Generalized quantum entropies: a definition and some properties

Steeve Zozor

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Cagliari, November 3, 2016
1 Motivations & goals

2 Classical \((h, \phi)\)-entropies
   • Definition
   • Properties

3 Quantum \((h, \phi)\)-entropies
   • Definition
   • Basic properties

4 Composite quantum systems
   • Bipartite systems – (sub)additivity, pure state
   • \((h, \phi)\)-entropy and entanglement

5 Relative \((h, \phi)\)-entropies
   • Classical context
   • Quantum context

6 Conclusions
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Motivations

- Increasing field of investigation on quantum information processing or transmission.
- Necessitate the use of quantum information measures, or of quantum entropies.
- There exist some definitions: von Neumann, quantum versions of Rényi, Tsallis, Kaniadakis types, ...
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Note

In the classical context, there exists a generalized family proposed by Salicrú (Csiszár); Contains the Shannon entropy, that of Rényi, Havrda-Charvát (Daróczy, Vajda, Tsallis, ...) among others.
Goals

- To define a generalized family of quantum entropies.
- To study their properties (common or specific).
- To apply them in quantum information processing.
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**Definition**

Let \(p = [p_1 \, \cdots \, p_N] \in [0 ; 1]^N\), \(\sum_k p_k = 1\)

\[
H_{(h,\phi)} (p) = h \left( \sum_k \phi(p_k) \right)
\]

\(\phi : [0 ; 1] \rightarrow \mathbb{R}\) and \(h : \mathbb{R} \rightarrow \mathbb{R}\),

- \(\phi\) is concave and \(h\) is increasing, or
- \(\phi\) is convex and \(h\) is decreasing


S. Zozor *et al.*
*Generalized quantum entropies: a definition and some properties*
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\( \phi : [0 ; 1] \rightarrow \mathbb{R} \) y \( h : \mathbb{R} \rightarrow \mathbb{R} \),

- \( \phi \) is concave and \( h \) is increasing, or
- \( \phi \) is convex and \( h \) is decreasing

**Moreover**

- \( \phi(0) = 0 \) (no elementary uncertainty associated to the probability 0)
- \( h(\phi(1)) = 0 \) (no uncertainty associated to a deterministic state)


S. Zozor et al., *Generalized quantum entropies: a definition and some properties*
## Famous Examples

<table>
<thead>
<tr>
<th></th>
<th>$\phi$</th>
<th>$h$</th>
<th>$H_{(h,\phi)} (p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shannon</td>
<td>$-x \ln x$</td>
<td>$x$</td>
<td>$- \sum_k p_k \ln p_k$</td>
</tr>
<tr>
<td>Rényi</td>
<td>$x^\alpha$</td>
<td>$\ln x / (1-\alpha)$</td>
<td>$\ln(\sum_k p_k^\alpha) / (1-\alpha)$</td>
</tr>
<tr>
<td>HCT</td>
<td>$x^\alpha$</td>
<td>$x-1 / (1-\alpha)$</td>
<td>$\sum_k p_k^\alpha - 1 / (1-\alpha)$</td>
</tr>
<tr>
<td>Unified</td>
<td>$x^r$</td>
<td>$x^{s-1} / (1-r)^s$</td>
<td>$(\sum_k p_k^r)^s - 1 / (1-r)^s$</td>
</tr>
<tr>
<td>Kaniadakis</td>
<td>$\frac{x^{1-\kappa} - x^{1+\kappa}}{2\kappa}$</td>
<td>$x$</td>
<td>$\sum_k \left( p_k^{1-\kappa} - p_k^{1+\kappa} \right) / 2\kappa$</td>
</tr>
</tbody>
</table>
Basic properties

For any pair of entropic functional \((h, \phi)\),

- Invariance to a permutation of the \(p_k\)'s

- Expansibility: 
  \[
  H_{(h,\phi)}([p_1 \cdots p_N \ 0]) = H_{(h,\phi)}([p_1 \cdots p_N])
  \]
  (consequence of \(\phi(0) = 0\))

- Fusion: 
  \[
  H_{(h,\phi)}([p_1 \ p_2 \cdots p_N]) \geq H_{(h,\phi)}([p_1 + p_2 \cdots p_N])
  \]
  (Petković’s inequality \(\phi(a + b) \leq \phi(a) + \phi(b)\) for concave \(\phi\) with \(\phi(0) = 0\))
Majorization

