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A FREE BOUNDARY NUMERICAL METHOD FOR SOLVING AN OVERDETERMINED ELLIPTIC PROBLEM

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Main objective: footbridge

AUTOCAD representation

Real picture: prototype (Spain 2004)

Project: *Prototipo de pasarela peatonal con estructura de membrana portante: estudios previos, proyecto, construcción y pruebas in situ*

CONTENTS

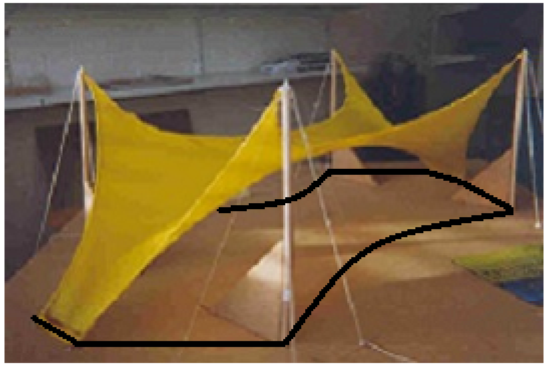
- General overview about membrane structures
- Mathematical formulation of the equilibrium problem
- Analysis of the free boundary problem
- A numerical example

Membranes Overview

- Surface with minimum thickness (no bending stiffness). Tensions
- Stress tensor. Tangent plane (tensions: positive tensor)
- Closed/Open ↔ boundaries and support external structure (positive or negative Gaussian curvature)
- Boundaries: rigid (bending stiffness) or cables (tension)
- Common applications: covers, airbags, kites,...
- Bearing applications ↔ to bear external loads (footbridges)

OPEN

CLOSED



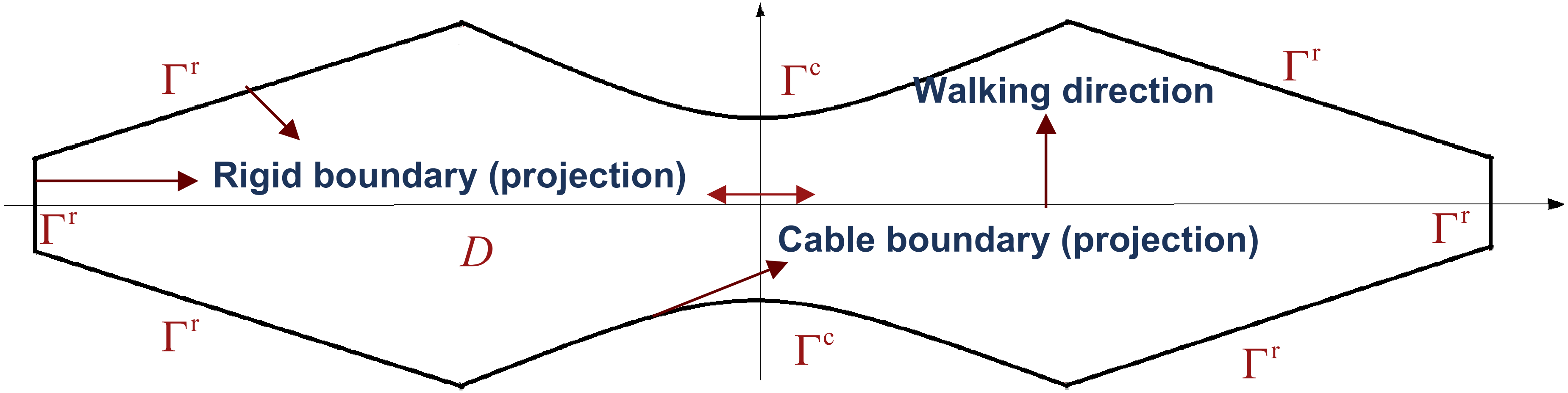
Projected domain and boundaries of a membrane for footbridge

$$\Gamma = \partial D \quad \Gamma = \Gamma^r \cup \Gamma^c$$

- **Cable: geometric restrictions ↔ convex shape (catenary)**

  Γ^c **depends on the stress tensor**

- **Rigid boundary: NO restriction.** Γ^r **not depending**



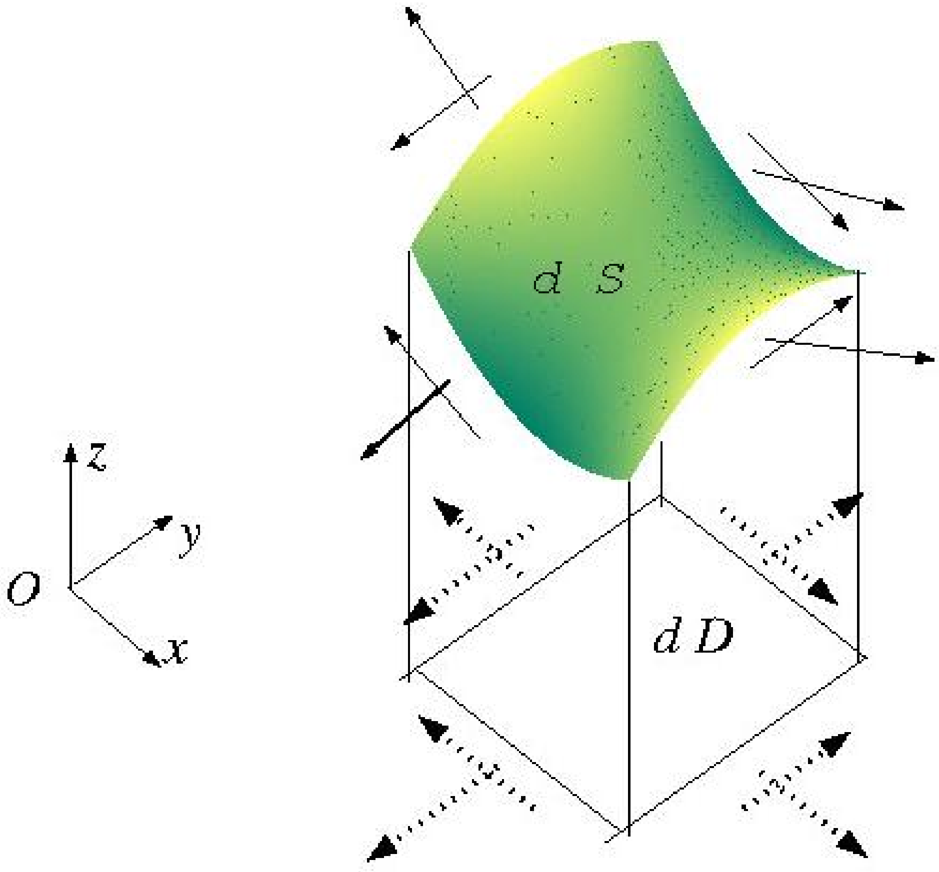
→ Natural stress tensor

⇢ Projected stress tensor

$$\underbrace{S \leftrightarrow z(x,y); z_{,xx} z_{,yy} - z_{,xy}^{2} < 0}$$

$$\Downarrow$$

Negative gaussian curvature surface



$$\sigma = N_{\alpha\beta}$$ ($\alpha,\beta = 1,2$ or x,y)

$$N_{\alpha\beta} = N_{\alpha\beta}(x,y)$$ ➡ **PROJECTED STRESS TENSOR**

Positive tensor

Problem formulation: EQUILIBRIUM EQUATIONS-MEMBRANE

Weight neglected + No external load (Prestressing phase)

