Quantum Probability and The Problem of Pattern Recognition

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Outline

Introduction

- We discuss a possible generalization of the problem of pattern recognition to arbitrary probabilistic models.

[F. Holik, G. Sergioli, H. Freytes, A. Plastino, “Pattern Recognition In Non-Kolmogorovian Structures”, arxiv:1609.06340, (2016)]

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- We discuss a possible generalization of the problem of pattern recognition to arbitrary probabilistic models.
- We discuss how to deal with the problem of recognizing an individual pattern among a family of different species or classes of objects which obey probabilistic laws which do not comply with Kolmogorov's axioms.
- Our framework allows for the introduction of non-trivial correlations (as entanglement or discord) between the different species involved, opening the door to a new way of harnessing these physical resources for solving pattern recognition problems.

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- In his approach, probabilities are considered as measures defined over boolean sigma algebras of a sample space.
- Interestingly enough, states of classical statistical theories can be described using Kolmogorov's axioms, because they define measures over the sigma algebra of measurable subsets of phase space.

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- It is natural to associate to A_Δ which can be represented as the measurable set $f^{-1}(\Delta)$ (the set of all states which make the proposition true).
- If the probabilistic state of the system is given by μ, the corresponding probability of occurrence of f_Δ will be given by $\mu(f^{-1}(\Delta))$.

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$$\mu : \Sigma \rightarrow [0, 1] \tag{1}$$

which satisfies

$$\mu(\emptyset) = 0 \tag{2}$$

$$\mu(A^c) = 1 - \mu(A), \tag{3}$$

where $(\ldots)^c$ means set-theoretical-complement. For any pairwise disjoint denumerable family $\{A_i\}_{i \in I}$,

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- A state of a classical probabilistic theory will be defined as a Kolmogorovian measure with $\Sigma = \mathcal{P}(\Gamma)$.

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- As is well known, projection operators can be used to describe elementary experiments (the analogue of this in the classical setting are the subsets of phase space).
- In this way, a comparison between quantum states and classical probabilistic states can be traced in formal and conceptual grounds.

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- Due to this fact, empirical propositions associated to quantum systems are represented by projection operators, which are in one to one correspondence to closed subspaces related to the projective geometry of a Hilbert space.
- Thus, empirical propositions associated to quantum systems form a non-distributive —and thus non-Boolean— lattice.

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- If $\mathbf{A}$ represents the self adjoint operator of an observable associated to a quantum particle, the proposition "the value of $\mathbf{A}$ lies in the interval Δ" will define a testeable experiment represented by the projection operator $\mathbf{P_A}(\Delta) \in \mathcal{P}(\mathcal{H})$, i.e., the projection that the spectral measure of $\mathbf{A}$ assigns to the Borel set Δ.

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- The probability assigned to the event $\mathbf{P_A}(\Delta)$, given that the system is prepared in the state ρ, is computed using Born's rule:

$$p(\mathbf{P_A}(\Delta)) = \mathrm{tr}(\rho \mathbf{P_A}(\Delta)). \tag{5}$$

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- Born's rule defines a measure on $\mathcal{P}(\mathcal{H})$ with which it is possible to compute all probabilities and mean values for all physical observables of interest.

Quantum Probabilistic Models

Due to Gleason's theorem a quantum state can be defined by a measure s over the orthomodular lattice of projection operators $\mathcal{P}(\mathcal{H})$ as follows

$$s : \mathcal{P}(\mathcal{H}) \rightarrow [0;1] \tag{6}$$

such that:

$$s(\mathbf{0}) = 0 \ (\mathbf{0} \text{ is the null subspace}). \tag{7}$$

$$s(P^{\perp}) = 1 - s(P), \tag{8}$$

and, for a denumerable and pairwise orthogonal family of projections P_j

$$s(\sum_j P_j) = \sum_j s(P_j). \tag{9}$$

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- In the quantum case, the Boolean algebra Σ is replaced by $\mathcal{P}(\mathcal{H})$, and the other conditions are the natural generalizations of the classical event structure to the non-Boolean setting.
- The fact that $\mathcal{P}(\mathcal{H})$ is not Boolean lies behind the peculiarities of probabilities arising in quantum phenomena.

