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# EXACT SOLUTIONS OF INTEGRABLE NONLINEAR EVOLUTION EQUATIONS

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CONTENTS:

"... I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stop – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, arounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still on a rate of some eight or nine miles an hour, preserving its original figure .... in the month of August 1834 was my first chance inteview with that singular and beatiful phenomenon which I have called the Wave of Translation.....The first day I saw it it was the happiest day of my life" [Scott Russell, 1834]

The experiment was conducted on the Union Canal between Edinburgh and Glasgow and, scaled down, in Scott Russell's garden/garage.

Empirical formula: $c^2 = g(h+\eta)$, where $g$ is gravity, $h$ the depth of the channel, and $\eta$ the maximal height of the wave.

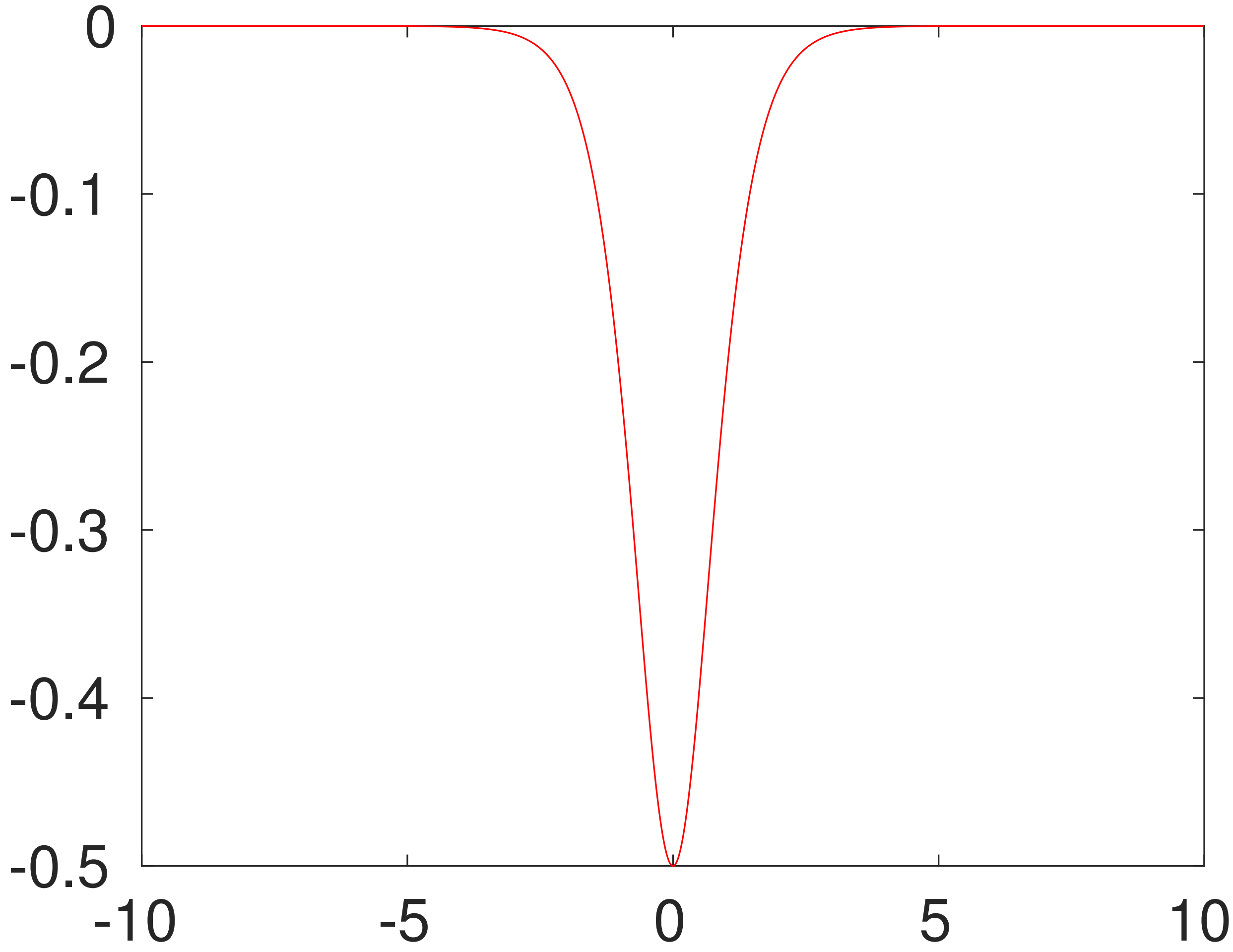
The phenomenon was generally dismissed [e.g., by Airy (1845)]. A theoretical explanation was given by Boussinesq (1871) and by Korteweg and De Vries (1895). The latter derived the (dimensionless) equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

The traveling wave solution found by Boussinesq and by Korteweg and de Vries has the form

$$u(x,t) = \frac{-\frac{1}{2}c}{\cosh^2(x - x_0 - ct)},$$

where $c > 0$ is the speed as well as half the amplitude and $x_0 \in \mathbb{R}$ is the position of the extreme value.