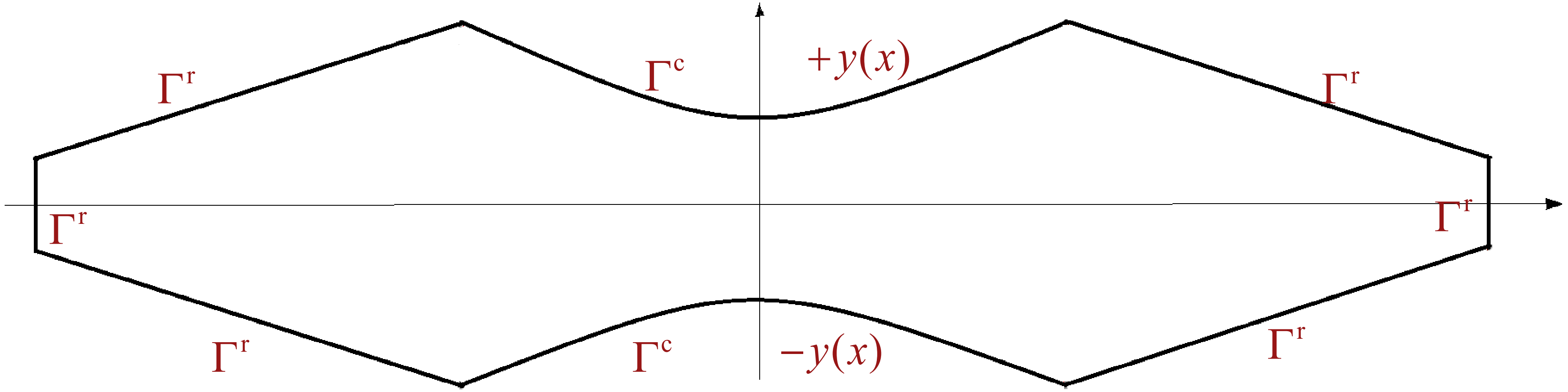
$$\left\{\begin{array}{ll} N_{xx,x} + N_{xy,y} = 0 & \text{direction } x \\ N_{xy,x} + N_{yy,y} = 0 & \text{direction } y \\ N_{xx} z_{,xx} + 2N_{xy} z_{,xy} + N_{yy} z_{,yy} = 0 & \text{direction } z \end{array}\right\} \operatorname{div}(\sigma \cdot \nabla z) = 0$$

SHAPE FINDING PROBLEM (Elliptic)

Stresses $N_{\alpha\beta} = N_{\alpha\beta}(x, y)$ Given

Surface $z = z(x, y)$ Unknown

Problem formulation: EQUILIBRIUM EQUATIONS-BOUNDARY



$z = g \quad \text{on} \quad \Gamma^r$ ⟷ **Dirichlet condition**

$$\boxed{z_{,xx} + 2y' z_{,xy} + y'^2 z_{,yy} = 0 \quad \text{on} \quad \Gamma^c}$$

Cable-membrane compatibility equation

↳ **NOT COMMON BOUNDARY CONDITION**

Osculator plane cable ⟷ **Tangent plane surface**

MATHEMATICAL FORMULATION

GIVEN

$\sigma = N_{\alpha\beta}$ **stress tensor such that** $\sum_{\beta=1}^{2} N_{\alpha\beta,\beta} = 0 \quad (\alpha,\beta=1,2)$

$y \leftrightarrow \Gamma^{c}$ **projection of the cable, depending on** σ

D **domain** **(** $\partial D = \Gamma = \Gamma^{r} \cup \Gamma^{c}$**)**

g **shape of** z **on** Γ^{r}

FIND

$z = z(x, y)$ **solving**

$$\begin{cases} \operatorname{div}(\sigma \cdot \nabla z) = 0 \quad \text{in} \quad D \\ z = g \quad \text{on} \quad \Gamma^{r} \\ z_{,xx} + 2y' z_{,xy} + y'^{2} z_{,yy} = 0 \quad \text{on} \quad \Gamma^{c} \end{cases}$$

GENERAL EQUILIBRIUM PROBLEM

Remark

$$\begin{cases} \mathrm{div}(\sigma \cdot \nabla z) = 0 \quad \text{in} \quad D & \textbf{(1)} \\ z = g \quad \text{on} \quad \Gamma^{\mathrm{r}} & \textbf{(2)} \\ z_{,xx} + 2y' z_{,xy} + y'^2 z_{,yy} = 0 \quad \text{on} \quad \Gamma^{\mathrm{c}} & \textbf{(3)} \end{cases}$$

$$+ \ \boxed{z = h \quad \text{on} \quad \Gamma^{\mathrm{c}}} \qquad \textbf{(4)}$$

(1),(2) and (4): Dirichlet Problem

↓

Unique solution z;

$\cancel{\downarrow}$

$z_{,xx} + 2y' z_{,xy} + y'^2 z_{,yy} \neq 0$ **(NOT (3)!!!)**

(1),(2), (3) and (4) ⟺ **overdetermined Problem**

$z(x, y(x)) =: h(x)$ **unknown** ⟺ **FREE BOUNDARY PROBLEM**

↕ (by differentiation)

$$z_{,xx} + 2y' z_{,xy} + y'^2 z_{,yy} = 0$$

↕

$$z_{,y} y'' = h''$$

(1),(2), (3) and (4) ⟺

$$\begin{cases} \mathrm{div}(\sigma \cdot \nabla z) = 0 \quad \text{in} \ D \\ z = g \quad \text{on} \ \Gamma^{\mathrm{r}} \\ z = h \quad \text{on} \ \Gamma^{\mathrm{c}} \\ z_{,y} y'' = h'' \quad \text{on} \ \Gamma^{\mathrm{c}} \end{cases}$$