Definition

$p, p'$ proba. vectors, with components increasingly arranged,

$p \prec p'$  \hspace{10pt} (p \text{ is majorized by } p')$

if \hspace{10pt} \sum_{k=1}^{n} p_k \leq \sum_{k=1}^{n} p'_k \hspace{10pt} \forall \hspace{5pt} n < \max(N, N') \hspace{10pt} \& \hspace{10pt} \sum_{k=1}^{\max(N, N')} p_k = \sum_{k=1}^{\max(N, N')} p'_k$

Majorization is a partial order relationship
**Majorization**

**Definition**

$p, p'$ proba. vectors, with components increasingly arranged,

$$
p < p' \quad (p \text{ is majorized by } p')
$$

if

$$
\sum_{k=1}^{n} p_k \leq \sum_{k=1}^{n} p'_{k} \quad \forall \; n < \max(N,N') \quad \& \quad \sum_{k=1}^{\max(N,N')} p_k = \sum_{k=1}^{\max(N,N')} p'_{k}
$$

Majorization is a partial order relationship

**Examples**

For any $p$, of dimension $N$,

$$
\begin{bmatrix}
\frac{1}{N} & \cdots & \frac{1}{N}
\end{bmatrix} \prec \begin{bmatrix}
\frac{1}{\|p\|_0} & \cdots & \frac{1}{\|p\|_0} & 0 & \cdots & 0
\end{bmatrix} \prec \begin{bmatrix}
1 & 0 & \cdots & 0
\end{bmatrix}
$$
For any pair of entropic functionals \((h, \phi)\),

**Schur-concavity**

- \( p \prec p' \Rightarrow H_{(h,\phi)}(p) \geq H_{(h,\phi)}(p') \) (equality iif \( p \equiv p' \))

  (consequence of the Karamata’s theorem)

- Reciprocal if for all pairs \((h, \phi)\)
Properties linked to the majorization

For any pair of entropic functionals \((h, \phi)\),

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- Reciprocal if for all pairs \((h, \phi)\)

**Bounds**

\[
0 \leq H_{(h, \phi)}(p) \leq h \left( \|p\|_0 \phi \left( \frac{1}{\|p\|_0} \right) \right) \leq h \left( N \phi \left( \frac{1}{N} \right) \right)
\]

(certainty and uniform)  
(consequence of majorization relationships)
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Quantum \((h, \phi)\)-entropy: definition

Let \(\rho\) be a density operator acting on \(\mathcal{H}^N\) (\(\rho \geq 0\) hermitian, with \(\text{Tr} \rho = 1\))

**Definition**

\[
H_{(h,\phi)}(\rho) = h(\text{Tr} \phi(\rho))
\]

with \(\phi : [0; 1] \rightarrow \mathbb{R}, \phi(0) = 0\) \& \(h : \mathbb{R} \rightarrow \mathbb{R}, h(\phi(1)) = 0\),

- \(\phi\) is concave and \(h\) is increasing, or
- \(\phi\) is convex and \(h\) is decreasing

(for \(\rho = \sum_k \lambda_k \vert e_k \rangle \langle e_k \vert\), \(\phi(\rho) = \sum_k \phi(\lambda_k) \vert e_k \rangle \langle e_k \vert\))
Quantum vs classical \((h, \phi)\)-entropy

**Diagonal form**

\[
\rho = \sum_k \lambda_k \ket{e_k}\bra{e_k}
\]

where

- \(\{\ket{e_k}\}\) is the orthonomonal base of \(\mathcal{H}^N\) that diagonalizes \(\rho\),
- \(\lambda = [\lambda_1 \cdots \lambda_N] \in [0; 1]^N, \quad \sum_k \lambda_k = 1\) the eigenvalues of \(\rho\)
Quantum vs classical \((h, \phi)\)-entropy

**Diagonal form**

\[ \rho = \sum_k \lambda_k |e_k\rangle\langle e_k| \]

where

- \(\{|e_k\rangle\}\) is the orthonormal base of \(\mathcal{H}^N\) that diagonalizes \(\rho\),
- \(\lambda = [\lambda_1 \cdots \lambda_N] \in [0; 1]^N\), \(\sum_k \lambda_k = 1\) the eigenvalues of \(\rho\)