KdV one-soliton as a function of $x - x_0 - ct$ for $c = 1$

In 1954 Fermi, Pasta and Ulam [plus the programmer Tsingou] studied numerically a system of 64 springs, each of which is connected in a nonlinear way to its neighbours. The system is as follows:

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1})[1 + \alpha(x_{j+1} - x_j - 1)], \qquad j = 0, 1, \ldots, 63,$$

expecting to find equipartition of the energy between the springs. Instead a travelling wave was found. The Los Alamos report disappeared in an archive for eight years.

In 1965 Kruskal and Zabusky observed that, by taking the limit in an appropriate way, the difference equation gives rise to the Korteweg-de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

These authors introduced the word soliton.

In 1967 Gardner, Greene, Kruskal, and Miura (GGKM) presented the so-called inverse scattering transform (IST) method to solve the Korteweg-de Vries (KdV) equation

$$Q_t - 6QQ_x + Q_{xxx} = 0, \qquad (x,t) \in \mathbb{R}^2.$$

$$\begin{tikzcd}
Q(x,0) \arrow[r, "\text{direct scattering}"] \arrow[d, "\text{KdV}"] & \{R(k,0), \{\kappa_s\}_{s=1}^N, \{C_s(0)\}_{s=1}^N\} \arrow[d, "\text{time evolution}"] \\
Q(x,t) & \{R(k,t), \{\kappa_j\}_{s=1}^N, \{C_s(t)\}_{s=1}^N\} \arrow[l, "\text{inverse scattering}"']
\end{tikzcd}$$

where

$$R(k,t) = e^{8ik^3t} R(k,0), \qquad C_s(t) = e^{8\kappa_s^3 t} C_s(0).$$

Consider the Schrödinger equation on the line

$$-\psi_{xx}(k,x,t)+Q(x,t)\psi(k,x,t)=k^2\psi(k,x,t),$$

where $\operatorname{Im} k \geq 0$. Then the scattering data consist of the Jost solution from the right

$$f_r(k,x,t) \simeq \begin{cases} e^{-ikx}, & x\to -\infty, \\ \frac{1}{T(k)}e^{-ikx}+\frac{R(k,t)}{T(k)}e^{ikx}, & x\to +\infty, \end{cases}$$

the (finitely many and simple) poles $i\kappa_s$ of the transmission coefficient, and

the (positive) norming constants $C_s(t)=\left[\int_{-\infty}^{\infty} dx\, f_r(i\kappa_s,x,t)^2\right]^{-1}$.

The potential $Q(x,t)$ is to be Faddeev class in the sense that it is real-valued and satisfies $\int_{-\infty}^{\infty} dx\,(1+|x|)|Q(x,t)|<+\infty$.

The direct and inverse scattering theory of the Schrödinger equation on the line for Faddeev class potentials was largely developed by Faddeev (1964).

In 1972 Zakharov and Shabat (ZS) presented the inverse scattering transform (IST) method to solve the nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} \pm 2|u|^2 u = 0, \qquad (x,t) \in \mathbb{R}^2,$$

where the plus sign corresponds to the focusing case and the minus sign to the defocusing case. In the focusing case we have

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{direct scattering}"] \arrow[d, "\text{NLS}"] & \{R(k,0), \{a_s\}_{s=1}^N, \{C_s(0)\}_{s=1}^N\} \arrow[d, "\text{time evolution}"] \\
u(x,t) & \{R(k,t), \{a_s\}_{s=1}^N, \{C_s(t)\}_{s=1}^N\} \arrow[l, "\text{inverse scattering}"']
\end{tikzcd}$$

where

$$R(k,t) = e^{4ik^2 t} R(k,0), \qquad C_s(t) = e^{-4ia_s^2 t} C_s(0).$$

Consider the Zakharov-Shabat system

$$\boldsymbol{v}_x = \begin{pmatrix} -ik & u(x,t) \\ \mp u(x,t)^* & ik \end{pmatrix} \boldsymbol{v}.$$

Then the scattering data consist of the Jost solution from the right

$$\phi(k,x,t) \simeq \begin{cases} e^{-ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, & x \to -\infty, \\ \begin{pmatrix} \frac{1}{T(k)} e^{-ikx} \\ \frac{R(k,t)}{T(k)} e^{ikx} \end{pmatrix}, & x \to +\infty, \end{cases}$$

the (finitely many and simple) poles $ia_s$ of the transmission coefficient, and the (complex nonzero) norming constants $C_s(t)$.

The complex potential $u(x,t)$ is to belong to $L^1(\mathbb{R})$.

In the defocusing case the scattering data only consist of the reflection coefficient $R(k,t)$.

Nonlinear evolution equations are called integrable if their initial-value problem can be solved by a suitable inverse scattering transform. This means in particular that this equation is associated with a linear eigenvalue problem. The IST translates the time evolution of the potential into that of the scattering data associated with the linear eigenvalue problem.

PROPERTIES OF INTEGRABLE SYSTEMS:

- Admitting a class of exact solutions, many of soliton or breather type.
- Being an integrable Hamiltonian system in the sense that the IST constitutes a canonical transformation from physical variables to action-angle variables.
- Having infinitely many conserved quantities.