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- The new algebras are known today as von Neumann algebras, and their elementary components can be classified as Type I, Type II and Type III factors.
- It can be shown that, the projective elements of a factor form an orthomodular lattice. Classical models can be described as commutative algebras.
- The models of standard quantum mechanics can be described by using Type I factors (Type I_n for finite dimensional Hilbert spaces and Type I_∞ for infinite dimensional models). These are algebras isomorphic to the set of bounded operators on a Hilbert space.

Quantum Probabilistic Models

- Further work revealed that a rigorous approach to the study of quantum systems with infinite degrees of freedom needed the use of more general von Neumann algebras, as is the case in the axiomatic formulation of relativistic quantum mechanics. A similar situation holds in algebraic quantum statistical mechanics.

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- In these models, States are described as complex functionals satisfying certain normalization conditions, and when restricted to the projective elements of the algebras, define measures over lattices which are not the same to those of standard quantum mechanics.

Maximal Boolean Subalgebras: Contextual probabilistic Models

Contextuality rules

- It is important to mention that an arbitrary orthomodular lattice $\mathcal{L}$ can be written as a sum:

$$\mathcal{L} = \bigvee_{\mathcal{B} \in \mathfrak{B}} \mathcal{B}$$

where $\mathfrak{B}$ is the set of all possible Boolean subalgebras of $\mathcal{L}$.

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where $\mathfrak{B}$ is the set of all possible Boolean subalgebras of $\mathcal{L}$.
- A state s on $\mathcal{L}$ defines a classical probability measure on each $\mathcal{B}$. In other words, $s_{\mathcal{B}}(\ldots) := s|\mathcal{B}(\ldots)$ is a Kolmogorovian measure over $\mathcal{B}$.

Examples: Q-bit

Qbit

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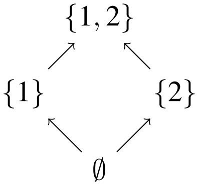
- Notice that when $\mathcal{H}$ is finite dimensional, its maximal Boolean subalgebras will be finite.
- $\mathcal{P}(\mathbb{C}^2) \Longrightarrow \{\mathbf{0}, \mathbf{P}, \neg\mathbf{P}^{\perp}, \mathbf{1}_{\mathbb{C}^2}\}$ with $\mathbf{P} = |\varphi\rangle\langle\varphi|$ for some unit norm vector $|\varphi\rangle$ and $\mathbf{P}^{\perp} = |\varphi^{\perp}\rangle\langle\varphi^{\perp}|$.

Boolean algebra

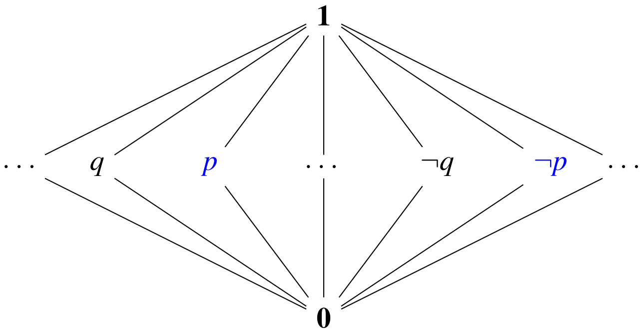
Figure: Hasse diagram of $\mathcal{B}_2$



B_2

Skeleton of a qbit

$$\mathcal{P}(\mathbb{C}^2)$$



Examples: Q-trit

Qtrit-contextuality

- $\mathcal{P}(\mathbb{C}^3) \Longrightarrow$
  $\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

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  $\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$
- Given $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\varphi_3\rangle \Longrightarrow$

$$\{\mathbf{0}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_{12}, \mathbf{P}_{13}, \mathbf{P}_{23}, \mathbf{1}_{\mathbb{C}^3}\}$$

$P_i = |\varphi_i\rangle\langle\varphi_i|$ $(i = 1,2,3)$ and $P_{ij} := |\varphi_i\rangle\langle\varphi_i| + |\varphi_j\rangle\langle\varphi_j|$ $(i,j = 1,2,3)$.

