# Implementing Łukasiewicz operations in quantum computational with mixed states

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# Infinite-valued Łukasiewicz logic

- Since the Eighties, the interest in many-valued logics has increased enormously. In particular, the logics with truth values in $[0, 1]$ emerged as a consequence of the 1965 proposal of fuzzy set theory by Zadeh called *Fuzzy logic*.

- There exists a deep connection between the infinite-valued Łukasiewicz logic, Ulam Games and its application to classical communication with feedback.

- It suggest a potential implementations of Łukasiewicz logic to the quantum error correction theory.

There are two basic operations on $[0,1]$ defining the Łukasiewicz logic

1. $\neg x = 1 - x,$ [Negation]
2. $x \oplus y = \min\{x + y, 1\}$ [Łukasiewicz truncated sum]

- In the framework of quantum computation with mixed states we introduce a probabilistic type representation for the Łukasiewicz truncated sum.
- More precisely, it will be represented as a quantum operation built from a polynomial that approximates $x \oplus y = \min\{x + y, 1\}$

# 1 Basic notions in quantum computation

Idea and concept of quantum computing was introduced way back in 1970s and 1980s, by Richard Feynmann, David Deutsch and Paul Benioff.

- In a classical computer, information is encoded in a series of bits and these bits are manipulated via Boolean logic gates like *NOT*, *OR*, *AND*.
- Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, specially $\mathbb{C}^2$, the two-dimensional space of a *qbit*.
- Similarly to the classical computing case, we can introduce and study the behavior of a number of *quantum logical gates* (hereafter quantum gates for short) operating on qbits.

Quantum computing can simulate all computations which can be done by classical systems; however, one of the main advantages of quantum computation and quantum algorithms is that they can speed up computations

- The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$ where $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ is called the *logical basis*.

- The two basis-elements $|0\rangle$ and $|1\rangle$ are usually taken as encoding the classical bit-values 0 and 1, respectively.

- Thus, qbits $|\varphi\rangle$ in $\mathbb{C}^2$ are superpositions of the basis vectors with complex coefficients

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \quad \text{with} \quad |c_0|^2 + |c_1|^2 = 1$$

Recalling the Born rule, any qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ may be regarded as a piece of information, where

- $|c_0|^2$ corresponds to the probability-value of the information described by the basic state $|0\rangle$;

- $|c_1|^2$ corresponds to the probability-value of the information described by the basic state $|1\rangle$.

**Definition 1.1** Let $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ be a qbit. Then its *probability value* is

$$p(|\psi\rangle) = |c_1|^2$$

The quantum states of interest in quantum computation lie in the tensor product

$$\otimes^n \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \quad n - times$$

A special basis, called the $2^n$-*computational basis*, is chosen for $\otimes^n \mathbb{C}^2$.

- it consists of the $2^n$ orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$ where $\iota$ is in binary representation and $|\iota\rangle$ can be seen as tensor product of states (Kronecker product) $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$ where $\iota_j \in \{0, 1\}$.
- A $n$-qbit $|\psi\rangle \in \otimes^n \mathbb{C}^2$ is a superposition

$$|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota |\iota\rangle \;\; with \;\; \sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of **quantum gates**, i.e. **unitary operators** acting on $n$-qbits of $\otimes^n \mathbb{C}^2$

$$\mathcal{L}(\otimes^n \mathbb{C}^2) \ni U \quad s.t. \quad U \cdot U^\dagger = U^\dagger \cdot U = I$$

The standard model for quantum computation is given by

$$\{\text{qbits}, \quad \text{unitary operators}\}$$

## 2 Quantum computation with mixed states

- In general, a quantum system is not in a pure state. This may be caused, for example, by the non complete efficiency in the preparation procedure or by the fact that systems cannot be completely isolated from the environment, undergoing decoherence of their states.
- There are interesting processes that cannot be encoded in unitary evolutions, for example, at the end of the computation a non-unitary operation, a measurement, is applied, and the state becomes a probability distribution over pure states, or what is called a *mixed state*.

