The Inverse Scattering Transform for the Defocusing Nonlinear Schrödinger Equation with Non-zero Boundary Conditions

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Nonlinear Evolution Equations and Linear Algebra.
A Conference to celebrate the 60th birthday of Cornelis van der Mee
Cagliari, Italy
September 2-5, 2013

Introduction

We study the Inverse Scattering Transform (IST) for the defocusing nonlinear Schrödinger (NLS) equation

$$iq_t = q_{xx} - 2|q|^2 q$$

with non-zero boundary conditions (NZBCs)

$$q(x,t) \to q_\pm(t) = q_0 e^{2iq_0^2 t + i\theta_\pm} \qquad x \to \pm\infty$$

$q_0 > 0$ and $0 \le \theta_\pm < 2\pi$ are arbitrary constants.

Defocusing NLS is important in describing many nonlinear phenomena:

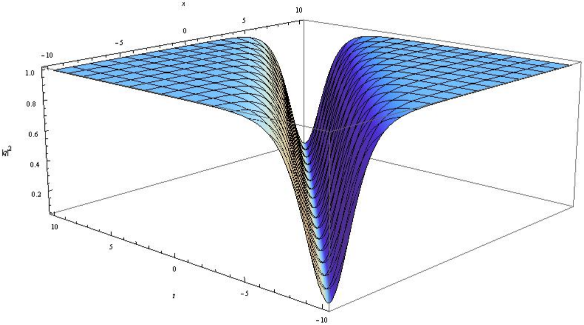
- surface waves in deep water
- plasma physics
- nonlinear fiber optics
- Bose-Einstein condensation

Interest in NLS as a prototypical integrable system: most dispersive energy preserving systems give rise, in appropriate limits, to the scalar NLS equation.

The class of nonvanishing potentials q as $|x| \to \infty$ for defocusing NLS includes soliton solutions with NZBCs, called dark/gray solitons

$$q(x,t) = q_0 e^{2iq_0^2 t} \left[\cos\alpha + i(\sin\alpha) \tanh\left[q_0(\sin\alpha)\left(x - 2q_0\, t\cos\alpha - x_0\right)\right]\right],$$

q_0, α and x_0 arbitrary real parameters. Dark soliton solutions appear as localized dips of intensity $q_0^2 \sin^2\alpha$ on the background field q_0:



The IST for defocusing NLS equation with NZBCs was studied by:

- 1973: Zakharov and Shabat
- 1977-1978: Kawata and Inoue
- 1978-1985: Gerdjikov and Kulish
- 1980-1984: Leon, Boiti and Pempinelli; Asano and Kato
- 1987: Faddeev and Takhtajan

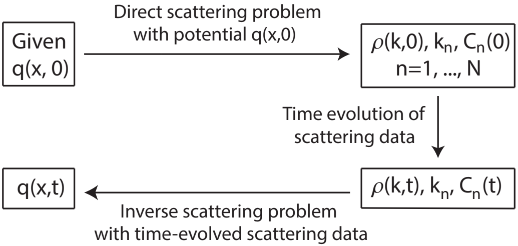
but many open issues remain to be addressed, such as:

- Identify the most suitable functional class of non-decaying potentials where the direct and inverse scattering problems are well-posed
- Rigorously establish analyticity properties of eigenfunctions and scattering data
- Investigate the well-posedness of the Riemann-Hilbert problem

We address these problems and indicate some improvements.

Inverse Scattering Transform

IST is a nonlinear version of the Fourier transform to solve the initial-value problem for certain nonlinear integrable PDEs.



- **Direct Problem**: The initial data $q(x, 0)$ are transformed into scattering data (reflection coefficient, discrete eigenvalues, and norming constants).
- **Time Evolution**: The time dependence of the scattering data is determined.
- **Inverse Problem**: The solution $q(x, t)$ is recovered from the evolved scattering data.

The scattering problem

Defocusing NLS can be associated to the following scattering problem

$$\frac{\partial X}{\partial x}(x,k) = (-ik\sigma_3 + Q(x))\,X(x,k)\,, \qquad x \in \mathbb{R} \tag{1}$$

(Ablowitz-Kaup-Newell-Segur scattering problem) where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} 0 & q(x) \\ q^*(x) & 0 \end{pmatrix},$$

$q(x)$ is the potential, $q_\pm$ are the NZBCs, $q(x) - q_\pm \in L^1(\mathbb{R}^\pm)$, k is the complex spectral parameter.