Qtrit Boolean subalgebras:

Figure: Maximal Boolean subalgebras of $\mathbb{C}^3$

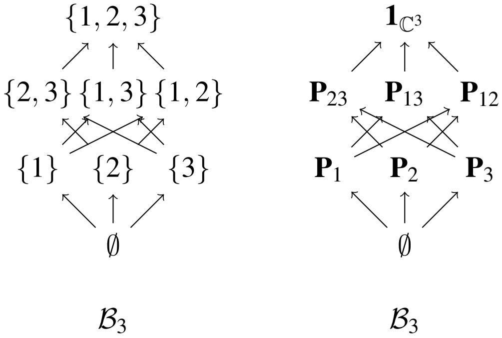
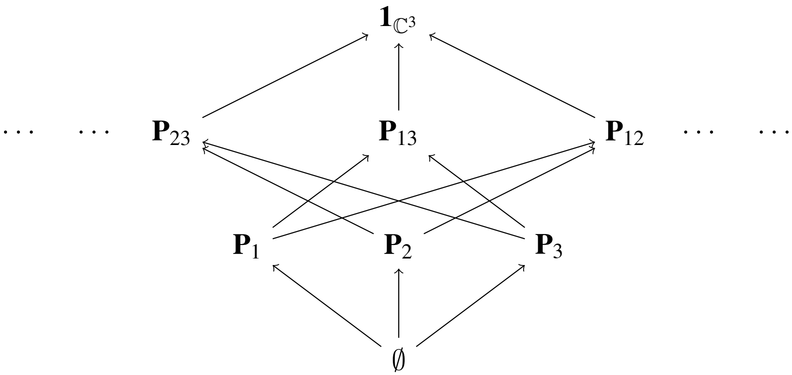


Figure: Skeleton of $\mathbb{C}^3$



$\mathcal{P}(\mathbb{C}^3)$

Partial Kolmogorovian measures

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- A quantum state ρ defines a Kolmogorovian probability distribution on each maximal subalgebra of an orthomodular lattice.
- Something completely analogous occurs for more general physical theories of importance.
- But these Kolmogorovian measures are pasted in a coherent way: Born's rule (von Neumann's axioms).

Generalized Probabilistic Models

This opens the door to a meaningful generalization of Kolmogorov's axioms to a wide variety of orthomodular lattices.
Let $\mathcal{L}$ be an orthomodular lattice. Then, define

$$s : \mathcal{L} \rightarrow [0; 1],$$

($\mathcal{L}$ standing for the lattice of all events) such that:

$$s(\mathbf{0}) = 0. \tag{10}$$

$$s(E^{\perp}) = 1 - s(E),$$

and, for a denumerable and pairwise orthogonal family of events E_j

$$s(\sum_j E_j) = \sum_j s(E_j).$$

where $\mathcal{L}$ is a general orthomodular lattice (with $\mathcal{L} = \Sigma$ and $\mathcal{L} = \mathcal{P}(\mathcal{H})$ for the Kolmogorovian and quantum cases respectively).

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- While classical systems can be described as simplexes, non-classical theories can display a more involved geometrical structure.
- These models can go far beyond classical and quantum mechanics, and can be used to described different theories (such as, for example, Popescu Rorlich boxes).

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- The structure of these measurable properties imposes severe restrictions on the interpretation of the probabilities defined by the states, depending on the algebraic and geometric features of the underlying event structure.
- This has implications for the different interpretations of probability theory.

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- In this way, each state ν defines a concrete probability for each possible experiment.

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- The real numbers $p(E_i,\nu)$ must satisfy $\sum_{i=1}^{n} p(E_i,\nu) = 1$; otherwise, the probabilities would not be normalized.
- Notice that each possible outcome E_i of each possible experiment E, induces a linear functional $E_i(...) : \mathcal{C} \longrightarrow [0,1]$, with $E_i(\nu) := \nu(E_i)$. Functionals of this form are usually called *effects*.

