329 | 0.9230 | 53 | 0.6104 | 0.9880 |

Table: computation of $\rho(TG^{[p,p]}_{n,n})$, for $p = 1, 2, 3, 4$ and increasing values of $n$, with $\omega_{1,1} = 1/3$, $\omega_{2,2} = 1.1022$, $\omega_{3,3} = 1.3739$, $\omega_{4,4} = 1.3991$.

From the table, $\rho(TG^{[p,p]}_{n,n})$ is bounded by a constant $< 1$ independent of $n$ (indeed, an optimality theorem analogous to the one in slide 15 was proved for $1 \leq p_1, p_2 \leq 3$ and $\nu \in \mathbb{Q}$, $\nu > 0$). However $\rho(TG^{[p,p]}_{n,n}) \approx 1$ for $p = 3, 4$ (and every $n$); as in the 1D case, possible solutions to this 'bad convergence rate for large $p$' can be found by changing the smoothers and/or adopting a multi-iterative strategy.

In the table below, the system $K_{n,n}^{[4,4]}\mathbf{u} = \mathbf{g}$ resulting from the IgA approximation of

$$\begin{cases} -\Delta u = 1, & \text{on } (0,1)^2, \\ u = 0, & \text{on } \partial((0,1)^2), \end{cases} \tag{5}$$

is solved for different values of $n$, using first $TG_{n,n}^{[4,4]}$ (with relaxation parameter of the (Richardson) smoother equal to 1.3991), then $\widetilde{TG}_{n,n}^{[4,4]}$ (with relaxation parameter of the (Gauss-Seidel) smoother equal to 1.3000), and finally a multi-iterative method with a pre-smoothing step by the gradient and a post-smoothing step by the relaxed Richardson method (with relaxation parameter 0.6700). These three methods are the 2D counterparts of the three methods in the table at slide 17 (with a different post-smoother for the multi-iterative method). The methods start with $\mathbf{u}_0 = \mathbf{0}$ and stop at the first term $\mathbf{u}_{c(n)}$ satisfying $\|\mathbf{g} - K_{n,n}^{[4,4]}\mathbf{u}_{c(n)}\|_\infty \le 10^{-8}$.

| $n$ | $c(n)$ [ $TG_{n,n}^{[4,4]}$ ] | $c(n)$ [ $\widetilde{TG}_{n,n}^{[4,4]}$ ] | $c(n)$ [ multi-iterative method ] |
|---|---|---|---|
| 21 | 481 | 40 | 40 |
| 29 | 633 | 34 | 25 |
| 37 | 720 | 29 | 19 |
| 45 | 769 | 27 | 17 |
| 53 | 798 | 25 | 15 |
| 61 | 814 | 23 | 14 |
| 69 | 823 | 22 | 13 |
| 77 | 827 | 21 | 13 |

| $n$ | $c(n)$ $[p=1]$ | $n$ | $c(n)$ $[p=2]$ | $n$ | $c(n)$ $[p=3]$ | $n$ | $c(n)$ $[p=4]$ |
|---|---|---|---|---|---|---|---|
| 16 | 10 | 15 | 11 | 14 | 29 | 13 | 76 |
| 32 | 10 | 31 | 9 | 30 | 21 | 29 | 44 |
| 64 | 9 | 63 | 7 | 62 | 13 | 61 | 30 |
| 128 | 8 | 127 | 6 | 126 | 11 | 125 | 16 |
| 256 | 7 | 255 | 6 | 254 | 8 | 253 | 11 |
| 512 | 6 | 511 | 5 | 510 | 7 | 509 | 8 |

Table: V-cycle for solving the system $K_{n,n}^{[p,p]}\mathbf{u}=\mathbf{g}$ resulting from the IgA approximation of (5). The method starts with $\mathbf{u}_0=\mathbf{0}$ and stops at the first term $\mathbf{u}_{c(n)}$ such that $\|\mathbf{g}-K_{n,n}^{[p,p]}\mathbf{u}_{c(n)}\|_\infty \le 10^{-8}$.