The scattering problem (1) can be written in the equivalent form:

$$\frac{\partial X}{\partial x}(x,k) = A(x,k)X(x,k) + (Q(x) - Q_f(x))X(x,k)\,,$$

where we have defined

$$A(x,k) = \theta(x)A_+(k) + \theta(-x)A_-(k)\,, \qquad Q_f(x) = \theta(x)Q_+ + \theta(-x)Q_-\,,$$

$$A_\pm(k) = -ik\sigma_3 + Q_\pm \equiv \begin{pmatrix} -ik & q_\pm \\ q_\pm^* & ik \end{pmatrix}, \qquad Q_\pm = \begin{pmatrix} 0 & q_\pm \\ q_\pm^* & 0 \end{pmatrix}.$$

Our contributions

We will indicate some steps forward with respect to the results in the existing literature. In particular:

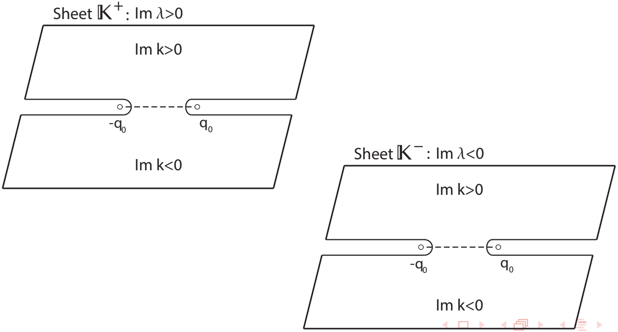
- We show that the direct problem is well defined when $q - q_\pm \in L^{1,2}(\mathbb{R}^\pm)$, i.e., $(1+|x|)^2[q(x) - q_\pm] \in L^1(\mathbb{R}^\pm)$
- We derive integral representations for the scattering coefficients
- We establish rigorously the analyticity properties of eigenfunctions and scattering data for potentials in this functional class
- We prove that, if $q - q_\pm \in L^{1,4}(\mathbb{R}^\pm)$, the discrete eigenvalues are *finite* in number and belong to the spectral gap $k \in (-q_0, q_0)$
- We formulate and solve the inverse problem as a Riemann-Hilbert problem and via Marchenko integral equations

Two-sheeted Riemann surface

Asymptotic eigenvalues and eigenvectors of the scattering problem depend on the spectral variable $\lambda = \sqrt{k^2 - q_0^2}$.
The variable k is then thought of as belonging to a Riemann surface $\mathbb{K}$ consisting of a sheet $\mathbb{K}^+$ and a sheet $\mathbb{K}^-$ which both coincide with the complex plane cut along the semilines

$$\Sigma = (-\infty, -q_0] \cup [q_0, \infty)$$

with edges glued such that $\lambda(k)$ is continuous through the cut:



Direct problem: Fundamental eigenfunctions

Consider the scattering problem

$$\frac{\partial X}{\partial x}(x,k) = A(x,k)X(x,k) + (Q(x) - Q_f(x))X(x,k)\,, \qquad (2)$$

$$A(x,k) = \theta(x)A_+(k) + \theta(-x)A_-(k)\,, \qquad Q_f(x) = \theta(x)Q_+ + \theta(-x)Q_-\,,$$
$$A_\pm(k) = -ik\sigma_3 + Q_\pm \equiv \begin{pmatrix} -ik & q_\pm \\ q_\pm^* & ik \end{pmatrix}, \qquad Q_\pm = \begin{pmatrix} 0 & q_\pm \\ q_\pm^* & 0 \end{pmatrix}.$$

We define, for $k \in \Sigma$, the *fundamental eigenfunctions* as solutions to (2) satisfying

$$\tilde{\Psi}(x,k) = e^{xA_+(k)}[I_2 + o(1)], \qquad x \to +\infty,$$
$$\tilde{\Phi}(x,k) = e^{xA_-(k)}[I_2 + o(1)], \qquad x \to -\infty,$$

I_2 being the 2×2 identity matrix.