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- The model will be said to be finite dimensional if and only if $V(\mathcal{C})$ is finite dimensional. As in the quantum and classical cases, extreme points of the convex set of states will represent pure states.

Generalized Probabilistic Models

- It is important to remark the generality of the framework described above: all possible probabilistic models with finite outcomes can be described in such a way.

Generalized Probabilistic Models

- It is important to remark the generality of the framework described above: all possible probabilistic models with finite outcomes can be described in such a way.
- Furthermore, if suitable definitions are made, it is possible to include continuous outcomes in this setting.

Outline

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- There are formulations of the problem for the particular case of non-relativistic quantum mechanics and quantum optics.
- We look for a setting capable of describing generalized probabilistic models.

A General Framework

- Given a collection of classes of objects O_i, let us assume that the state of each object o^i_j (i.e., object j of class O_i) is represented by a state $\nu^i_j \in \mathcal{C}_i$, where $\mathcal{C}_i$ is the convex operational model representing object o^i_j.

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- Suppose that weights p_j^i are assigned to the objects o_j^i, representing the rational agent's knowledge about the importance of object o_j^i as a representative of class $\mathcal{C}_i$ (if all objects are equally important, the weights are chosen as $p_j^i = \frac{1}{N_i}$).

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- This means that the probabilistic state of the whole class O_i can be represented by a mixture $\nu_i = \sum_j p_j \nu^i_j \in \mathcal{C}_i$.

We Allow Correlations

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- But we notice that under these conditions, the states ν_i will be *improper mixtures*, and then, no consistent ignorance interpretation can be given for them [**?**] (and this means that the weights loss their ignorance interpretation).

Comparison

- A particular object o must be identified and compared with the information given by the generalized states of the classes represented by ν_i (or more generally, by $\tilde{\nu}$), obtained in the learning process.

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- We will assume, as usual, that knowledge about o is represented by a generalized state ν. Notice that, in order to obtain ν, several copies of the unknown object o may be needed, whenever the probabilistic character of the model is irreducible.

Quantum Pattern Recognition

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- The collection of chosen properties can be non-commutative. Thus, the properties of object q^i_j (object j of class C_i) will be represented by operators (representing the class Q_i).
- The only thing that we can do, is to assign probabilities for each property coordinate using the quantum state ρ^i_j of each object q^i_j. Thus, if —as in the classical case— we assign weights p^i_j to each object q^i_j, knowledge about the class Q_i can now be represented by a mixture $\rho_i = \sum_j p^i_j \rho^i_j$.

Quantum Pattern Recognition

Given the fact that in general, interaction between physical systems represented by classes Q_i can be non-negligible, and thus, non-trivial correlations may be involved, we will assume that the states ρ_i are arbitrary states of the Hilbert space $\mathcal{H}_i$ (i.e., the ρ_i are not necessarily proper mixtures). We call $\tilde{\rho}$ the global state of the whole set of classes.

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- Notice however, that the state ρ could, in the general case, be unknown to the agent, and he may have only access to a sample of values $\{a_j\}$ of the operators σ_j.
- Thus, for the classification problem, he should be able to, either reconstruct the unknown state ρ using quantum statistical inference methods, or just directly compare the sampled values with the information provided by the global state $\tilde{\rho}$.

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- Suppose now that at an initial state, the agent has an information $\rho_i(0)$ for each class Q_i, and he is confronted with an individual of which it has information $\rho(0)$, and a global state $\tilde{\rho}(0)$.

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- A *quantum learning operator* will be thus a family of quantum operations $\{\Lambda(t_1), ..., \Lambda(t_n)\}$. Hence, a *quantum learning process* will be a succession of global states $\{\tilde{\rho}(0), \Lambda(t_1)\tilde{\rho}(0), \Lambda(t_2)\tilde{\rho}(t_1), ..., \Lambda(t_n)\tilde{\rho}(t_{n-1})\}$.