## HOW TO GENERATE INTEGRABLE SYSTEMS: LAX PAIRS

Lax (1968): Let the associated linear eigenvalue problem be $Lu = \lambda u$. Starting from an additional linear operator $A$, we get

$$L_t + LA - AL = 0.$$

EXAMPLE:

$$L = -\frac{d^2}{dx^2} + u(x,t), \qquad A = -4\frac{d^3}{dx^3} + 6u\frac{d}{dx} + 3u_x.$$

Then we get the KdV equation $u_t + u_{xxx} - 6uu_x = 0$.

## HOW TO GENERATE INTEGRABLE SYSTEMS: AKNS PAIRS

Ablowitz, Kaup, Newell, and Segur (1974): Consider the pair of differential equations

$$V_x = XV, \qquad V_t = TV,$$

where $X$ and $T$ are square matrices depending on $(x, t, \lambda)$, $\lambda$ being as-pectral parameter, and $\det V(x, t, \lambda) \not\equiv 0$. Then

$$(X_t + XT)V = (XV)_t = (V_x)_t = (V_t)_x = (TV)_x = (T_x + TX)V,$$

implying the so-called zero curvature condition

$$X_t - T_x + XT - TX = 0.$$

The inverse scattering transform (IST) method consists of three major steps:

- DIRECT SCATTERING: Compute the scattering data from the initial solution (potential). These scattering data can be "summarized" as the initial Marchenko integral kernel.

- (Usually trivial) time evolution of the scattering data, including the (usually trivial) time evolution of the Marchenko integral kernel.

- INVERSE SCATTERING: Solve the Marchenko integral equation at time $t$ and apply the formula to get the potential from its solution.

Instead of solving the Marchenko integral equation, we can alternatively solve a Riemann-Hilbert problem.

The Marchenko integral equation has the form

$$\boldsymbol{K}(x,y;t)+\boldsymbol{F}(x+y;t)+\int_x^\infty dz\,\boldsymbol{K}(x,z;t)\boldsymbol{F}(z+y;t)=\boldsymbol{0},$$

and the potential $u(x,t)$ follows directly from $\boldsymbol{K}(x,x;t)$.

In this talk we focus on situations where

$$\boldsymbol{F}(x+y;t)=\boldsymbol{F}_1(x;t)\boldsymbol{F}_2(y;t)$$

for suitable matrix functions $\boldsymbol{F}_1(x;t)$ and $\boldsymbol{F}_2(y;t)$.

Now consider the matrix triplet $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}; \boldsymbol{H})$, where $\boldsymbol{A}$ has only eigenvalues with positive real part, $\boldsymbol{H}$ commutes with $\boldsymbol{A}$, and

$$\boldsymbol{F}(x+y;t) = \boldsymbol{C}e^{-(x+y)\boldsymbol{A}}e^{t\boldsymbol{H}}\boldsymbol{B} = \underbrace{\boldsymbol{C}e^{-x\boldsymbol{A}}}_{=\boldsymbol{F}_1(x;t)}\ \underbrace{e^{-y\boldsymbol{A}}e^{t\boldsymbol{H}}\boldsymbol{B}}_{=\boldsymbol{F}_2(y;t)}.$$

Then

$$\boldsymbol{G}(x;t) = e^{-x\boldsymbol{A}}e^{t\boldsymbol{H}}\int_0^\infty dz\, e^{-z\boldsymbol{A}}\boldsymbol{B}\boldsymbol{C}e^{-z\boldsymbol{A}}e^{-x\boldsymbol{A}} = e^{-x\boldsymbol{A}}e^{t\boldsymbol{H}}\boldsymbol{P}e^{-x\boldsymbol{A}}.$$

Consequently,

$$\boldsymbol{K}(x,y;t) = -\boldsymbol{C}e^{-x\boldsymbol{A}}\left[I + e^{-x\boldsymbol{A}}e^{t\boldsymbol{H}}\boldsymbol{P}e^{-x\boldsymbol{A}}\right]^{-1}e^{-y\boldsymbol{A}}e^{t\boldsymbol{H}}\boldsymbol{B}.$$

Here $\boldsymbol{P}$ is the (unique) solution to the Sylvester equation

$$\boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}.$$

Focusing NLS: Here the Marchenko integral kernel has the $2 \times 2$ matrix form

$$\boldsymbol{F}(x+y;t) = \begin{pmatrix} 0 & -F(x+y;t)^* \\ F(x+y;t) & 0 \end{pmatrix},$$

where $F_t + 8F_{xxx} = 0$. More precisely,

$$F(x+y;t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x+y)} e^{8ik^3 t} R(k,0) + \sum_{s=1}^{N} \sum_{j=0}^{n_s-1} \frac{(x+y)^j}{j!} C_{sj}(t).$$

Putting

$$F(x+y;t) = Ce^{-(x+y)A} e^{tH} B, \qquad H = -4iA^2,$$

we write

$$\boldsymbol{F}(x+y;t) = \begin{pmatrix} B^\dagger & 0_{1\times p} \\ 0_{1\times p} & C \end{pmatrix} \begin{pmatrix} e^{-(x+y)A^\dagger + tH^\dagger} & 0_{p\times p} \\ 0_{p\times p} & e^{-(x+y)A+tH} \end{pmatrix} \begin{pmatrix} 0_{p\times 1} & -C^\dagger \\ B & 0_{p\times 1} \end{pmatrix}.$$

