Fuzzy Representation of Quantum Fredkin Gate

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in collaboration with
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Quantum Computation : Notions and Notations

- In Quantum Computation, information is encoded into and processed by means of quantum systems.
- A *Qubit* is a quantum-bit of information. It corresponds to a pure qunatum state representable by a ray-vector of the 2d Hilbert space $\mathbb{C}^2$.
- The *standard orthonormal basis* $\{|0\rangle, |1\rangle\}$ of the 2-d Hilbert Space ($\mathbb{C}^2$) is generally taken as the *quantum computational basis*.
- The projection of a Qubit state vector on to $|1\rangle$ is taken to be related to the *logical truth value* of the corresponding Qubit, and $|0\rangle$ to the *logical falsity*.

. . . Notations

To stress that an operator A is defined on a Hilbert space of the form $H^{(n)} \in \otimes^{2^n}\mathbb{C}^2$, we denote it as $A^{(n)}$.

A quantum state vector $|x_1\rangle \otimes |x_2\rangle \otimes \ldots \otimes |x_n\rangle \equiv |x_1, \ldots, x_n\rangle$ is taken to be a Q-register encoding

- the logical TRUENESS with a probability $\langle x_n|1\rangle$, and
- the logical FALSITY with a probability $\langle x_n|0\rangle$.

Logical Gates for Universal Computation

Gates of Classical Computation

- A Logical Gate is a circuit-element that performs on its input states an elementary logical operation like NOT, AND, OR, XOR etc.
- Universal Gates: One Gate to emulate them all...
  $e.g.$, NAND, Toffoli, Fredkin.
- Logical reversibility: one-to-one relation between input and output. E.g., Fredkin, Tofolli etc

Logical Gates for Universal Computation

Gates of Quantum Computation

- Representable as unitary operators upon Hilbert spaces.
- Possible to construct infinitely many quantum gates.
- Quantum Universality?
  One finite set of Quantum gates to approximately mimic any possible Quantum gate. E.g., Tofolli, Fredkin . . .
- Quantum Gates, represented as unitary operators, acting on pure state vectors, are therefore reversible -by construction.

3-bit Gates with Reversible Logic

Toffoli Gate

- It implements a Controlled-Controlled-Not operation: $T(x,y,z) = (x,\ y,\ xy \mathbin{\widehat{+}} z)$, where, $\widehat{+}$ is addition modulo 2.
- It is logically *reversible* but not conservative: the *bit-parity* of its output is not same as that of its input - in general.

Fredkin Gate

- It implements a Controlled-Swap operation:

$$F(x,y,z) = (x,\ y \mathbin{\widehat{+}} x(y \mathbin{\widehat{+}} z),\ z \mathbin{\widehat{+}} x(y \mathbin{\widehat{+}} z))$$

- It is *logically reversible*, *conserves parity* as well.

The Conservative Logic

- The number of 1's present in the output of the gate is the same as the number of $1'$s as was in its input.
- In other words, the *parity of bits* remains unchanged during the operation of logically-conservative gates like the Fredkin Gate.
- E.g., if the bits are to be encoded by the spin-half systems, the logical conservativity of a gate implies that the number of spin-up (or, equivalently the spin-down) states would remain unchanged during the operational cycles of that gate.

Reversibility, Conservativity and Thermo-economy

- Landauer type of heat generation in physical systems: *1 bit of information lost irreversibly would irrefutably amount to a heat generation of* $KT \ln 2$ - *at the least*.
- Logical Reversibility: If inputs of a logical gates are recoverable by using its outputs. i.e., one-one correspondence between output and input.
- If a gate-module in a given circuit is logically irreversible, then, it must be the case that some information about the input states is lost from the gate-module in question.
- This mysterious part of information may either be irreversibly lost –resulting in heat-dissipation, or
- be just *hidden away* (in a *deterministically retrievable* manner) in some other module of the physical circuit, –in which case it may not be resulting in a heat generation, but perhaps costing a memory-resource overhead.

- In general, *logical reversibility* does not automatically guarantee *thermodynamic reversibility*.
- E.g., in atomic / quantum optical systems, say in two-state systems (i.e., qubit) if, during an operation, the ground-state $|1\rangle$ could be flipped to the excited state $|0\rangle$, an additional re-pumping of populations would be required to maintain the excited state – to counter the dissipations due to spontaneous emission.
- With parity conservation between the input and output of a gate – in addition to the logical reversibility, however, there could be more room for a circumvention of Landauer type of heat generation in physical implementations.
- It is possible to deduce the amount of information lost during a gate operation using concepts of information-entropy.

Quantum Operations upon Density of States

- It is hard to find /prepare perfectly pure quantum states, due to a variety of reasons such as the limitations in preparation procedures, the *decoherence* due to interactions with environment, etc.

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- Density matrices are better choice to represent quantum states.
- Not all quantum processes are representable as Unitary operators; exceptions include *quantum measurements*. They are better modeled as *quantum operations* using operator-sums due to Kraus.