**Quantum vs classical**

\[ H_{(h, \phi)}(\rho) = H_{(h, \phi)}(\lambda) \]
By definition, $\rho \prec \rho'$ means that $\lambda \prec \lambda'$. 
Properties linked to the majorization

By definition, $\rho \prec \rho'$ means that $\lambda \prec \lambda'$

For any pair of entropic functionals $(h, \phi)$,

\textbf{Schur-concavidad (& recip.)}

$$\rho \prec \rho' \implies H_{(h,\phi)}(\rho) \geq H_{(h,\phi)}(\rho')$$

equality iif $\rho' = U\rho U^\dagger$ or $\rho = U\rho' U^\dagger$ with $U$ isometry ($U^\dagger U = I$)
By definition, \( \rho \prec \rho' \) means that \( \lambda \prec \lambda' \)

For any pair of entropic functionals \((h, \phi)\),

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**Bounds**

\[
0 \leq H_{(h,\phi)}(\rho) \leq h\left(\text{rank } \rho \phi \left(\frac{1}{\text{rank } \rho}\right)\right) \leq h\left(N\phi \left(\frac{1}{N}\right)\right)
\]

pure state \(|\psi\rangle\langle\psi|\) max. mixed \(\frac{I}{N}\)
Concavity

If $h$ is concave, then $H_{(h,\phi)}(\cdot)$ is concave,

$$H_{(h,\phi)}(\omega \rho + (1 - \omega) \rho') \geq \omega H_{(h,\phi)}(\rho) + (1 - \omega) H_{(h,\phi)}(\rho')$$

(Peierls’s inequality, $\text{Tr}(\rho) \leq \sum_k \phi(\langle f_k | \rho | f_k \rangle) \ & \ & \phi$ concave)
**Properties specific to the quantum context**

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If $h$ is concave, then $H_{(h,\phi)}(\cdot)$ is concave,

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**Mixture**

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \implies H_{(h,\phi)}(\rho) \leq H_{(h,\phi)}(p)$$

(Schrödinger’s mixture $p = B\lambda$, $B$ bistoch., Hardy-Littlewood-Pólya $p \prec \lambda$)
## Properties Specific to the Quantum Context

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If $h$ is concave, then $H_{(h,\phi)}(\cdot)$ is concave,

$$H_{(h,\phi)}(\omega \rho + (1 - \omega) \rho') \geq \omega H_{(h,\phi)}(\rho) + (1 - \omega) H_{(h,\phi)}(\rho')$$

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### Entropy vs Diagonal

$p^E(\rho)$ diag. $\rho$ in $E = \{e_k\}$ orth. base:

$$H_{(h,\phi)}(\rho) \leq H_{(h,\phi)}(p^E(\rho))$$

(Schur-Horn’s theorem: $p^E(\rho) \prec \lambda$)
Effect of a transform or a measure

Transform

- Invariance to a unitary transf. $U$ (e.g., time evolution)

$$H_{(h,\phi)} \left( U \rho U^\dagger \right) = H_{(h,\phi)} (\rho)$$

- Decrease s.t. bistochastic operation (e.g., general measure):

$E(\rho) = \sum_k A_k \rho A_k^\dagger, \quad \sum_k A_k^\dagger A_k = I = \sum_k A_k A_k^\dagger$ (complete)

$$H_{(h,\phi)} (E(\rho)) \geq H_{(h,\phi)} (\rho) \quad \text{ (information degradation)}$$

Equality iif $E(\rho) = U \rho U^\dagger, \quad U$ unitary

(Hardy-Littlewood-Pólya: $E(\rho) \prec \rho$)
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$$\sum_k A_k^\dagger A_k = I = \sum_k A_k A_k^\dagger$$ (complete)

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Equality iif $\mathcal{E}(\rho) = U\rho U^\dagger$, $U$ unitary

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**Consequence**

$$\{E_k\} \in \mathbb{E} \text{ rank one POVM, } p^E(\rho) = \text{Tr}(E_k\rho),$$

$$H_{(h,\phi)}(\rho) = \min_{\mathbb{E}} H_{(h,\phi)}(p^E(\rho))$$
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Let $\mathcal{H}^A \otimes \mathcal{H}^B$, $\rho^{AB}$, $\rho^A = \text{Tr}_B \rho^{AB}$, $\rho^B = \text{Tr}_A \rho^{AB}$