In view of these facts, several authors have paid attention to a more general model of quantum computational processes, qbits states are replaced by mixed states. This model is known as *quantum computation with mixed states*.

[1 ] D. Aharanov, A. Kitaev and N. Nisan: *Quantum circuits with mixed states*, Proc. 13th Annual ACM Symp. on Theory of Computation, 20-30 (STOC,1997).

[2 ] V. Tarasov: *Quantum computer with Mixed States and Four-Valued Logic*, J. Phys. A **35**, 5207-5235 (2002).

Let $H$ be a complex Hilbert space.

- In the model of quantum computation with mixed states, we regard a quantum state in a Hilbert space $H$ as a *density operator* $\rho$ i.e.

    1. $\rho$ is Hermitian ($\rho = \rho^\dagger$),
    2. $0 \leq \langle \rho(v), v \rangle$ for each $v \in H$,
    3. $tr(\rho) = 1$

- Quantum gates are represented by *quantum operations*. They are linear maps $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ s.t.

    1. $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$ where $A_i$ are linear operators,
    2. $\sum_i A_i^\dagger A_i = I$.
    3. We can prove that quantum operations preserve density operators.

In the representation of quantum computational processes based on mixed states, a quantum circuit is a circuit whose inputs and outputs are labeled with density operators and whose quantum gates are labeled with quantum operations.

In this powerful model we can extend the notion of probability assigned to a qbit.

- To each vector of the quantum computational basis of $\mathbb{C}^2$ we may associate two density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ that represent the standard basis in this framework.
- Let $P_1^{(2^n)}$ be the operator $P_1^{(2^n)} = (\otimes^{n-1} I) \otimes P_1$ on $\otimes^n \mathbb{C}^2$ where $I$ is the $2 \times 2$ identity matrix.
- By applying the Born rule, we consider the probability of a density operator $\rho$ as follows:

$$p(\rho) = \mathrm{tr}(P_1^{(2^n)} \rho).$$

Note that: in the particular case in which $\rho = |\psi\rangle\langle\psi|$ where

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

we obtain that $p(\rho) = |c_1|^2$.

**Proposition 1.1** *For each density operator*

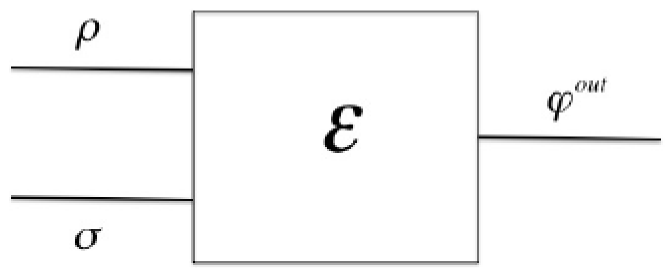
$$\rho = [\rho_{i,j}] = \begin{pmatrix} & & & & \cdots \\ & \rho_{2,2} & & & \cdots \\ & & - & & \cdots \\ & & & \rho_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \otimes^n \mathbb{C}^2$$

$$p(\rho) = Tr(P_1^{(n)} \rho) = \sum_{i=1}^{2^{n-1}} \rho_{2i,2i}$$

□

## 3 Łukasiewicz sum and quantum circuits

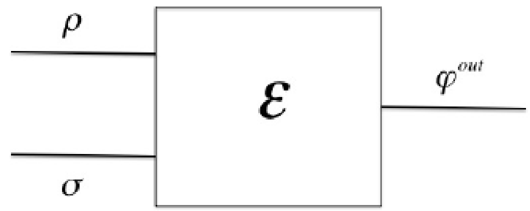
- In literature, several logical algebraic structures related to the infinite many valued Łukasiewicwicz logic and quantum computation were introduced and investigated.
- These structures are based on an ideal quantum circuit, represented by a quantum operation $\mathcal{E}$ of the form




such that $$p(\varphi^{out}) = p(\rho) \oplus p(\sigma) = \min\{p(\rho) + p(\sigma), 1\}$$

$\mathcal{E}$ is a quantum operation

$\rho$ and $\sigma$ are density operators in $\mathbb{C}^2$.