Fundamental matrix

We define the *fundamental matrix* $\mathcal{G}(x,y;k)$ for the scattering problem with generator $A(x,k)=\theta(x)A_+(k)+\theta(-x)A_-(k)$ by:

$$\mathcal{G}(x,y;k)=\begin{cases} e^{(x-y)A_+(k)}, & x,y\geq 0,\\ e^{(x-y)A_-(k)}, & x,y\leq 0,\\ e^{xA_+(k)}e^{-yA_-(k)}, & x,-y\geq 0,\\ e^{xA_-(k)}e^{-yA_+(k)}, & x,-y\leq 0. \end{cases}$$

$\mathcal{G}(x,y;k)$ solves the scattering problem with potential $Q(x)=Q_f(x)$, i.e.

$$\frac{\partial}{\partial x}\mathcal{G}(x,y;k)=A(x,k)\mathcal{G}(x,y;k)\,,$$
$$\mathcal{G}(x,x;k)=I_2\,.$$

$\mathcal{G}(x,y;k)$ depends continuously on $(x,y,k)\in\mathbb{R}^2\times\Sigma$, and it satisfies:

$$\|\mathcal{G}(x,y;k)\|\leq\begin{cases} C, & k<-q_0 \text{ or } k>q_0,\\ C(1+|x|)(1+|y|), & k=\pm q_0, \end{cases}$$

where $C\geq 1$ is independent of $(x,y)\in\mathbb{R}^2$.

Theorem 1

If $q(x) - q_\pm \in L^1(\mathbb{R}^\pm)$, then the Volterra integral equations

$$\tilde\Psi(x,k) = \mathcal{G}(x,0;k) - \int_x^\infty dy\, \mathcal{G}(x,y;k)[Q(y) - Q_f(y)]\tilde\Psi(y,k)\,,$$

$$\tilde\Phi(x,k) = \mathcal{G}(x,0;k) + \int_{-\infty}^x dy\, \mathcal{G}(x,y;k)[Q(y) - Q_f(y)]\tilde\Phi(y,k)\,,$$

have the fundamental eigenfunctions $\tilde\Psi$, $\tilde\Phi$ as their unique solutions and they are continuous for any $k \in \Sigma \setminus \{\pm q_0\}$.
If $q(x) - q_\pm \in L^{1,2}(\mathbb{R}^\pm)$, the result also holds for $k = \pm q_0$.

Moreover, for $k \in \Sigma$:

$$\tilde\Psi(x,k) = \mathcal{G}(x,0;k)[\mathbb{A}_l(k) + o(1)], \qquad x \to -\infty,$$

$$\tilde\Phi(x,k) = \mathcal{G}(x,0;k)[\mathbb{A}_r(k) + o(1)], \qquad x \to +\infty,$$

with *transition coefficient matrices* $\mathbb{A}_l(k)$ and $\mathbb{A}_r(k)$ given by

$$\mathbb{A}_l(k) = I_2 - \int_{-\infty}^\infty dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\tilde\Psi(y,k),$$

$$\mathbb{A}_r(k) = I_2 + \int_{-\infty}^\infty dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\tilde\Phi(y,k).$$

Fundamental eigenfunctions

The fundamental eigenfunctions can also be derived as perturbations of $e^{xA_\pm(k)}$ as $x \to \pm\infty$, via the integral equations

$$\tilde{\Psi}(x,k) = e^{xA_+(k)} - \int_x^\infty dy\, e^{(x-y)A_+(k)}[Q(y) - Q_+]\tilde{\Psi}(y,k)\,,$$
$$\tilde{\Phi}(x,k) = e^{xA_-(k)} + \int_{-\infty}^x dy\, e^{(x-y)A_-(k)}[Q(y) - Q_-]\tilde{\Phi}(y,k)\,.$$

The integral equations for $\tilde{\Psi}$ coincide for $x \geq 0$, whereas the ones for $\tilde{\Phi}$ coincide for $x \leq 0$.
The latter, however, are not suitable for investigating the behavior of the eigenfunctions as $x \to \mp\infty$, since their iterates are continuous functions of $x \in \mathbb{R}$ which converge uniformly to $\tilde{\Psi}(x,k)$ (resp., $\tilde{\Phi}(x,k)$) for $x \geq x_0 > -\infty$ (resp., $x \leq x_0 < +\infty$), but nothing can be said about the limit as $x \to -\infty$ (resp. $x \to +\infty$).