(Sub)additivity

- If (i) $\phi(ab) = \phi(a)b + a\phi(b)$ and $h(x + y) = h(x) + h(y)$, or
- (ii) $\phi(ab) = \phi(a)\phi(b)$ and $h(xy) = h(x) + h(y)$, then

$$H_{(h,\phi)}(\rho^A \otimes \rho^B) = H_{(h,\phi)}(\rho^A) + H_{(h,\phi)}(\rho^B)$$

(e.g., von Neuman, Rényi)

- $H_{(h,\phi)}(\rho^{AB}) \leq H_{(h,\phi)}(\rho^A \otimes \rho^B) \Rightarrow \phi(x) = -x \ln x$ (counterexample, except if $\phi$ satisfies a functional eq...)

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(counterexample, except if $\phi$ satisfies a functional eq...)
**ADDITIVITIES, PURE STATE**

Let $\mathcal{H}^A \otimes \mathcal{H}^B$, $\rho^{AB}$, $\rho^A = \text{Tr}_B \rho^{AB}$, $\rho^B = \text{Tr}_A \rho^{AB}$

**Sub)additivity**

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(counterexample, except if $\phi$ satisfies a functional eq....)

**Pure states**

$$\rho^{AB} = |\psi\rangle\langle\psi| \implies H_{(h,\phi)}(\rho^A) = H_{(h,\phi)}(\rho^B)$$

(Schmidt’s decomposition)
Separable states:

\[ \rho^{AB} = \sum_m \omega_m |\Psi_m^A\rangle\langle \Psi_m^A| \otimes |\Psi_m^B\rangle\langle \Psi_m^B| \quad \omega_m \geq 0, \sum_m \omega_m = 1 \]
Separable states:

\[ \rho^{AB} = \sum_m \omega_m |\Psi_m^A \rangle \langle \Psi_m^A| \otimes |\Psi_m^B \rangle \langle \Psi_m^B| \quad \omega_m \geq 0, \sum_m \omega_m = 1 \]

Separability inequality

If \( \rho^{AB} \) is separable, then

\[ H_{(h,\phi)}(\rho^{AB}) \geq \max \{ H_{(h,\phi)}(\rho^A), H_{(h,\phi)}(\rho^B) \} \]

(\( \rho^{AB} \prec \rho^A \) \& \( \rho^{AB} \prec \rho^B \))

Generalizable to multipartite systems and totally separable states

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Entanglement Detection: an example

Werner: \( \rho^{AB} = \omega |\Psi^-\rangle \langle \Psi^-| + (1 - \omega) \frac{I}{4} \), \( |\Psi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \)

Entangled iif \( \omega > \frac{1}{3} \); \( \rho^A = \rho^B = \frac{I}{2} \)
**Entanglement Detection: an example**

Werner: \( \rho^{AB} = \omega \ket{\Psi^-} \bra{\Psi^-} + (1 - \omega) \frac{I}{4}, \quad \ket{\Psi^-} = \frac{\ket{00} - \ket{11}}{\sqrt{2}} \)

Entangled iif \( \omega > \frac{1}{3} \); \( \rho^A = \rho^B = \frac{I}{2} \)

\( \phi(x) = x^\alpha, \quad h(x) = \frac{f(x)}{1 - \alpha} \)

**Detection**

Criterion: \( \frac{f \left( 3 \left( \frac{1 - \omega}{4} \right)^\alpha + \left( \frac{1 + 3\omega}{4} \right)^\alpha \right) - f \left( 2^{1 - \alpha} \right)}{\alpha - 1} > 0 \Rightarrow \text{entangled} \)
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Motivations & goals
Classical \((h, \phi)\)-entropies
Quantum \((h, \phi)\)-entropies
Composite quantum systems
Relative \((h, \phi)\)-entropies
Conclusions

PROGRAMA

1. Motivations & goals
2. Classical \((h, \phi)\)-entropies
   - Definition
   - Properties
3. Quantum \((h, \phi)\)-entropies
   - Definition
   - Basic properties
4. Composite quantum systems
   - Bipartite systems – (sub)additivity, pure state
   - \((h, \phi)\)-entropy and entanglement
5. Relative \((h, \phi)\)-entropies
   - Classical context
   - Quantum context
6. Conclusions