... but a such quantum operation $\mathcal{E}$ **does not exist!!**... indeed

- the above circuit with two inputs, is mathematically represented by an expression of the form:

$$\mathcal{E}(\rho, \sigma) = \sum_i A_i (\rho \otimes \sigma) A_i^\dagger .$$

- if $\rho = [x_{j,k}]$, $\sigma = [y_{l,m}]$ and $A_i = [a_{r_i,s_i}]$, $p(\mathcal{E}(\rho, \sigma))$ can be seen as a polynomial in the variables $x_{2,2}$, $y_{2,2}$ assuming the following form:

$$p(\mathcal{E}(\rho, \sigma)) = \sum_{\alpha,\beta} f_{\alpha,\beta}(a_{r_i,s_i}, x_{j,k}, y_{l,m}) x_{2,2}^{\alpha} y_{2,2}^{\beta}$$

  where $\alpha, \beta \in \{0, 1\}$.

- Since Łukasiewicz sum is not a polynomial, it is not representable as a quantum operation!!!

However the expression

$$p(\mathcal{E}(\rho,\sigma)) = \sum_{\alpha,\beta} f_{\alpha,\beta}(a_{r_i,s_i}, x_{j,k}, y_{l,m}) x_{2,2}^{\alpha} y_{2,2}^{\beta}$$

**suggests to look for a quantum operation $\mathcal{E}$ such that the polynomial $p(\mathcal{E}(\rho,\sigma))$ approximates the value $p(\rho) \oplus p(\sigma)$**

In order to follow this strategy, we have to deal with the following two issues:

a The probability value $p(\mathcal{E}(\rho,\sigma))$ does not only depend on $p(\rho)$ and $p(\sigma)$ but it can also depend on the anti-diagonal elements of $\rho$ and $\sigma$ among the coefficients $f_{\alpha,\beta}(a_{r_i,s_i},\rho_{j,k},\sigma_{l,m})$.

To solve this problem we need to introduce a quantum operation that delete the anti diagonal elements of a density operator in $\mathbb{C}^2$ preserving its probability value. The following proposition provides this quantum operation for an arbitrary density operator.

**Proposition 3.1** *The Matrices* $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *define a quantum operation*

$$\mathcal{A}nt(\rho) = D_1 \rho D_1^\dagger + D_2 \rho D_2^\dagger$$

*such that*

$$\rho = \begin{bmatrix} 1-x & r \\ r^\dagger & x \end{bmatrix} \mapsto \mathcal{A}nt(\rho) = \begin{bmatrix} 1-x & 0 \\ 0 & x \end{bmatrix}$$

□

b. The polynomial degree of $p(\mathcal{E}(\rho, \sigma))$ is equal to 1. This could not guarantee a good approximation for the Łukasiewicz sum.

- To obtain a polynomial $p(\mathcal{E}(\rho, \sigma))$ of an arbitrary degree, we need to increase the power of the variables $x_{2,2}$ and $y_{2,2}$ in the polynomial
- this can be achieved by involving tensorial powers of $\rho$ and $\sigma$. Indeed,

for an arbitrary density operator $\rho = \begin{bmatrix} 1-x & r \\ r^{\dagger} & x \end{bmatrix}$ in $\mathbb{C}^2$, by induction on $n$, we can prove that

$$Diag(\otimes^n \rho) = \{(1-x)^{\alpha} x^{\beta} : \alpha + \beta = n\}$$

Taking into account the two items above and in order to obtain a good approximation for the Łukasiewicz sum, our strategy will be

- to consider a quantum operation of the form $\mathcal{E}(\otimes^n \rho, \otimes^n \sigma)$ (instead of $\mathcal{E}(\rho, \sigma)$) where, $\rho$ and $\sigma$ can be considered as diagonal density operators in $\mathbb{C}^2$ and $p(\mathcal{E}(\otimes^n \rho, \otimes^n \sigma)) \approx p(\rho) \oplus p(\sigma)$.