FREE BOUNDARY STRATEGY

Problem A (z unknown)

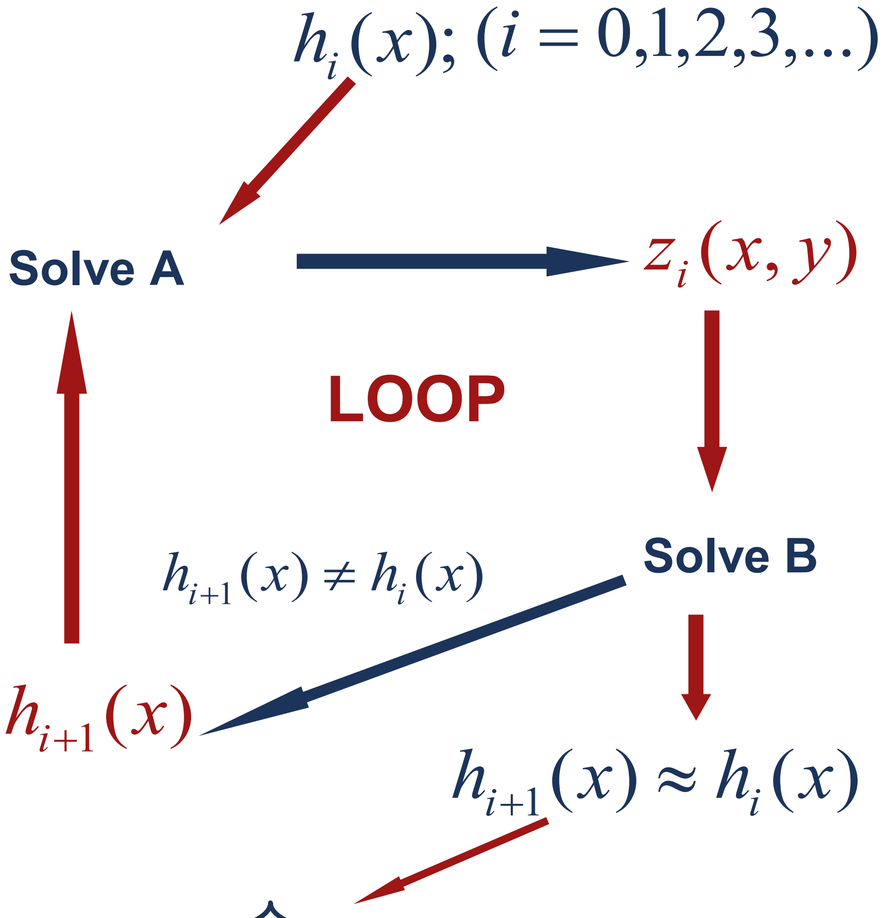
$$\begin{cases} \mathrm{div}(\sigma \cdot \nabla z) = 0 & \text{in } D \\ z = g & \text{on } \Gamma^{r} \\ z = h & \text{on } \Gamma^{c} \end{cases}$$

Problem B (h unknown)

$$\begin{cases} z_{,y} y'' = h'' \\ h(\mp a) = b & \text{(Boundary conditions)} \end{cases}$$

Generally

$$z(x, y(x)) \neq h(x)$$



$h_i(x); (i = 0,1,2,3,\ldots)$

Solve A

$z_i(x, y)$

LOOP

Solve B

$h_{i+1}(x) \neq h_i(x)$

$h_{i+1}(x)$

$h_{i+1}(x) \approx h_i(x)$

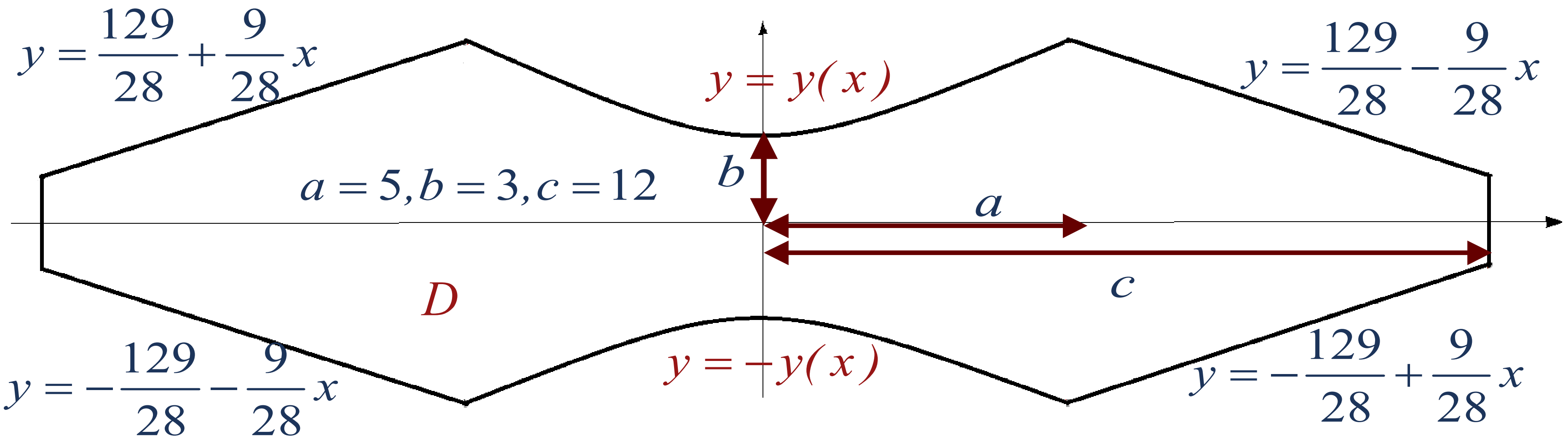
$$\overbrace{\textbf{Solution} \quad z_i(x, y); \quad h_i(x) = z_i(x, y(x))}$$

A NUMERICAL EXAMPLE: DOMAIN and STRESS TENSOR

$$N_{xx} = 10\ \text{kN/m},\ N_{xy} = 0\ \text{kN/m},\ N_{yy} = 4\ \text{kN/m}$$

Convex curve ← $y(x) = \frac{9}{2} - \sqrt{\frac{49}{4} - \frac{2}{5}x^2}$

DEFINING THE DOMAIN (lengths in metres)