S. Zozor et al.
Generalized quantum entropies: a definition and some properties
Conditional probability: \( p^{A|B=b} = \frac{p^{AB}_{a,b}}{p^B_b} \)
Relative entropy and mutual information

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**From the conditional probability**

Relative entropy: \( H^J_{(h,\phi)} (A|B) = \sum_b p^B_b H_{(h,\phi)} (p^{A|B=b}) \)

Mutual information: \( J_{(h,\phi)} (A; B) = H_{(h,\phi)} (A) - H^J_{(h,\phi)} (A|B) \)

\( h \) concave guarantees that \( J_{(h,\phi)} \geq 0 \ldots \) \( J_{(h,\phi)} \) not symmetrical...
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$h$ concave guarantees that $J_{(h,\phi)} \geq 0$... $J_{(h,\phi)}$ not symmetrical...

From the chain rule

Relative entropy: $H^\mathcal{I}_{(h,\phi)}(A|B) = H_{(h,\phi)}(A, B) - H_{(h,\phi)}(B)$

Mutual information: $\mathcal{I}_{(h,\phi)}(A; B) = H_{(h,\phi)}(A) - H^\mathcal{I}_{(h,\phi)}(A|B)$

$\mathcal{I}_{(h,\phi)}$ symmetrical, but with no guarantee that $\mathcal{I}_{(h,\phi)} \geq 0$...
\{ \Pi^B \} \text{ local projective measurement:}
\begin{align*}
p^B_j &= \text{Tr} \left( I \otimes \Pi^B_j \rho^{AB} \right), \quad \rho^{A|\Pi^B_j} = \frac{I \otimes \Pi^B_j \rho^{AB} \otimes \Pi^B_j}{p^B_j}.
\end{align*}
{Π^B} local projective measurement:

\[ p_j^B = \text{Tr} \left( I \otimes \Pi_j^B \rho^{AB} \right), \quad \rho^{A|\Pi_j^B} = \frac{I \otimes \Pi_j^B \rho^{AB} I \otimes \Pi_j^B}{p_j^B} \]

From the conditional state

Relative entropy vs Π^B:  \( H^J_{(h,\phi)} (A|\Pi^B) = \sum_j p_j^B H_{(h,\phi)} (\rho^{A|\Pi_j^B}) \)

Relative entropy vs B:  \( H^J_{(h,\phi)} (A|B) = \min \{\Pi^B\} H^J_{(h,\phi)} (A|\Pi^B) \)
Relative entropy and mutual information

\[ \{ \Pi^B \} \text{ local projective measurement:} \]

\[
p^B_j = \text{Tr} \left( I \otimes \Pi_j^B \rho^{AB} \right), \quad \rho^{A|\Pi_j^B} = \frac{I \otimes \Pi_j^B \rho^{AB} I \otimes \Pi_j^B}{p_j^B}
\]

From the conditional state

Relative entropy vs \( \Pi^B \): 

\[
H_{(h, \phi)}^J (A|\Pi^B) = \sum_j p^B_j H_{(h, \phi)} \left( \rho^{A|\Pi_j^B} \right)
\]

Relative entropy vs \( B \): 

\[
H_{(h, \phi)}^J (A|B) = \min_{\{\Pi^B\}} H_{(h, \phi)}^J (A|\Pi^B)
\]

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Relative entropy: 

\[
H_{(h, \phi)}^J (A|B) = H_{(h, \phi)} (A, B) - H_{(h, \phi)} (B)
\]
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S. Zozor et al.
Generalized quantum entropies: a definition and some properties
Summary

- We proposed an extension of the \((h, \phi)\)-entropies for the quantum systems (that extends the trace-entropies).

- These extensions are based on two entropic functionals \(\phi \& h\), and encompass various famous entropies such that the von Neuman’s, Tsallis’s, Rényi’s (thanks to \(h\)), unified, trace entropies or not.

- We proposed possible associated measures such that relative entropies and mutual informations; a unified point of view is still missing; there properties remain to be investigated.
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We studied various properties shared by the whole family; the main ones rely on the notion of majorization.

In particular, the Schur-concavity appears to be crucial in the quantum context.

We studied the effect of quantum operations (unitary transform, measures) on these entropies.

We studied their properties for composite systems: they allow to propose entanglement detection criteria.
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