- From a physical point of view, to implement a quantum operation $\mathcal{E}(\otimes^n \rho, \otimes^n \sigma)$ for a large $n$ turns out to be inefficient, because it requires many copies of the involved states.

For this reason and in order to keep a reasonable phisical efficiency during the implementation, we confine our attention in 2-degrees polynomial approximant.

## 4 Polynomial approximation for Łukasiewicz sum

The key idea will be to reduce the problem to a one-variable approximation function.

- Let us consider the function

$$[0,2] \ni z \mapsto h(z) = \begin{cases} \frac{z}{2}, & \text{if } z \in [0,1], \\ 1 - \frac{z}{2}, & \text{if } z \in (1,2], \end{cases}$$

- and we define

$$g(z) = \frac{z}{2} + h(z).$$

- Note that if $z = x + y$ and $z \in [0,2]$ then:

$$g(x+y) = x \oplus y$$

- $h(z)$ is a symmetric function with respect to the point $z = 1$, in other words, $h(2 - z) = h(z)$.
- Consequently we can approximate $h(z)$ by using the symmetric functions $z^i (2 - z)^i$.

In this way, we can prove that the approximants $P_n(x, y) = g_n(x + y)$ for the Łukasieqicz $x \oplus y$ assume the form:

$$P_n(x, y) = \frac{x + y}{2} + \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2} \binom{1/2}{i} (x + y)^i ((1 - x) + (1 - y))^i.$$

In particular

$$\widehat{P}_2(x,y) = \frac{5}{12}(x+y)(1-x) + \frac{5}{12}(x+y)(1-y) + \frac{1}{2}(x+y) + \frac{1}{120}$$

$$E(P_2) = \max_{(x,y)\in[0,1]^2} |x \oplus y - \widehat{P}_2| \leq \frac{3}{40}$$

But!!

- we can see that $\max_{(x,y)\in[0,1]^2} \widehat{P}_2(x,y) = \frac{43}{40} > 1$
- Thus, there is not a quantum operation $\mathcal{E}$ such that $p(\mathcal{E}(\rho\otimes\sigma)) = \widehat{P}_2(p(\rho), p(\sigma))$.

Thus, in order to avoid this problem, if we consider the product $\frac{40}{43}\widehat{P}_2$ we finally obtain the following polynomial that approximates the Łukasiewicz sum:

$$P_{\text{Ł}_2}(x,y) = \frac{40}{43}\left(\frac{5}{12}(x+y)(1-x) + \frac{5}{12}(x+y)(1-y) + \frac{1}{2}(x+y) + \frac{1}{120}\right)$$

We have to built a quantum operation $\text{Ł}_2$ in $\otimes^4\mathbb{C}^2$ such that

$$p(\text{Ł}_2(\otimes^2\rho, \otimes^2\sigma)) \approx p(\rho) \oplus p(\sigma)$$

where $\rho$ and $\sigma$ are density operators in $\mathbb{C}^2$ with the null element in the anti-diagonal entries.

# 5 Łukasiewicz sum as 16-dimensional quantum operation

- Let $\rho = \begin{bmatrix} 1-x & 0 \\ 0 & x \end{bmatrix}$ $\sigma = \begin{bmatrix} 1-y & 0 \\ 0 & y \end{bmatrix}$ two density operators in $\mathbb{C}^2$ (hence, $p(\rho) = x$ and $p(\sigma) = y$).

- the matrix $(z_{i,j})_{1 \leq i,j \leq 16} = (\rho \otimes \rho) \otimes (\sigma \otimes \sigma)$ is the diagonal matrix whose diagonal coefficients are given by

$$
\begin{aligned}
z_1 &= (1-x)^2(1-y)^2, & z_9 &= (1-x)x(1-y)^2, \\
z_2 &= (1-x)^2(1-y)y, & z_{10} &= (1-x)x(1-y)y, \\
z_3 &= (1-x)^2(1-y)y, & z_{11} &= (1-x)x(1-y)y, \\
z_4 &= (1-x)^2y^2, & z_{12} &= (1-x)xy^2, \\
z_5 &= (1-x)x(1-y)^2, & z_{13} &= x^2(1-y)^2, \\
z_6 &= (1-x)x(1-y)y, & z_{14} &= x^2(1-y)y, \\
z_7 &= (1-x)x(1-y)y, & z_{15} &= x^2(1-y)y, \\
z_8 &= (1-x)xy^2, & z_{16} &= x^2y^2.
\end{aligned}
$$