Then

$$
\boldsymbol{P} = \int_0^\infty dz \begin{pmatrix} e^{-zA^\dagger} & 0_{p\times p} \\ 0_{p\times p} & e^{-zA} \end{pmatrix} \begin{pmatrix} 0_{p\times 1} & -C^\dagger \\ B & 0_{p\times 1} \end{pmatrix} \begin{pmatrix} B^\dagger & 0_{1\times p} \\ 0_{1\times p} & C \end{pmatrix} \begin{pmatrix} e^{-zA^\dagger} & 0_{p\times p} \\ 0_{p\times p} & e^{-zA} \end{pmatrix}
$$

$$
= \begin{pmatrix} 0_{p\times p} & -Q \\ N & 0_{p\times p} \end{pmatrix},
$$

where

$$
N = \int_0^\infty dz\, e^{-zA} B B^\dagger e^{-zA^\dagger}, \qquad Q = \int_0^\infty dz\, e^{-zA^\dagger} C^\dagger C e^{-zA},
$$

solve the Lyapunov equations

$$
AN + NA^\dagger = BB^\dagger, \qquad A^\dagger Q + QA = C^\dagger C.
$$

Observe that

$$
\langle N\boldsymbol{x}, \boldsymbol{x}\rangle = \int_0^\infty dz \left\| B^\dagger e^{-zA^\dagger} \boldsymbol{x} \right\|^2, \qquad \langle Q\boldsymbol{x}, \boldsymbol{x}\rangle = \int_0^\infty dz \left\| C e^{-zA} \boldsymbol{x} \right\|^2,
$$

are both nonnegative.

Writing

$$\boldsymbol{K}(x,y;t) = \begin{pmatrix} \overline{K}^{\mathsf{up}}(x,y;t) & K^{\mathsf{up}}(x,y;t) \\ \overline{K}^{\mathsf{dn}}(x,y;t) & K^{\mathsf{dn}}(x,y;t) \end{pmatrix},$$

we obtain

$$\begin{aligned}
\overline{K}^{\mathsf{up}}(x,y;t) &= -B^\dagger e^{-2xA^\dagger} e^{tH^\dagger} Q e^{-xA} \tilde{\Gamma}(x,t)^{-1} e^{-yA} e^{tH} B, \\
\overline{K}^{\mathsf{dn}}(x,y;t) &= -C e^{-xA} \tilde{\Gamma}(x,t)^{-1} e^{-yA} e^{tH} B, \\
K^{\mathsf{up}}(x,y;t) &= B^\dagger e^{-xA^\dagger} \Gamma(x,t)^{-1} e^{-yA^\dagger} e^{tH^\dagger} C^\dagger, \\
K^{\mathsf{dn}}(x,y;t) &= -C e^{-2xA} e^{tH} N e^{-xA^\dagger} \Gamma(x,t)^{-1} e^{-yA^\dagger} e^{tH^\dagger} C^\dagger,
\end{aligned}$$

where

$$\begin{aligned}
\Gamma(x,t) &= I_p + e^{-xA^\dagger} e^{tH^\dagger} Q e^{-2xA} e^{tH} N e^{-xA^\dagger} \\
\tilde{\Gamma}(x,t) &= I_p + e^{-xA} e^{tH} N e^{-2xA^\dagger} e^{tH^\dagger} Q e^{-xA}.
\end{aligned}$$

The focusing NLS solution is given by

$$u(x,t) = -2K^{\text{up}}(x,x;t) = -2B^\dagger e^{-xA^\dagger}\Gamma(x,t)^{-1}e^{-xA^\dagger}e^{tH^\dagger}C^\dagger,$$
$$u(x,t)^* = 2\overline{K}^{\text{dn}}(x,x;t) = -2Ce^{-xA}\tilde{\Gamma}(x,t)^{-1}e^{-xA}e^{tH}B.$$

Alternatively,

$$|u(x,t)|^2 = \frac{d^2}{dx^2}\log[\det\Gamma(x,t)],$$

where

$$\Gamma(x,t) = I_p + e^{-xA^\dagger}e^{tH^\dagger}Qe^{-2xA}e^{tH}Ne^{-xA^\dagger}.$$

## MODIFICATIONS TO THE FOCUSING NLS:

Instead of $H = -4iA^2$, we choose another matrix $\boldsymbol{H}$ commuting with $\boldsymbol{A}$, thus another time factor.