Density Matrices for Quantum Computation

- Corresponding to the quantum computational basis vectors $\{|0\rangle, |1\rangle\}$ in $\mathbb{C}^2$, we associate the density operators:

$$P_0 = |0\rangle\langle 0| \text{ and } \mathrm{P}_1 = |1\rangle\langle 1| \ .$$

- Generalizing to encodings in higher dimensions, the n-qubit-basis density operators in $\otimes^n \mathbb{C}^2$ are given by

$$P_i^{(n)} \equiv (\otimes^{2^{(n-1)}} I) \otimes P_i \ ,$$

where $i = \{0, 1\}$, and I is the 2×2 identity matrix.

- The logical truth and false probabilities are then obtainable via the Born rule:

$$p(\rho) = \mathrm{tr}\left[P_1^{(2^n)} \rho\right] \ .$$

Kraus-Representation of the Quantum Operations

Kraus-Representation

- A *quantum operation* $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ is a linear operator taking density matrices in H_1 to density matrices in H_2.
- It is representable as $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$, with the operators A_i satisfying $\sum_i A_i^\dagger A_i = I$.
- Each unitary operator U gives rise to a quantum operation $\mathcal{O}_\mathcal{U}$ such that $\mathcal{O}_\mathcal{U}(\rho) = \mathcal{U}\rho\mathcal{U}^\dagger$ for any density operator ρ.

This model of *quantum operations* acting on *density operators* is referred to as *quantum computation with mixed states*.

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- A set of operations $\langle \oplus, \cdot, \neg \rangle$ over the interval $[0, 1]$, as in above, forms an algebraic structure called *Product Multi-Valued Algebra* (PMV-algebra).

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- A set of operations $\langle \oplus, \cdot, \neg \rangle$ over the interval $[0, 1]$, as in above, forms an algebraic structure called *Product Multi-Valued Algebra* (PMV-algebra).
- In the present work, we extend this analysis towards Fredkin Gate, especially for its bit-parity conservation property.

Fuzzy Quantum Toffoli Gate : Definition

For any natural numbers $n, m, l \geq 1$ and for any vectors of the standard orthonormal basis $|x\rangle = |x_1, x_2, \ldots, x_n\rangle \in \otimes^n \mathbb{C}^2$, $|y\rangle = |y_1, y_2, \ldots, y_m\rangle \in \otimes^m \mathbb{C}^2$ and $|z_1, z_2 \ldots, z_l\rangle \in \otimes^l \mathbb{C}^2$, the quantum Tofolli gate $T^{(n,m,l)}$ on $\otimes^{n+m+l}\mathbb{C}^2$ is defined to satisfy

$$T^{(m,n,l)}(|x\rangle \otimes |y\rangle \otimes |z\rangle) = |x\rangle \otimes |y\rangle |z_1, \ldots z_{l-1}\rangle \otimes |x_m y_n \widehat{+} z_l\rangle.$$

Fuzzy Quantum Toffoli Gate : Definition

Quantum Toffoli Gate: Matrix Form

For any natural number $n, m, l \geq 1$,

$$\begin{aligned} T^{(m,n,l)} &\equiv P_0^{(n)} \otimes P_0^{(m)} \otimes I^{(l)} \; + \; P_1^{(n)} \otimes P_1^{(m)} \otimes Not^{(l)} \\ &= I^{(n+m+l)} \; + \; P_1^{(n)} \otimes P_1^{(n)} \otimes P_1^{(m)} \otimes (Not - I)^{(l)} \\ &= I^{(n-1)} \otimes \left[\begin{array}{c|c} I^{(m+1)} & \mathbf{0} \\ \hline \mathbf{0} & I^{(m-1)} \otimes Xor^{(l)} \end{array} \right] \end{aligned}$$

- Matrix form of a quantum gate is encoding dependent.
- The above form is readily seen to be Unitary, leading to the Toffoli quantum operation: $\mathbb{T}^{(m,n,1)}(\rho) \equiv T^{(m,n,l)} \rho T^{(m,n,l)}$.

Quantum Toffoli Gate : Fuzzy properties

- The truth-probability of Quantum Toffoli Operation $(p(\mathbb{T}^{(m,n,l)(\rho_n \otimes \rho_m \otimes \rho_l)})$ turns out to be

$$(1 - p(\rho_l))p(\rho_n)p(\rho_m) + p(\rho_l)(1 - p(\rho_m)p(\rho_n)) \ .$$

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- The Quantum Tofolli Operation then has a representation in terms of $\langle \oplus, \cdot, \neg \rangle_3$ given by

$$\mathbb{T}^{(m,n,l)}(x, y, z) = \neg z \cdot x \cdot y \ \oplus \ z \cdot \neg(x \cdot y) \ .$$

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- At the opposite sides of spectrum are the probabilities for the implementation of AND and NAND.
- The probability expression picks up a *fuzzy-component* if the input states are non separable.