- A quantum operation $Ł_2(-) = \sum_k A_k(-)A_k^\dagger$ in $\otimes^4\mathbb{C}^2$ such that $p(Ł_2(\rho,\sigma)) = P_{Ł_2}(p(\rho), p(\sigma)) = P_{Ł_2}(x,y)$ will need to satisfy

$$P_{Ł_2}(x,y) = \sum_{i=1}^{16} a_i(A_1,\ldots,A_k)z_i$$

- $a_i(A_1,\ldots,A_k)$ are real numbers depending on the elements of the matrices $A_k$ for $1 \leq k \leq 16$

Then we need to rewrite $P_{Ł_2}(x,y)$ in the base $(z_i)_{1\leq i\leq 16}$. Indeed ...

$$\begin{aligned} P_{Ł_2}(x,y) &= \frac{40}{43}[\frac{1}{120}z_1 + \frac{27}{20}z_2 + \frac{111}{120}z_4 + \frac{27}{120}z_5 + \frac{51}{20}z_6 + \frac{141}{60}z_8 \\ &+ \frac{111}{120}z_{13} + \frac{141}{60}z_{14} + \frac{121}{120}z_{16}]. \end{aligned}$$

*Let us consider the following $16 \times 16$ matrices for $1 \leq k \leq 8$:*

$[\mathbf{1}]_{(2k,j)}$ *having* $1$ *in the* $(2k, j)$*-entry and* $0$ *in any other entry,*

$[\mathbf{1}]_{(2k-1,j)}$ *having* $1$ *in the* $(2k-1, j)$*-entry and* $0$ *in any other entry.*

*Let us define the following family of matrices, for $1 \leq k \leq 8$*

$$\text{Ł}_{(2k,1)} = \sqrt{\tfrac{1}{129}}[\mathbf{1}]_{(2k,1)} \qquad \text{Ł}_{(2k-1,1)} = \sqrt{\tfrac{1}{16} - \tfrac{1}{129}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,2)} = \sqrt{\tfrac{54}{43}}[\mathbf{1}]_{(2k,2)} \qquad \text{Ł}_{(2k-1,2)} = \sqrt{\tfrac{1}{16} - \tfrac{54}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,4)} = \sqrt{\tfrac{37}{43}}[\mathbf{1}]_{(2k,4)} \qquad \text{Ł}_{(2k-1,4)} = \sqrt{\tfrac{1}{16} - \tfrac{37}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,5)} = \tfrac{3}{\sqrt{43}}[\mathbf{1}]_{(2k,5)} \qquad \text{Ł}_{(2k-1,5)} = \sqrt{\tfrac{1}{16} - \tfrac{9}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,6)} = \sqrt{\tfrac{102}{43}}[\mathbf{1}]_{(2k,6)} \qquad \text{Ł}_{(2k-1,6)} = \sqrt{\tfrac{1}{16} - \tfrac{102}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,8)} = \sqrt{\tfrac{94}{43}}[\mathbf{1}]_{(2k,8)} \qquad \text{Ł}_{(2k-1,8)} = \sqrt{\tfrac{1}{16} - \tfrac{94}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,13)} = \sqrt{\tfrac{37}{43}}[\mathbf{1}]_{(2k,13)} \qquad \text{Ł}_{(2k-1,13)} = \sqrt{\tfrac{1}{16} - \tfrac{37}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,14)} = \sqrt{\tfrac{94}{43}}[\mathbf{1}]_{(2k,14)} \qquad \text{Ł}_{(2k-1,14)} = \sqrt{\tfrac{1}{16} - \tfrac{94}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$\text{Ł}_{(2k,16)} = \tfrac{11}{\sqrt{129}}[\mathbf{1}]_{(2k,16)} \qquad \text{Ł}_{(2k-1,16)} = \sqrt{\tfrac{1}{16} - \tfrac{121}{129}}[\mathbf{1}]_{(2k-1,1)}$$