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- A *quantum learning operator* will be thus a family of quantum operations $\{\Lambda(t_1), ..., \Lambda(t_n)\}$. Hence, a *quantum learning process* will be a succession of global states $\{\tilde{\rho}(0), \Lambda(t_1)\tilde{\rho}(0), \Lambda(t_2)\tilde{\rho}(t_1), ..., \Lambda(t_n)\tilde{\rho}(t_{n-1})\}$.
- The goal of the learning process will be achieved if the uncertainty of the final state is reduced. The dispersion could be measured using the von Neumann entropy (or other quantum entropic measures).

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- Open sets are intended to represent local regions, and M models space-time with its symmetries. Local algebras are intended to represent local observables (such as particle detectors).
- For example, in ARQFT, M is Minkowski's four dimensional space-time, endowed with the Poincare group of transformations.

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- But in general, the local algebras of ARQFT will not be Type I factors as in standard quantum mechanics. For example, it can be shown that for a diamond region, a Type III factor must be assigned.
- This means that the orthomodular lattice involved will not be the lattice of projection operators of a Hilbert space, but a one with different properties.

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- In practical implementations, these states and the discrimination problem, could be restricted to a concrete space-time region.

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- In particular, a simpler but analogous version of the problem could be conceived by appealing to the Fock-space formalism, in order to describe the fields and the states involved.

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Pattern Recognition In AQSM:

- As in the quantum field theoretic example, a similar problem can be posed in the algebraic approach to quantum statistics.
- Here, a typical problem could be to discern a kind of atoms from a set of classes of gasses; now, the comparison will be between the state of the item and the classes involved.

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- Suppose that a machine has to solve a problem of recognizing handwritten digits. These drawings are first transformed into digitalized images of $n \times n$ pixels.
- This means that the information of each image is stored in a vector $\vec{x}$ of length $n \times n$.
- The goal is to build our automata in such a way that it takes a vector $\vec{x}$ as an input, and gives us as output the identity of the digit in question. In a real hardware, this vector should be stored using bits of a given length.

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- For each point $x \in L$ we have a Hilbert space $\mathcal{H}_x$, and for each subset of points $\Gamma \in L$, the associated Hilbert space is given by the tensor product $\mathcal{H}_\Gamma = \bigotimes_{x\in\Gamma} \mathcal{H}_x$.

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- Every subset $\Gamma \in L$ has associated an algebra $\mathcal{A}(\mathcal{H}_\Gamma)$. The norm completion of the collection $\mathcal{A} = \{\mathcal{A}_\Gamma\}_{\Gamma \in L}$ is a quasi-local $C^\star$-algebra when equipped with the net of $C^\star$-subalgebras $\mathcal{A}_\Gamma$.

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- Every subset $\Gamma \in L$ has associated an algebra $\mathcal{A}(\mathcal{H}_\Gamma)$. The norm completion of the collection $\mathcal{A} = \{\mathcal{A}_\Gamma\}_{\Gamma \in L}$ is a quasi-local $C^\star$-algebra when equipped with the net of $C^\star$-subalgebras $\mathcal{A}_\Gamma$.
- Thus, the classification problem must be done with respect to states defined in this algebra (such as KMS-states [**?**]), whose properties are different to that of a Type I factor.

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- This is the case for the most known quantum algorithms: Shor, Deutsch-Jozsa, etc.
- In these examples a pattern is to be found (for example, determining the period of a periodic function).
- Our framework could be useful to understand these processes.

Outline

Conclusions

- We propose a generalization of the pattern recognition problem to the non-commutative (or equivalently, non-Kolmogorovian) setting involving incompatible (non-simultaneously determinable) properties.

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- We propose a generalization of the pattern recognition problem to the non-commutative (or equivalently, non-Kolmogorovian) setting involving incompatible (non-simultaneously determinable) properties.
- In this way, we have shown that it is possible to find some important (and non-equivalent) examples of interest: standard quantum mechanics, algebraic relativistic quantum field theory, and algebraic quantum statistics.

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- The examples does not restrict only to these ones, but can include more general models, and particular, hybrid systems (classical and quantum).
- Our perspective could be useful to characterize some of the most important quantum computation algorithms (Shor, Simon and Jozsa-Deutsche) as quantum pattern recognition problems.

Some References

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