Fuzzy quantum Fredkin Gate : Definitions

For any natural numbers $n, m, l \geq 1$ and for any vectors of the standard orthonormal basis $|x\rangle = |x_1, x_2, \ldots, x_n\rangle \in \otimes^n \mathbb{C}^2$, $|y\rangle = |y_1, y_2, \ldots, y_m\rangle \in \otimes^m \mathbb{C}^2$ and $|z_1, z_2 \ldots, z_l\rangle \in \otimes^l \mathbb{C}^2$, the Fredkin quantum gate $F^{(n,m,l)}$ on $\otimes^{n+m+l} \mathbb{C}^2$ is defined to satisfy

$$F^{(n,m,l)}|x, y, z\rangle$$

$$= |x\rangle |y_1 \ldots y_{m-1},\ y_m \widehat{+} x_n (y_m \widehat{+} z_l)\rangle |z_1 \ldots z_{l-1},\ z_l \widehat{+} x_n (y_m \widehat{+} z_l)\rangle.$$

Fuzzy Quantum Fredkin Gate : Definitions

Quantum Fredkin Gate: Matrix Form

For any natural number $n, m, l \geq 1$, the generalized Fredkin Gate has the form

$$\begin{aligned} F^{(m,n,l)} &= P_0^{(n)} \otimes I^{(m+l)} \;+\; P_1^{(n)} \otimes SWAP^{(m,l)} \\ &= I^{(n+m+l)} \;+\; P_1^{(n)} \otimes (SWAP^{(m,l)} - I^{(m+l)}) \\ &= I^{(n-1)} \;\otimes\; \left[\begin{array}{c|c} I^{(m+l)} & \mathbf{0} \\ \hline \mathbf{0} & SWAP^{(m,l)} \end{array}\right] \end{aligned}$$

Here, $SWAP^{(m,l)}$ is a linear operator that is essentially a last-qubit swap gate, such that,

$$\mathrm{SWAP}^{(m,l)}|y_1, \ldots, y_m, z_1, \ldots, z_l\rangle = |y_1, \ldots, y_{m-1}, z_l, z_1, \ldots, z_{l-1}, y_m\rangle$$

Fuzzy Quantum Fredkin Gate : Definitions

The general matrix form (in computational basis) of the required $SWAP^{(m,l)}$ is as follows:

$$\begin{aligned} SWAP^{(m,l)} &= I^{(m-1)} \otimes \left[\begin{array}{c|c} \mathbf{P_0^{(l)}} & \mathbf{L_1^{(l)}} \\ \hline \mathbf{L_0^{(l)}} & \mathbf{P_1^{(l)}} \end{array} \right] \\ &= I^{(m-1)} \otimes \; SWAP^{(1,l)} \\ &= diag^{2^{(m-1)}} \left[SWAP^{(1,l)} \right] . \end{aligned}$$

Here, the operators L_1 and L_0 are the *bit-flip* operators with $L_1 \equiv |1\rangle\langle 0|$ ($i.e.$, the Ladder-raising operator) and $L_0 \equiv |0\rangle\langle 1|$ ($i.e.$, the Ladder-lowering operator), trivially extended to the higher dimensions as $\mathbf{L_1^{(l)}} = I^{(l-1)} \otimes L_1$ and $\mathbf{L_0^{(l)}} = I^{(l-1)} \otimes L_0$.

Quantum Fredkin Gate : Fuzzy properties

- The truth-probability of the generalized Quantum Fredkin Operation $(p(\mathbb{F}^{(m,n,l)(\rho_n \otimes \rho_m \otimes \rho_l)})$ turns out to be

$$\mathbb{F}_p^{(n,m,l)}\rho = (1 - p(\rho_n))\, p(\rho_l) + p(\rho_n)\, p(\rho_m)$$

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- Since $0 \leq p(\mathbb{F}^{(n,m,l)}(\rho_n \otimes \rho_m \otimes \rho_l)) \leq 1$, the above sum is a Łukasiewicz sum. And therefore, it can be rewritten as

$$p(\mathbb{F}^{(n,m,l)}(\rho_n \otimes \rho_m \otimes \rho_l)) \;=\; \neg p(\rho_n) \cdot p(\rho_l) \;\oplus\; p(\rho_n) \cdot p(\rho_m).$$

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- Therefore, $\mathbb{F}^{(m,n,l)}$ is $\langle \oplus, \cdot, \neg \rangle_3$-representable by $\neg \mathbf{x} \cdot \mathbf{z} \;\oplus\; \mathbf{x} \cdot \mathbf{y}$.

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- Therefore, $\mathbb{F}^{(m,n,l)}$ is $\langle \oplus, \cdot, \neg \rangle_3$-representable by $\neg \mathbf{x} \cdot \mathbf{z} \;\oplus\; \mathbf{x} \cdot \mathbf{y}$.
- The Conjunction derivable here is the same as the Holistic Conjunction derived previously using Tofolli gate.

Directions Further

- The incidence of non-separability in generalized quantum Fredkin gate and that of entanglement . . .
- Benchmarking the robustness and thermo-economicality of fuzzy-quantum-circuits, perhaps for some important family of quantum states.
- Forms of fuzzy-quantum gates for different types of encoding, OR,
- the possibility of Encoding-independent representation of Quantum Gates
- Building further implementation-friendly fuzzy-quantum circuits.
- Building Fuzzy logic based quantum games.

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