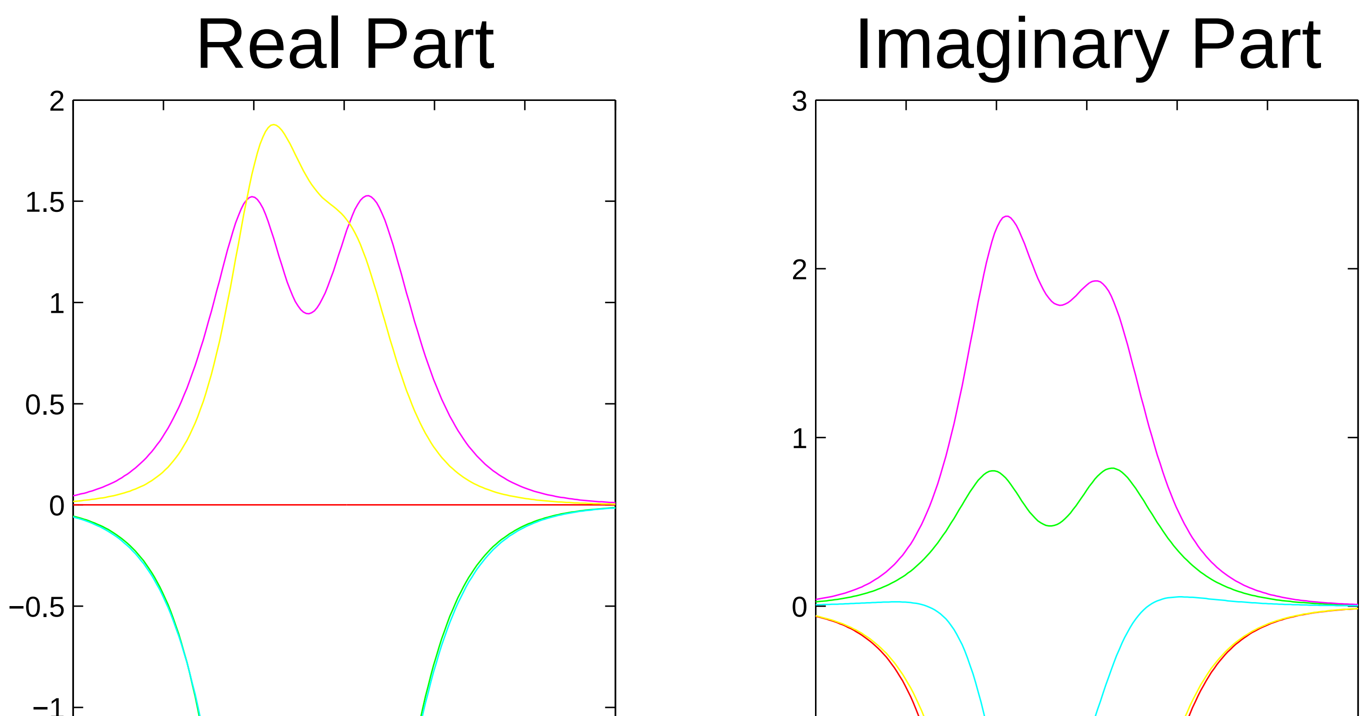
$$H = -4iA^2: \qquad u_t - iu_{xx} - 2i|u|^2 u = 0 \qquad \text{NLS},$$

$$H = 8A^3: \qquad u_t + u_{xxx} - 6u^2 u_x = 0 \qquad \text{mKdV},$$

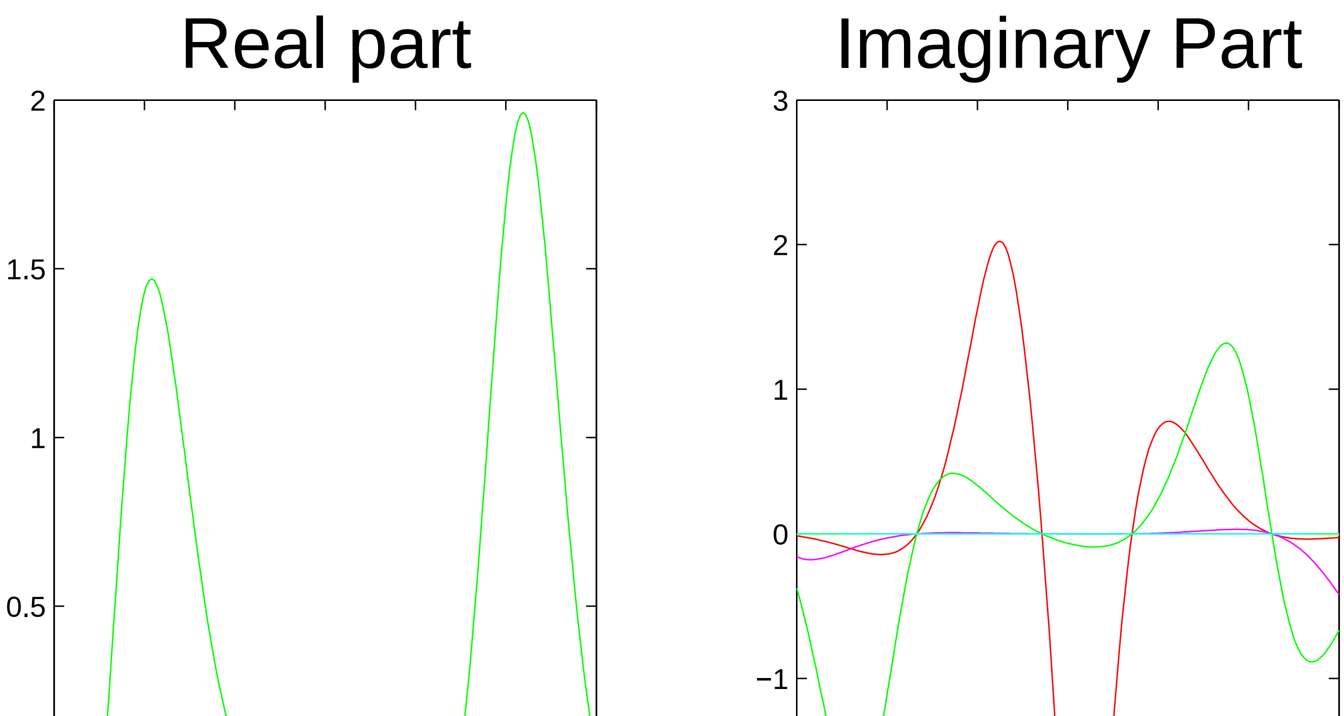
$$H = -\tfrac{1}{2}A^{-1}: \qquad v_{xt} = \sin(v), \quad u = \tfrac{1}{2}v_x \qquad \text{sine-Gordon},$$

$$H = [(4i\alpha_2 A^2 + 8\alpha_3 A^3)/(\alpha_3 - \alpha_2)]: \quad \text{Hirota},$$