| $n$ | $c(n)$ $[p=1]$ | $n$ | $c(n)$ $[p=2]$ | $n$ | $c(n)$ $[p=3]$ | $n$ | $c(n)$ $[p=4]$ |
|---|---|---|---|---|---|---|---|
| 16 | 9 | 15 | 11 | 14 | 29 | 13 | 78 |
| 32 | 8 | 31 | 9 | 30 | 21 | 29 | 45 |
| 64 | 7 | 63 | 7 | 62 | 13 | 61 | 30 |
| 128 | 6 | 127 | 6 | 126 | 10 | 125 | 16 |
| 256 | 6 | 255 | 5 | 254 | 7 | 253 | 11 |
| 512 | 5 | 511 | 5 | 510 | 7 | 509 | 9 |

Table: W-cycle for solving the system $K_{n,n}^{[p,p]}\mathbf{u}=\mathbf{g}$ resulting from the IgA approximation of (5). The method starts with $\mathbf{u}_0=\mathbf{0}$ and stops at the first term $\mathbf{u}_{c(n)}$ such that $\|\mathbf{g}-K_{n,n}^{[p,p]}\mathbf{u}_{c(n)}\|_\infty \le 10^{-8}$.

# Which symbol in general? A hint in [Beckermann, S., SINUM 2007]

$P_1$ FEM with vertices described by a geometric mapping $F$ for the problem

$$-\nabla(K\nabla^T u) + l.o.t. = g$$

on $\Omega$ L-shaped domain, $u$ given on $\partial\Omega$, $\Omega \subset \mathbb{R}^2$, $K$ piecewise continuous and pointwise symmetric positive definite. Then [see Beckermann, S., SINUM 2007]

$$\{A_n\} \sim_\lambda G(K(F(x)), J_F^{-1}(x), f(s)), \quad G(I, I, f(s)) = f(s), s \in (-\pi, \pi)^2, F(x) \in \Omega.$$

We expect that the formula is general i.e. valid for more general domains in $\mathbb{R}^d$, where richer spaces lead to a different trigonometric polynomials $f$ defined on the Fourier domain.
With $\circ =$ "Hadamard product" the elliptic operator can be written as

$$-\nabla(K(x)\nabla^T u(x)) + l.o.t. \equiv -e^T[K(x) \circ H_u(x)]e + l.o.t.$$

$H_u(x) =$ "Hessian of $u$", $e^T = (1, \ldots, 1)$. The resulting structure is

$$G(K(F(x)), J_F^{-1}(x), f(s)) = |\det(J_F(x))| \cdot e^T J_F^{-1}(x)[K(F(x)) \circ P(s)]J_F^{-T}(x)e$$

where $P(s)$ is the Finite Elements representation of the operator matrix $-H_u$ over the uniform triangulation $\mathcal{U}$, $f(s) = e^T P(s)e$, and $F$ is the transform s.t. $T = F(\mathcal{U})$.

# Conclusions

The symbol approach is useful for understanding the spectral features of the IgA matrices, depending on the various parameters, and for designing effective multigrid solvers... Here is list of issues to be studied:

- The role of different size-reduction strategies (Donatelli, S., Sesana, BIT 2012).
- Preconditioned Krylov methods to be studied with the help of the symbol (Kim, Parter, SINUM 1997).
- Mixed methods: multigrid as preconditioner and/or preconditioning as a smoother.
- Convection terms (i.e. losing the symmetry).
- Non-regular Geometry: the symbol? We have a reasonable hope using the notion of Generalized Locally Toeplitz (Tilli, LAA 1998, S., LAA 2003/06) plus Local Structures (Fried, Int.JSolidStruct 1973, Beckermann, S., SINUM 2007).
- An intriguing application: monument degradation by pollutants (Aregba, Diele, Natalini, SIAP 2004, Semplice, SISC 2010).

$$\Downarrow$$

A rich research program for the near future.