*Then,* $\L_2$ *defined as the operator in* $\otimes^4\mathbb{C}^2$ *given by*

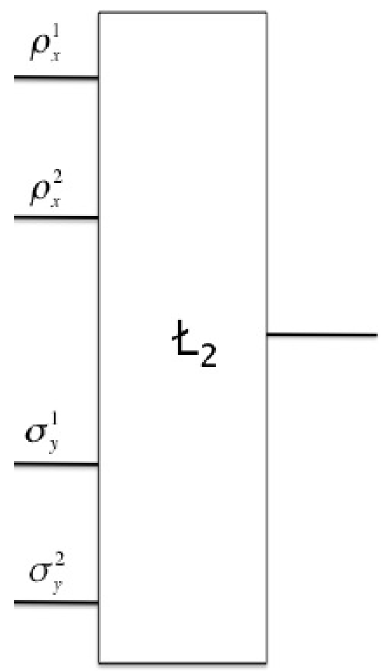
$$\begin{aligned}
\L_2(\varphi) &= \sum_{k=1}^{8}[\L_{(2k,1)}\varphi\L^{\dagger}_{(2k,1)} + \L_{(2k,2)}\varphi\L^{\dagger}_{(2k,2)} + \L_{(2k,4)}\varphi\L^{\dagger}_{(2k,4)} + \\
&+ \L_{(2k,5)}\varphi\L^{\dagger}_{(2k,5)} + \L_{(2k,6)}\varphi\L^{\dagger}_{(2k,6)} + \L_{(2k,8)}\varphi\L^{\dagger}_{(2k,8)} + \\
&+ \L_{(2k,13)}\varphi\L^{\dagger}_{(2k,13)} + \L_{(2k,14)}\varphi\L^{\dagger}_{(2k,14)} + \L_{(2k,16)}\varphi\L^{\dagger}_{(2k,16)} + \\
&+ \L_{(2k-1,1)}\varphi\L^{\dagger}_{(2k-1,1)} + \L_{(2k-1,2)}\varphi\L^{\dagger}_{(2k-1,2)} + \L_{(2k-1,4)}\varphi\L^{\dagger}_{(2k-1,4)} + \\
&+ \L_{(2k-1,5)}\varphi\L^{\dagger}_{(2k-1,5)} + \L_{(2k-1,6)}\varphi\L^{\dagger}_{(2k-1,6)} + \L_{(2k-1,8)}\varphi\L^{\dagger}_{(2k-1,8)} + \\
&+ \L_{(2k-1,13)}\varphi\L^{\dagger}_{(2k-1,13)} + \L_{(2k-1,14)}\varphi\L^{\dagger}_{(2k-1,14)} + \L_{(2k-1,16)}\varphi\L^{\dagger}_{(2k-1,16)}]
\end{aligned}$$

*is a quantum operation such that, for* $\rho, \sigma$ *diagonal density operators in* $\mathbb{C}^2$

$$p(\L_2((\otimes^2\rho) \otimes (\otimes^2\sigma))) = P_{\L_2}(p(\rho), p(\sigma)).$$

$$\begin{aligned} Ł_2(\psi) &= \sum_k \sum_{1=s}^{16} Ł_{(2k,s)} \psi Ł^{\dagger}_{(2k,s)} + \sum_k \sum_{1=s}^{16} Ł_{(2k-1,s)} \psi Ł^{\dagger}_{(2k-1,s)} \\ &= (\otimes^3 I) \otimes \begin{pmatrix} 1 - \sum_k a_k z_k & 0 \\ 0 & \sum_k a_k z_k \end{pmatrix} \end{aligned}$$

where $k \in \{1, 2, 4, 5, 6, 8, 13, 14, 16\}$.