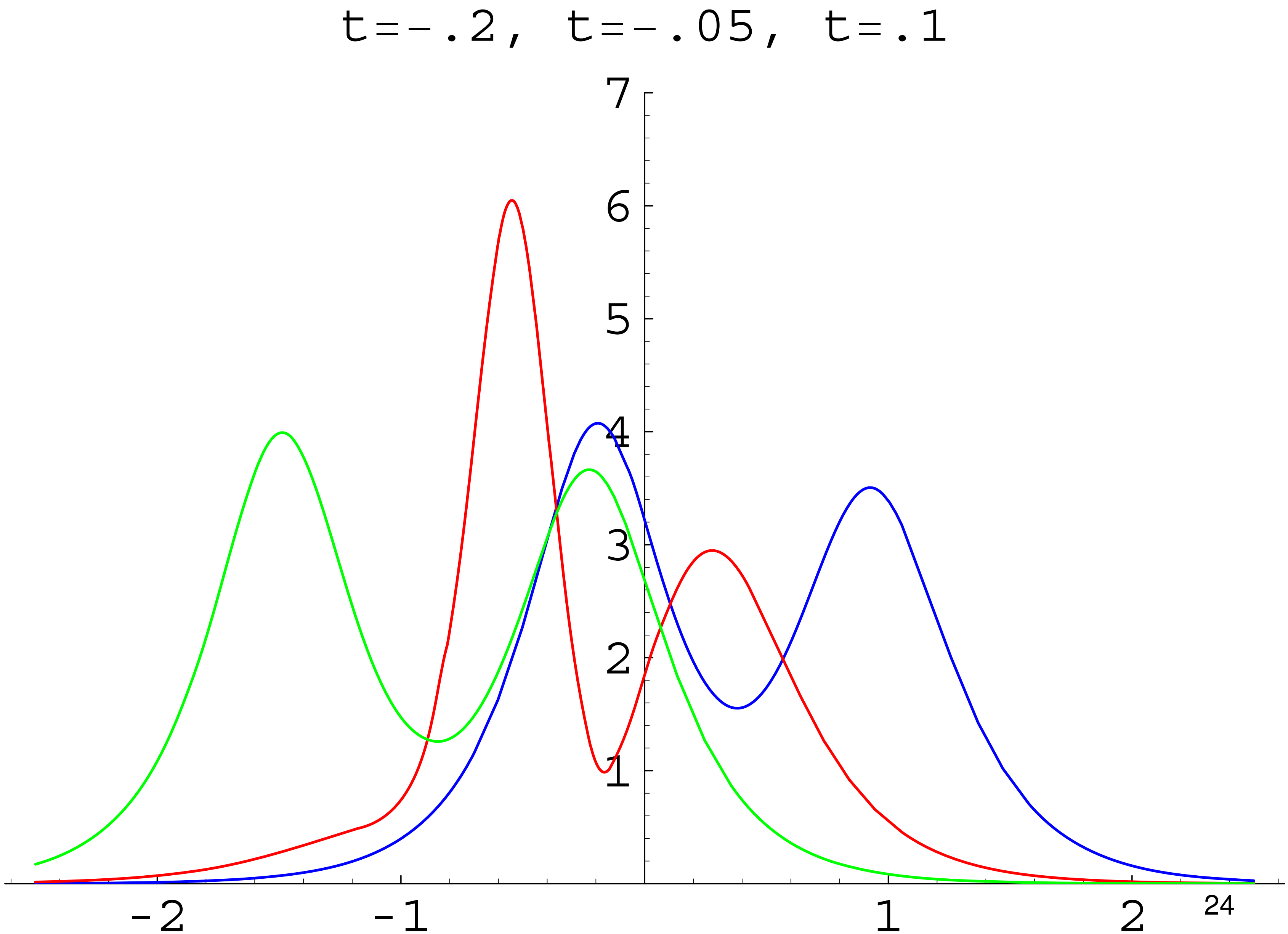
$$\frac{u_t}{\alpha_3 - \alpha_2} - i\alpha_2[u_{xx} + 2|u|^2 u] + \alpha_3[u_{xxx} + 6u^2 u_x] = 0.$$