Jost solutions

The *Jost solutions* from the right and the left, respectively, are defined as

$$\tilde{\Psi}(x,k)W_+(k) = \begin{pmatrix} \overline{\psi}(x,k) & \psi(x,k) \end{pmatrix} ,$$
$$\tilde{\Phi}(x,k)W_-(k) = \begin{pmatrix} \phi(x,k) & \overline{\phi}(x,k) \end{pmatrix} ,$$

where

$$W_\pm(k) = \begin{pmatrix} \lambda + k & \lambda - k \\ iq_\pm^* & -iq_\pm^* \end{pmatrix} , \quad A_\pm(k)W_\pm(k) = W_\pm(k)\mathrm{diag}(-i\lambda, i\lambda).$$

We then obtain for the Jost solutions the usual asymptotic behavior:

$$\overline{\psi}(x,k) \sim e^{-i\lambda x} \begin{pmatrix} \lambda + k \\ iq_+^* \end{pmatrix} , \qquad \psi(x,k) \sim e^{i\lambda x} \begin{pmatrix} \lambda - k \\ -iq_+^* \end{pmatrix} , \quad x \to +\infty,$$
$$\phi(x,k) \sim e^{-i\lambda x} \begin{pmatrix} \lambda + k \\ iq_-^* \end{pmatrix} , \qquad \overline{\phi}(x,k) \sim e^{i\lambda x} \begin{pmatrix} \lambda - k \\ -iq_-^* \end{pmatrix} , \quad x \to -\infty.$$

Since $\tilde{\Psi}(x,k)$ and $\tilde{\Phi}(x,k)$ are square matrix solutions of the scattering problem (a homogeneous first order system), we have

$$\tilde{\Phi}(x,k) = \tilde{\Psi}(x,k)\mathbb{A}_r(k), \qquad \tilde{\Psi}(x,k) = \tilde{\Phi}(x,k)\mathbb{A}_l(k),$$

where $\mathbb{A}_l(k)$ and $\mathbb{A}_r(k)$ are the transition coefficient matrices. Then

$$\begin{pmatrix}\phi(x,k) & \overline{\phi}(x,k)\end{pmatrix} = \begin{pmatrix}\overline{\psi}(x,k) & \psi(x,k)\end{pmatrix} S(k)\,,$$

where

$$S(k) = W_+^{-1}(k)\mathbb{A}_r(k)W_-(k) = \begin{pmatrix} a(k) & \overline{b}(k) \\ b(k) & \overline{a}(k)\end{pmatrix}.$$

Using the integral equations for the fundamental eigenfunctions, we get integral representations for the scattering coefficients:

$$\begin{pmatrix} a(k) & \overline{b}(k) \\ b(k) & \overline{a}(k)\end{pmatrix} = \int_0^\infty dy\, e^{i\lambda y\sigma_3} W_+^{-1}(k)[Q(y)-Q_+]\begin{pmatrix}\phi(y,k) & \overline{\phi}(y,k)\end{pmatrix}$$
$$+ W_+^{-1}(k)W_-(k)\left[I_2 + \int_{-\infty}^0 dy\, e^{i\lambda y\sigma_3} W_-^{-1}(k)[Q(y)-Q_-]\begin{pmatrix}\phi(y,k) & \overline{\phi}(y,k)\end{pmatrix}\right].$$