# 6 Łukasiewicz sum and quantum cloning machines for qubits

- Let $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ be a qubit in $\mathbb{C}^2$.
- Let us consider a quantum cloner providing the single copy density matrix

$$\rho_1 = \rho_2 = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{6}I = \begin{pmatrix} 1 - \frac{1}{6} - \frac{2}{3}|c_1|^2 & \frac{1}{6} + \frac{2}{3}c_1c_0 \\ \frac{1}{6} + \frac{2}{3}(c_1c_0)^\dagger & \frac{1}{6} + \frac{2}{3}|c_1|^2 \end{pmatrix}$$

- Let us start by making two approximate copies of a $\rho = a_0|0\rangle + a_1|1\rangle$ and $\sigma = b_0|0\rangle + b_1|1\rangle$.
- It provides the copies $\rho_x^{clon1} = \rho_x^{clon2}$ and $\sigma_y^{clon1} = \sigma_y^{clon2}$
- If we consider the following density matrices

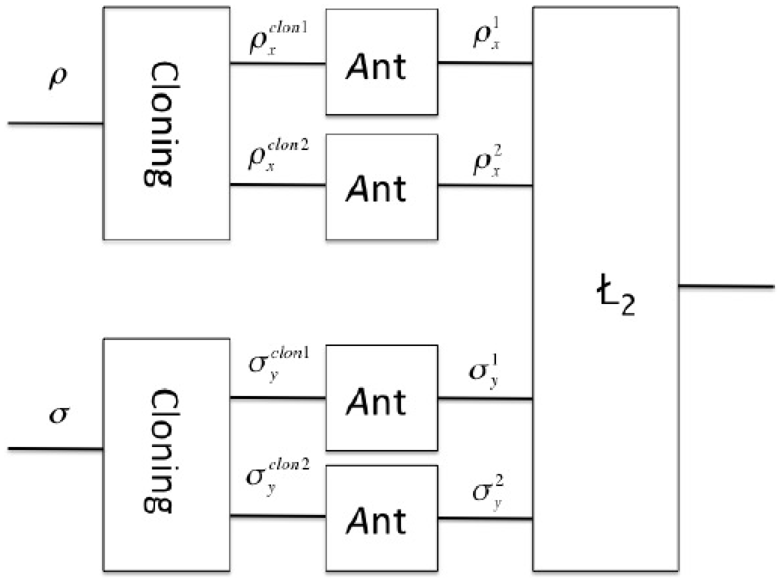
$$\rho_x^1 = \mathcal{A}nt(\rho_x^{clon1}) = \mathcal{A}nt(\rho_x^{clon2}) = \rho_x^2 = \begin{pmatrix} 1 - \frac{1}{6} - \frac{2}{3}x & 0 \\ 0 & \frac{1}{6} + \frac{2}{3}x \end{pmatrix}$$

$$\sigma_y^1 = \mathcal{A}nt(\sigma_y^{clon1}) = \mathcal{A}nt(\sigma_y^{clon2}) = \sigma_y^2 = \begin{pmatrix} 1 - \frac{1}{6} - \frac{2}{3}y & 0 \\ 0 & \frac{1}{6} + \frac{2}{3}y \end{pmatrix}$$

Then

$$p(\rho_x^i) \oplus p(\sigma_y^j) = \frac{1}{3} + \frac{2}{3}(p(\rho) \oplus p(\sigma))$$

In this way, we can show that the following circuit




satisfies

$$p(\mathcal{E}(\rho \otimes \sigma)) = \frac{2}{3} P_{\text{Ł}_2}(p(\rho), p(\sigma)) + \frac{1}{6} \approx \frac{2}{3}(p(\rho) \oplus p(\sigma)) + \frac{1}{6}$$

Finally, in order to obtain the best approximation we need to consider the following:

$$p(\rho) \oplus p(\sigma) \approx \frac{40}{40}\frac{3}{2}(p(\mathcal{E}) - \frac{1}{6}).$$