A NUMERICAL EXAMPLE: BOUNDARY CONDITIONS (step 0)

$$\begin{cases} 10z_{,xx}+4z_{,yy}=0 \ \text{ in } D \\ z=0 \ \text{ on } \Gamma_1^{\mathrm{r}} \\ z=g=6-\dfrac{26}{43}|x|+\dfrac{72}{43}|y| \ \text{ on } \Gamma_2^{\mathrm{r}} \\ z=h_0=1+\dfrac{3}{34}(x^2+y^2) \ \text{ on } \Gamma^{\mathrm{c}} \end{cases}$$

h

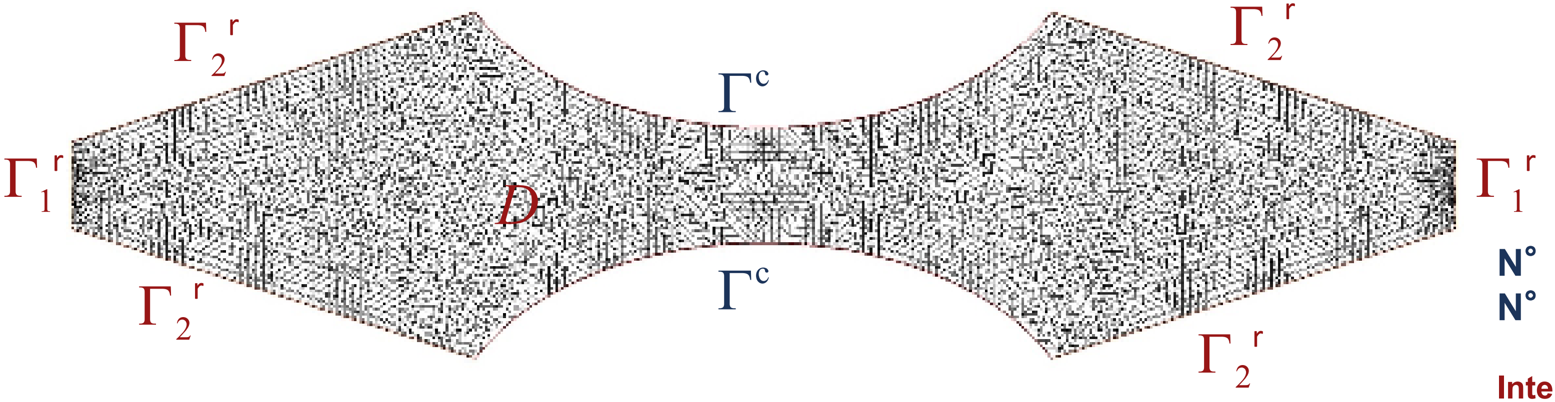
z

$$\begin{cases} z_{,y}y''=h'' \\ h(\mp a,\mp b)=g(\mp a,\mp b) \end{cases}$$

REMARK: h_0 **convex**

Finite Element Method (2D problem)

Finite Difference Method (1D problem)