Analiticity properties

Theorem 2

If $q(x) - q_{\pm} \in L^{1,2}(\mathbb{R}^{\pm})$, then

- *the Jost solutions $e^{-i\lambda x}\psi(x,k)$ and $e^{i\lambda x}\phi(x,k)$ are continuous for $k \in \overline{\mathbb{K}^{+}}$ and analytic for $k \in \mathbb{K}^{+}$;*
- *$e^{i\lambda x}\overline{\psi}(x,k)$ and $e^{-i\lambda x}\overline{\phi}(x,k)$ are continuous for $k \in \overline{\mathbb{K}^{-}}$ and analytic for $k \in \mathbb{K}^{-}$.*
- *The scattering coefficient $a(k)$ (resp $\overline{a}(k)$) is continuous in $k \in \overline{\mathbb{K}^{+}} \setminus \{\pm q_0\}$ (resp $k \in \overline{\mathbb{K}^{-}} \setminus \{\pm q_0\}$) and analytic in $k \in \mathbb{K}^{+}$ (resp $k \in \mathbb{K}^{-}$).*
- *The functions $b(k)$, $\overline{b}(k)$ are continuous in $k \in \Sigma \setminus \{\pm q_0\}$, but in general cannot be continued off Σ.*

Uniformization variable

Following (FT) we introduce a uniformization variable z defined by:

$$z = k + \lambda(k)\,.$$

- The two sheets $\mathbb{K}^+$, $\mathbb{K}^-$ of the R. surface are mapped respectively onto the upper and lower half-planes of the complex z-plane
- The cut Σ on the Riemann surface is mapped onto the real z axis
- The segments $-q_0 \le k \le q_0$ on $\mathbb{K}^+$ and $\mathbb{K}^-$ are mapped onto the upper and lower semicircles of radius q_0 and center at the origin of the z-plane.

Taking into account the symmetries in the eigenfunctions and scattering coefficients, the scattering data consist of:

- **Reflection coefficient** $\rho(z) = b(z)/a(z)$
- **Discrete eigenvalues** [zeros of $a(z)$] $\zeta_n = k_n + i\nu_n$, with $|k_n| < q_0$ and $\nu_n = \sqrt{q_0^2 - k_n^2}$. It is known that they are *simple* and belong to the spectral gap $k \in (-q_0, q_0)$. If $q - q_\pm \in L^{1,4}(\mathbb{R}^\pm)$, we proved that the discrete eigenvalues are *finite* in number.
- **Norming constants** C_n associated with the discrete eigenvalues ζ_n.

Inverse problem: Riemann-Hilbert problem

We formulate the inverse problem as a matrix Riemann-Hilbert problem on the real z-axis, with poles at the zeros of $a(z)$ in the upper half-plane of z and of $\overline{a}(z)$ in the lower half-plane:

$$\frac{\phi(x,z)}{a(z)}e^{i\lambda x} - \overline{\psi}(x,z)e^{i\lambda x} = \rho(z)e^{2i\lambda x}\psi(x,z)e^{-i\lambda x},$$
$$\frac{\overline{\phi}(x,z)}{\overline{a}(z)}e^{-i\lambda x} - \psi(x,z)e^{-i\lambda x} = \overline{\rho}(z)e^{-2i\lambda x}\overline{\psi}(x,z)e^{i\lambda x},$$

where $\rho(z)$ and $\overline{\rho}(z)$ are the reflection coefficients.

- We solve the Riemann-Hilbert problem by reducing it to a linear system of algebraic-integral equations.
- We study the asymptotic behavior of $\rho(z)$ and $\overline{\rho}(z)$ $(z \in \mathbb{R})$ as $z \to \infty$ and as $z \to 0$. It ensures that the algebraic-integral system of equations providing the solution of the inverse problem is well defined.

Inverse problem: Marchenko integral equations

We can also formulate the inverse problem in terms of the following Marchenko integral equations:

$$\boldsymbol{K}(x,y)+\mathbb{G}(x+y)+\int_x^\infty ds\,\boldsymbol{K}(x,s)\mathbb{G}(s+y)=0$$

where

$$\boldsymbol{K}(x,y)=\begin{pmatrix} K_{11}(x,y) & K_{12}(x,y)\\ K_{21}(x,y) & K_{22}(x,y)\end{pmatrix},\quad \mathbb{G}(s+y)\;=\begin{pmatrix} F_1(s+y) & F_2^*(s+y)\\ F_2(s+y) & F_1^*(s+y)\end{pmatrix},$$

$$F_1(x)=F_{1,c}(x)+iF'_{2,c}(x)-\frac{\zeta_n^*}{2}F_{1,d}(x),\quad F_2(x)=-iq_+^*\left[F_{2,c}(x)+\frac{1}{2}F_{1,d}(x)\right],$$