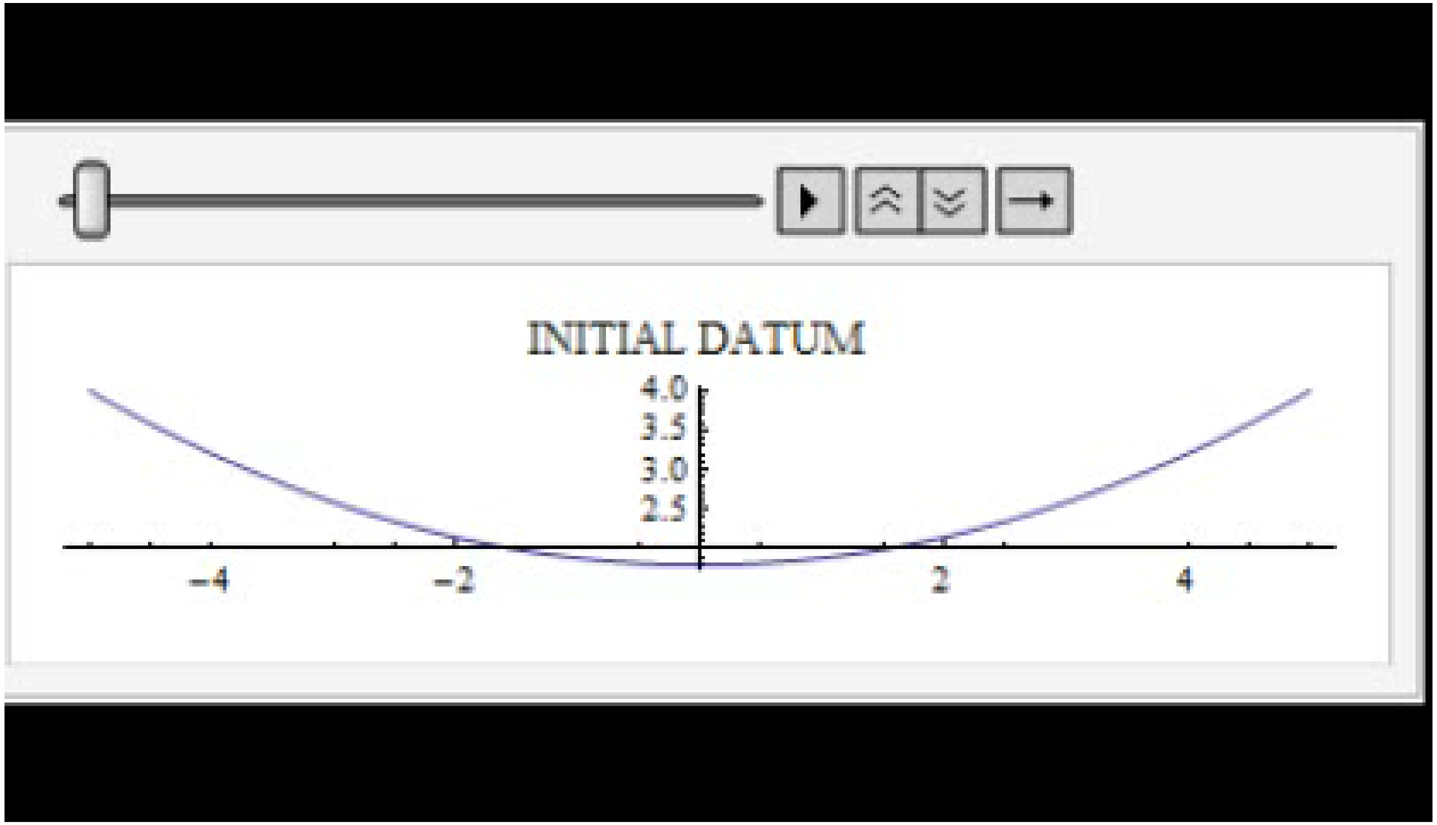
N° triangles: 8100
N° vertexes: 4251

Integration step size: 0.1

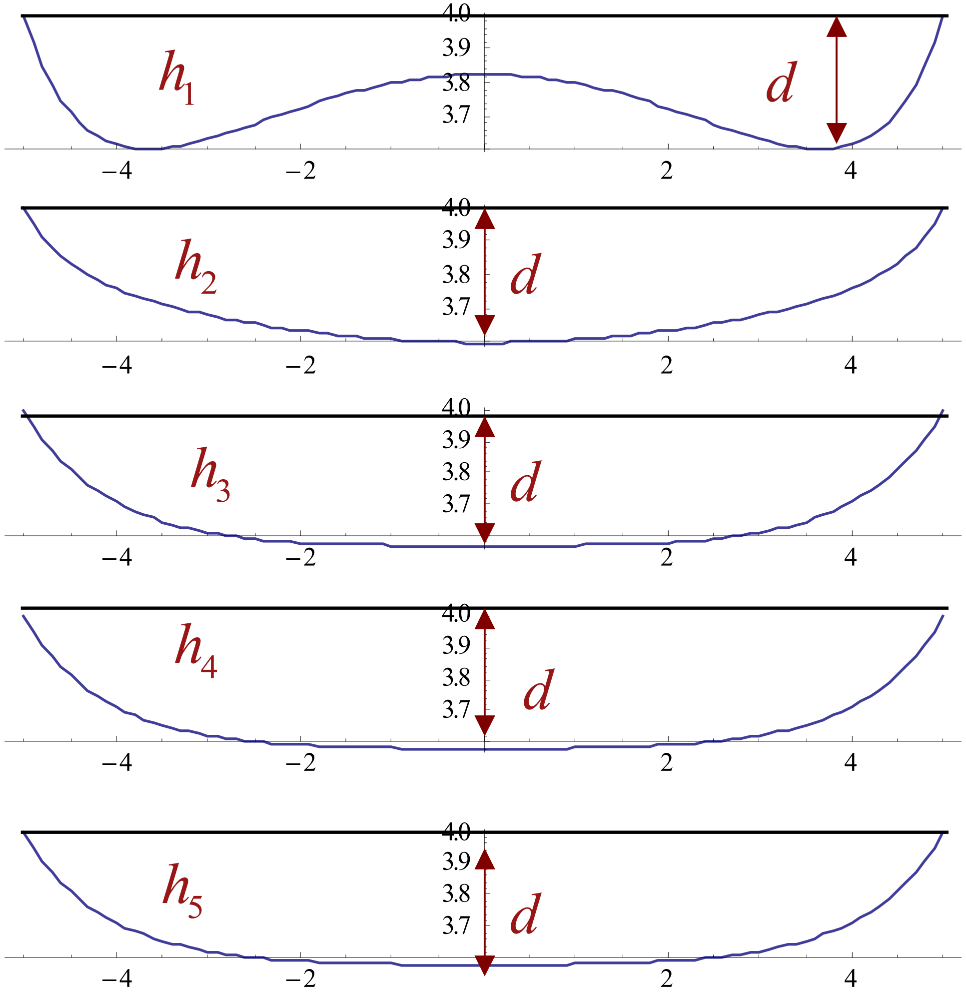
A NUMERICAL EXAMPLE: RESULTS-Cable

Evolution of h

Comparison h_i **and** h_{i+1}



A NUMERICAL EXAMPLE: DISPLACEMENT-Cable



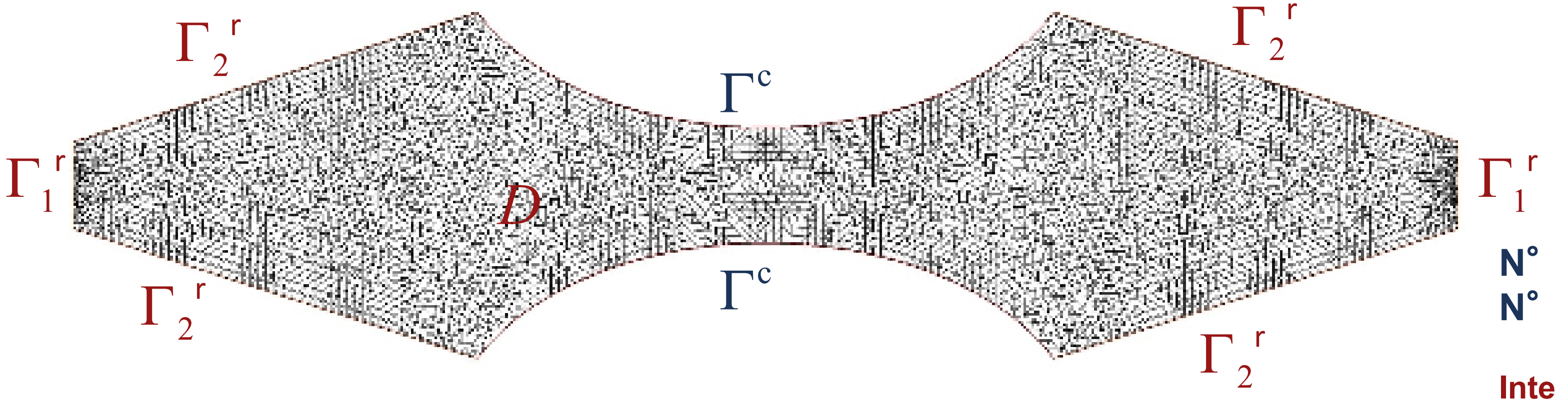
Iteration N°	Displacement d (m)
1	0.39844
2	0.401882
3	0.430081
4	0.420342
5	0.42058

A NUMERICAL EXAMPLE: BOUNDARY CONDITIONS (step 0)

$$\begin{tikzcd} \begin{cases} 10z_{,xx}+4z_{,yy}=0 \text{ in } D \\ z=0 \text{ on } \Gamma_1^r \\ z=g=6-\frac{26}{43}|x|+\frac{72}{43}|y| \text{ on } \Gamma_2^r \\ z=h_0=4 \text{ on } \Gamma^c \end{cases} \arrow[r, bend right, "z"'] & \begin{cases} z_{,y}y''=h'' \\ h(\mp a,\mp b)=g(\mp a,\mp b) \end{cases} \arrow[l, bend right, "h"'] \end{tikzcd}$$

REMARK: h_0 **straight**

Finite Element Method (2D problem)
Finite Difference Method (1D problem)



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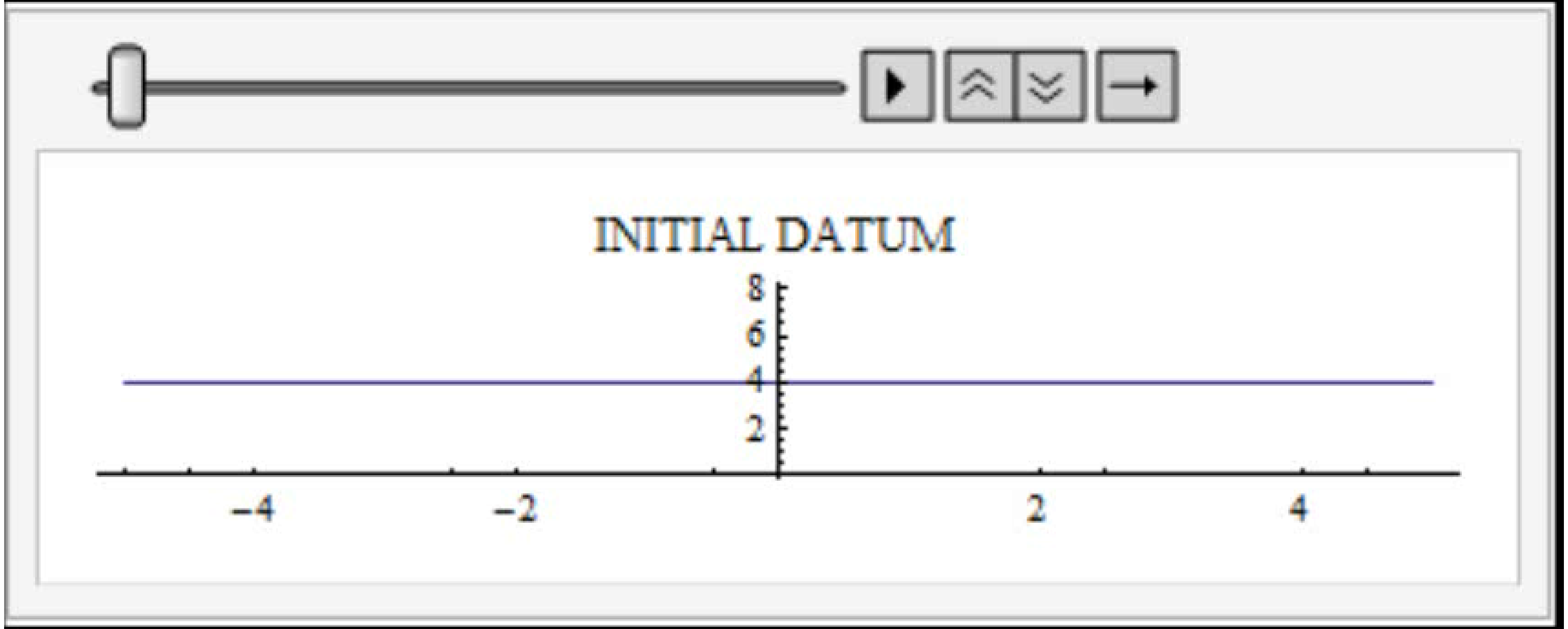
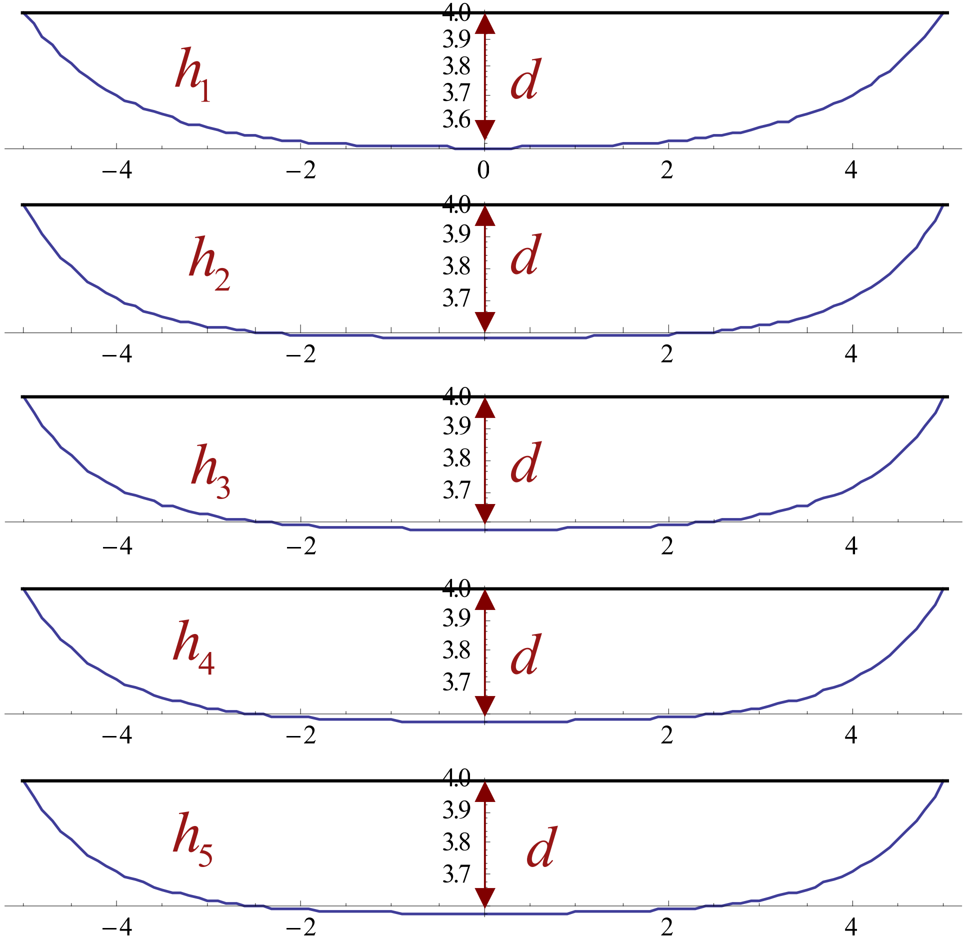
A NUMERICAL EXAMPLE: RESULTS-Cable

Evolution of h **Comparison h_i and h_{i+1}**



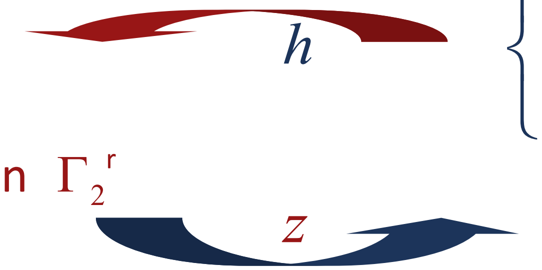
A NUMERICAL EXAMPLE: DISPLACEMENT-Cable



Iteration N°	Displacement d (m)
1	0.493791
2	0.418611
3	0.41909
4	0.42122
5	0.420892