$$F_{1,c}(x)=\frac{1}{2\pi}\int_{-\infty}^\infty d\zeta\, e^{i\zeta x}\,\frac{\rho(\sqrt{\zeta^2+q_0^2},\zeta)+\rho(-\sqrt{\zeta^2+q_0^2},\zeta)}{2},$$

$$F_{2,c}(x)=\frac{1}{2\pi}\int_{-\infty}^\infty d\zeta\, e^{i\zeta x}\,\frac{\rho(\sqrt{\zeta^2+q_0^2},\zeta)-\rho(-\sqrt{\zeta^2+q_0^2},\zeta)}{2\sqrt{\zeta^2+q_0^2}},$$

$$F_{1,d}(x)=-i\sum_{n=1}^N C_n\, e^{-\nu_n x}.$$

Triplet method

We have developed the **triplet method** as a tool to obtain explicit multisoliton solutions by solving the Marchenko integral equations via separation of variables.

In the *reflectionless case* ($\rho(z) \equiv 0$ for all $z \in \mathbb{R}$), we represent Marchenko kernel $\mathbb{G}$ as:

$$\mathbb{G}(z) = \boldsymbol{C} e^{-z\boldsymbol{A}} \boldsymbol{B} \,,$$

where

- $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is a minimal triplet
  The triplet yielding a minimal realization is unique up to a similarity transformation $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \to (\mathbf{SAS}^{-1}, \mathbf{SB}, \mathbf{CS}^{-1})$ for some unique invertible matrix $\mathbf{S}$
- $\boldsymbol{A}$ is a $p \times p$ matrix having only eigenvalues with positive real parts
- $\boldsymbol{B}$ is a $p \times 2$ matrix, $\boldsymbol{C}$ is a $2 \times p$ matrix

Taking into account the time evolution of the scattering data and

- $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}^{(1)} \\ \boldsymbol{C}^{(2)} \end{pmatrix}$ with $\boldsymbol{C}^{(1)}$ and $\boldsymbol{C}^{(2)}$ rows of length p
- $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}^{(1)} & \boldsymbol{B}^{(2)} \end{pmatrix}$ with $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ columns of length p
- $\mathbf{P}$ the unique solution of the Sylvester equation $\boldsymbol{A}\mathbf{P} + \mathbf{P}\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}$,

we recover the solution of defocusing NLS as

$$q(x,t) = q_+(t) + 2\boldsymbol{C}^{(1)}(t)[\mathbf{P}(t) + e^{2x\boldsymbol{A}}]^{-1}\boldsymbol{B}^{(2)}(t).$$

In order to have solutions of defocusing NLS with NZBCs, we have to assume

- the minimality of the triplet $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$
- the positivity of the real parts of the eigenvalues of the matrix $\boldsymbol{A}$
- the invertibility of the matrices $\mathbf{P} + e^{2x\boldsymbol{A}}$ and $\mathbf{P}$

Theorem 3

If $\mathbf{P}$ is an invertible matrix, then $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is a minimal triplet.

Unlike what happens with other NLEEs for which the triplet method has been applied, here the converse to Theorem 3 is not generally true.

Summary and overview

A rigorous theory of the IST for the defocusing NLS equation with (symmetric) NZBCs $q_\pm \equiv q_0 e^{i\theta_\pm}$ as $x \to \pm\infty$ has been presented.

- The direct problem is shown to be well-posed for potentials q such that $q - q_\pm \in L^{1,2}(\mathbb{R}^\pm)$, for which analyticity properties of eigenfunctions and scattering data are established.
- The inverse problem is formulated and solved both as a Riemann-Hilbert problem and via Marchenko integral equations in terms of a suitable uniform variable.
- The triplet method is developed as a tool to obtain explicit multisoliton solutions by solving the Marchenko integral equations.

We plan to extend the investigation to defocusing NLS with fully asymmetric NZBCs (different amplitudes as $x \to \pm\infty$) in order to study the long-time asymptotic behavior for the solutions of defocusing NLS.

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Thank you very much for your attention!

Happy birthday Prof. van der Mee!!!