A NUMERICAL EXAMPLE: BOUNDARY CONDITIONS (step 0)

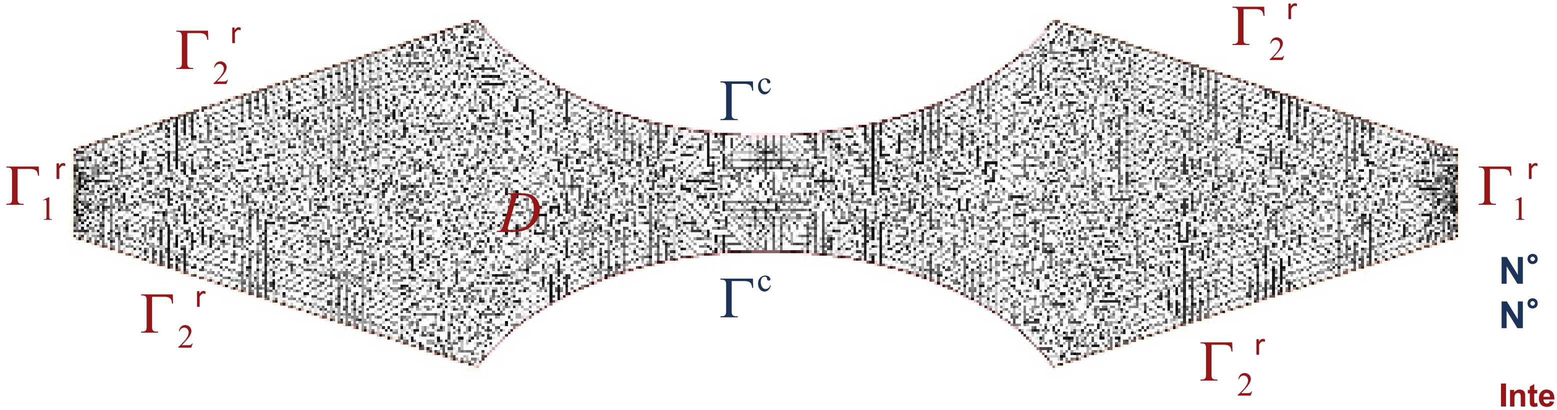
$$\begin{cases} 10z_{,xx}+4z_{,yy}=0 \text{ in } D \\ z=0 \text{ on } \Gamma_1^{r} \\ z=g=6-\frac{26}{43}|x|+\frac{72}{43}|y| \text{ on } \Gamma_2^{r} \\ z=h_0=7-\frac{3}{34}(x^2+y^2) \text{ on } \Gamma^{c} \end{cases}$$



$$\begin{cases} z_{,y}y''=h'' \\ h(\mp a,\mp b)=g(\mp a,\mp b) \end{cases}$$

REMARK: h_0 **concave**

Finite Element Method (2D problem)
Finite Difference Method (1D problem)

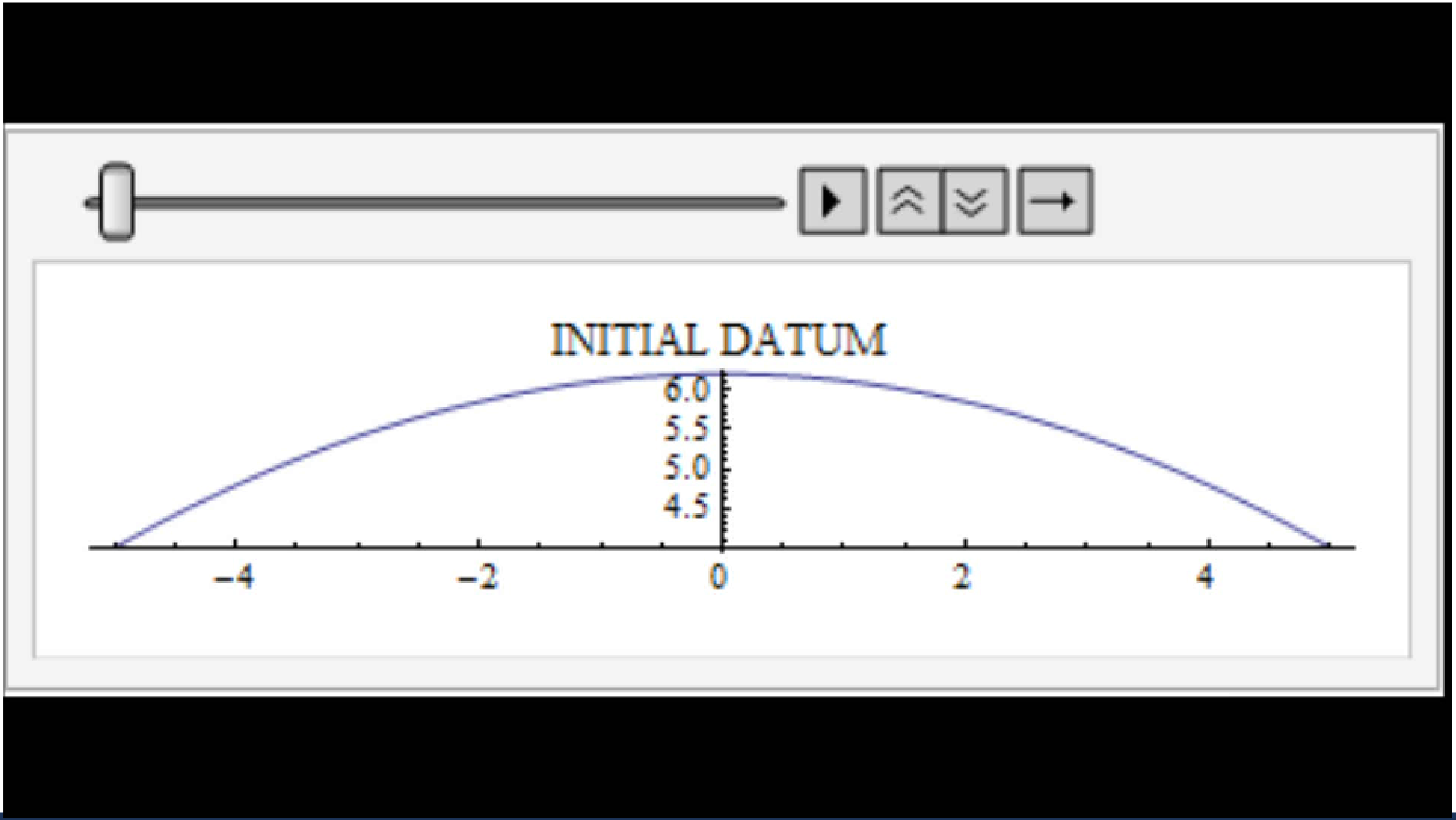


N° triangles: 8100
N° vertexes: 4251

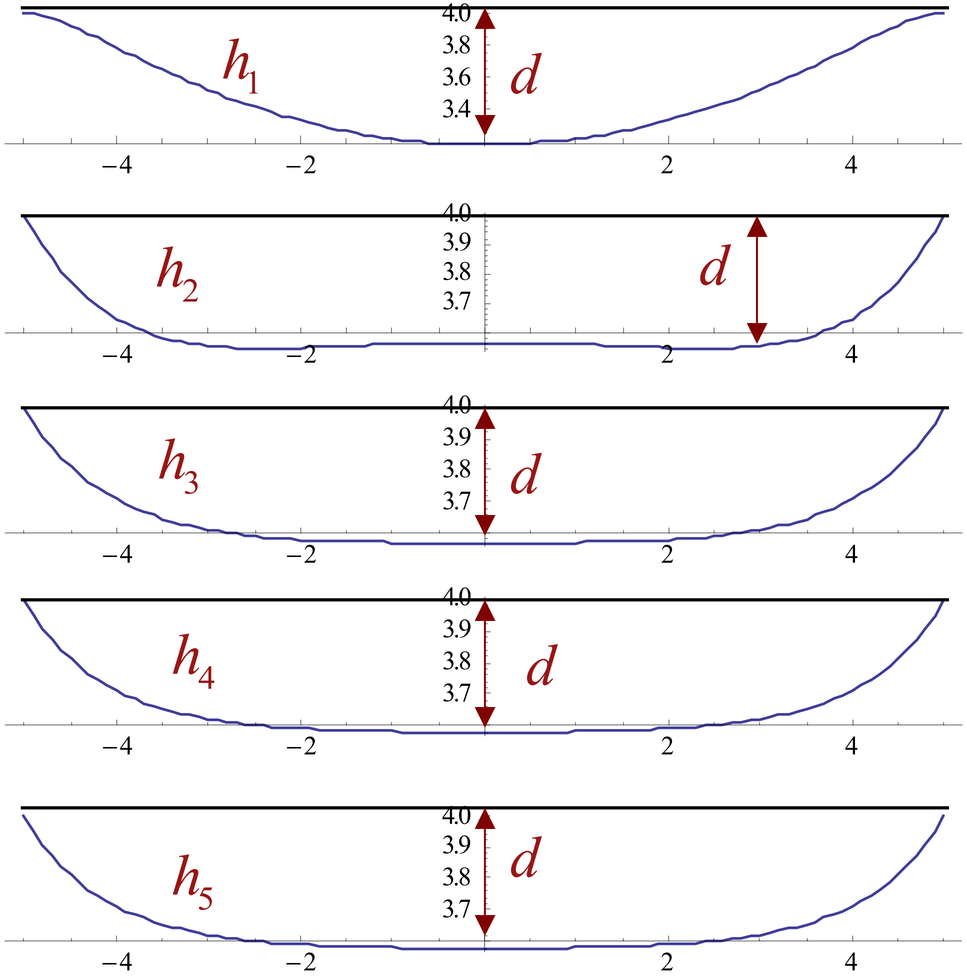
Integration step size: 0.1

A NUMERICAL EXAMPLE: RESULTS-Cable

Evolution of h **Comparison** h_i **and** h_{i+1}



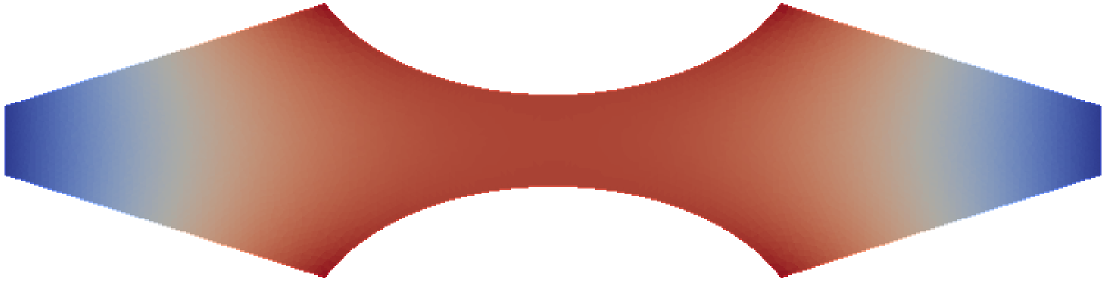
A NUMERICAL EXAMPLE: DISPLACEMENT-Cable

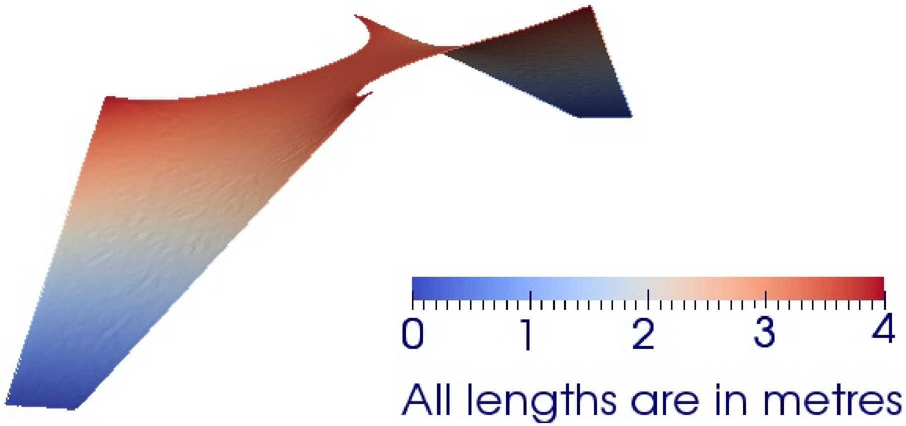


Iteration N°	Displacement d (m)
1	0.811832
2	0.455888
3	0.408099
4	0.422097
5	0.421205

... fast **and** unconditioned **method**

A NUMERICAL EXAMPLE: Final shape-Membrane





Qualitative final shape

Reference shape