Real Part
2
1.5
1
0.5
0
−0.5
−1
Imaginary Part
3
2
1
0



Real part
Imaginary Part
2
1.5
1
0.5
3
2
1
0
−1



t=-.2, t=-.05, t=.1
7
6
5
4
3
2
1
-2
-1
1
2

sine-Gordon:

$$u(x,t) = -4\int_x^\infty dr\, K(r,r;t) = -4\,\mathrm{Tr}[\arctan M(x,t)]$$
$$= 2i \log \frac{\det(I + iM(x,t))}{\det(I - iM(x,t))}$$
$$= 4 \arctan\left( i \frac{\det(I + iM(x,t)) - \det(I - iM(x,t))}{\det(I + iM(x,t)) + \det(I - iM(x,t))} \right),$$

where

$$M(x,t) = e^{-xA} e^{-\frac{1}{4}tA^{-1}} \int_0^\infty ds\, e^{-sA} BC e^{-sA} e^{-xA} e^{-\frac{1}{4}tA^{-1}}.$$

For $A = (a)$ with $a > 0$, $B = (1)$, and $C = (c)$ real nonzero,

$$u(x,t) = -4 \arctan\left( \frac{c}{2a} e^{-2a(x+[t/4a^2])} \right).$$

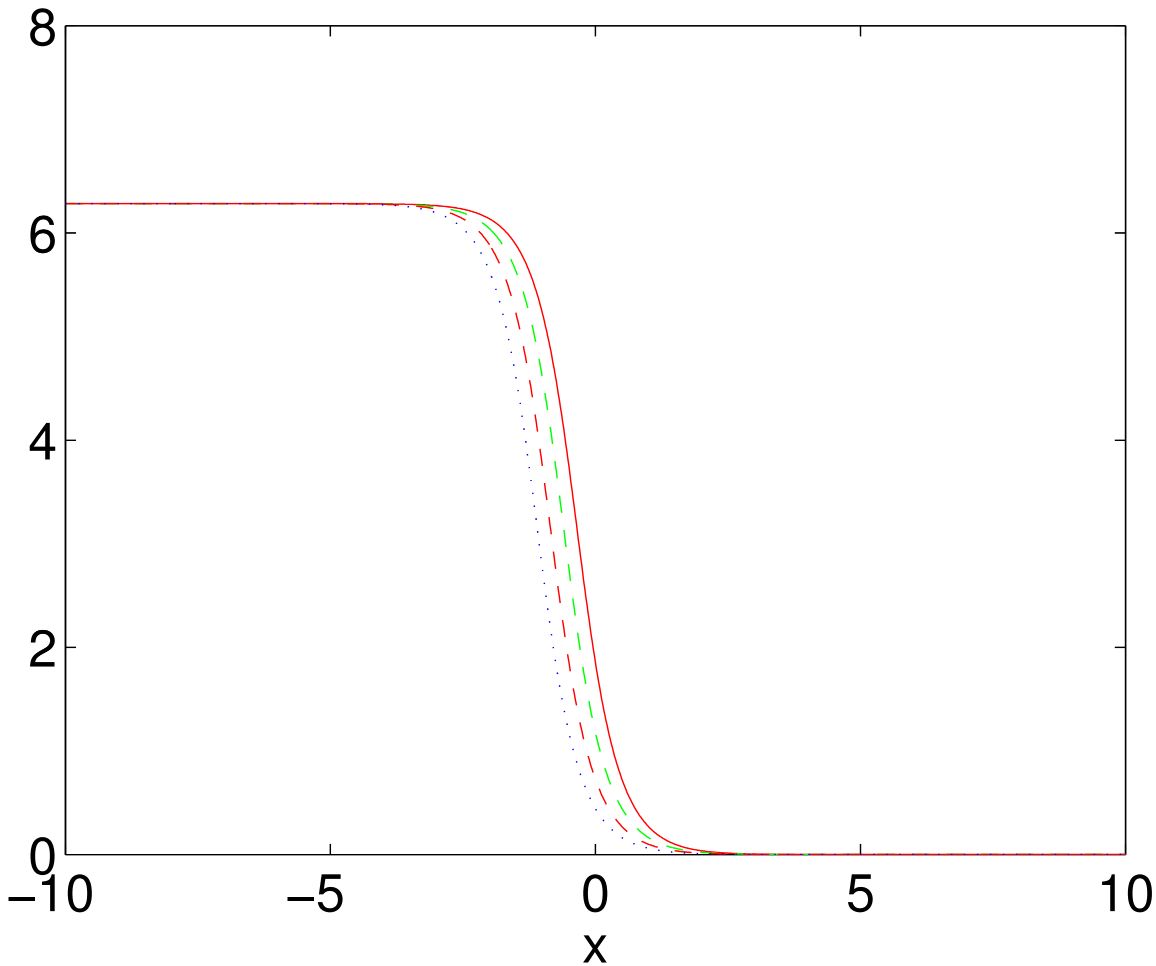



Fig. 4: antikink; $A = B = C = (1)$; $t \in \{0, 1, 2, 3\}$

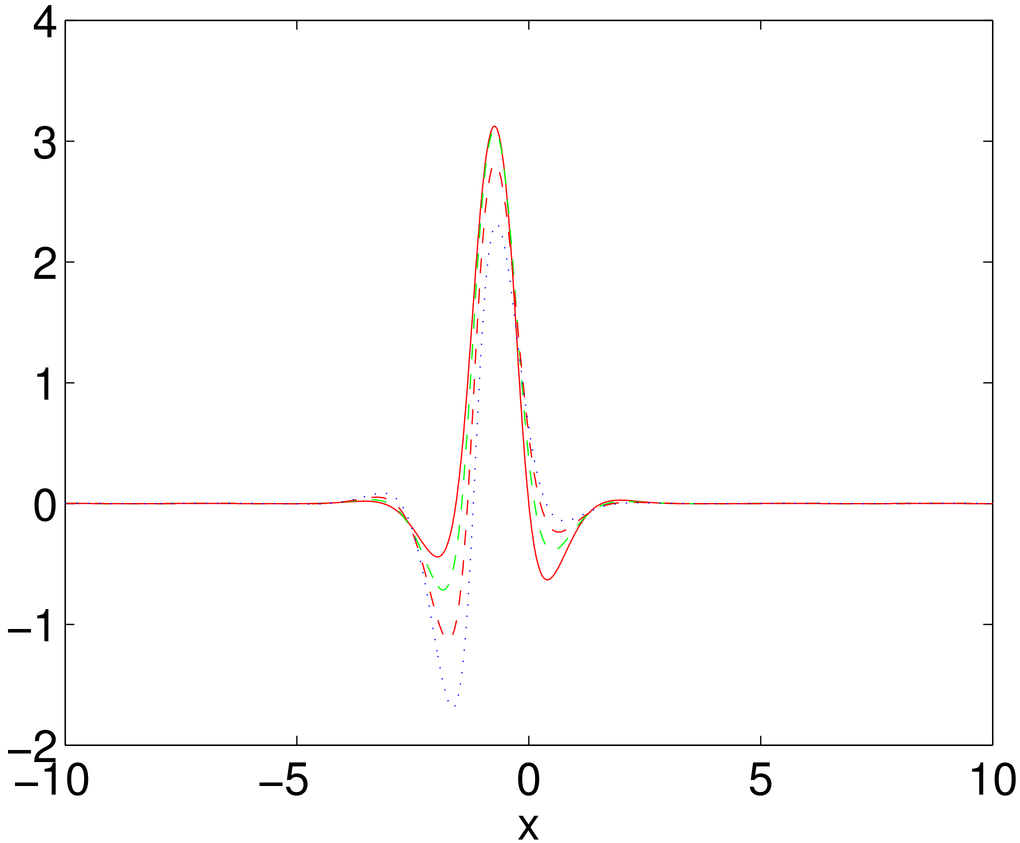



Fig. 5: Breather: $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$

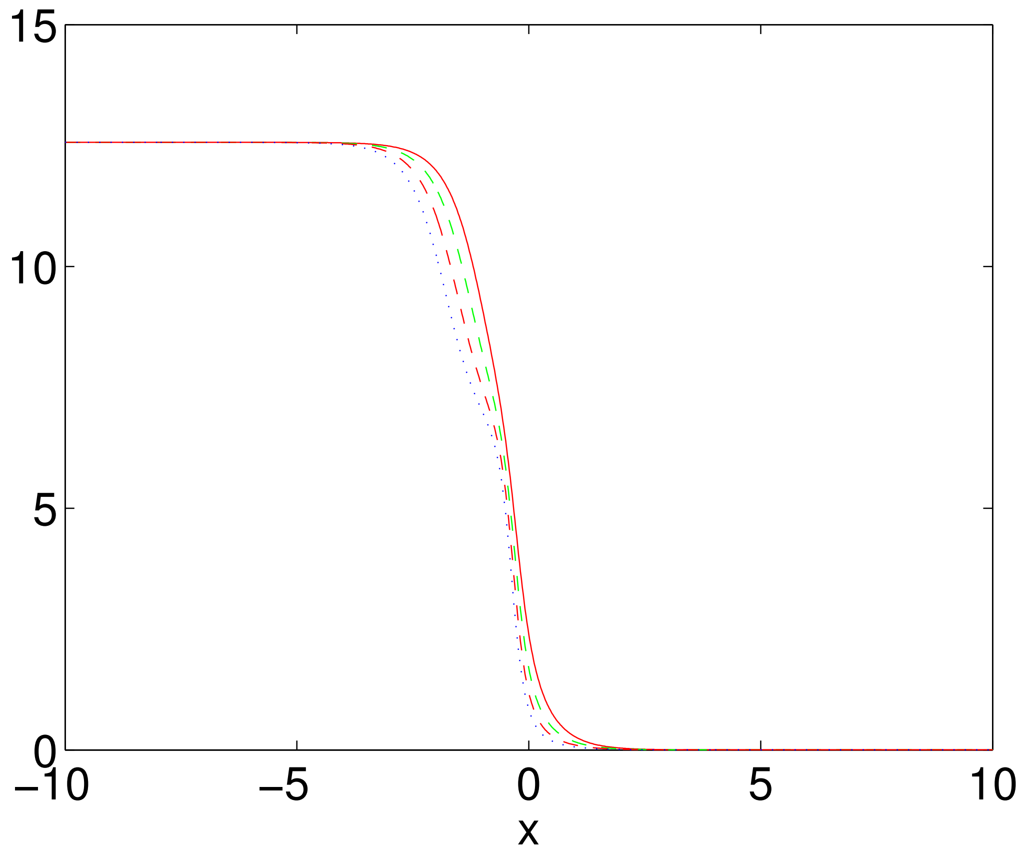



Fig. 6: antikink-kink collision

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

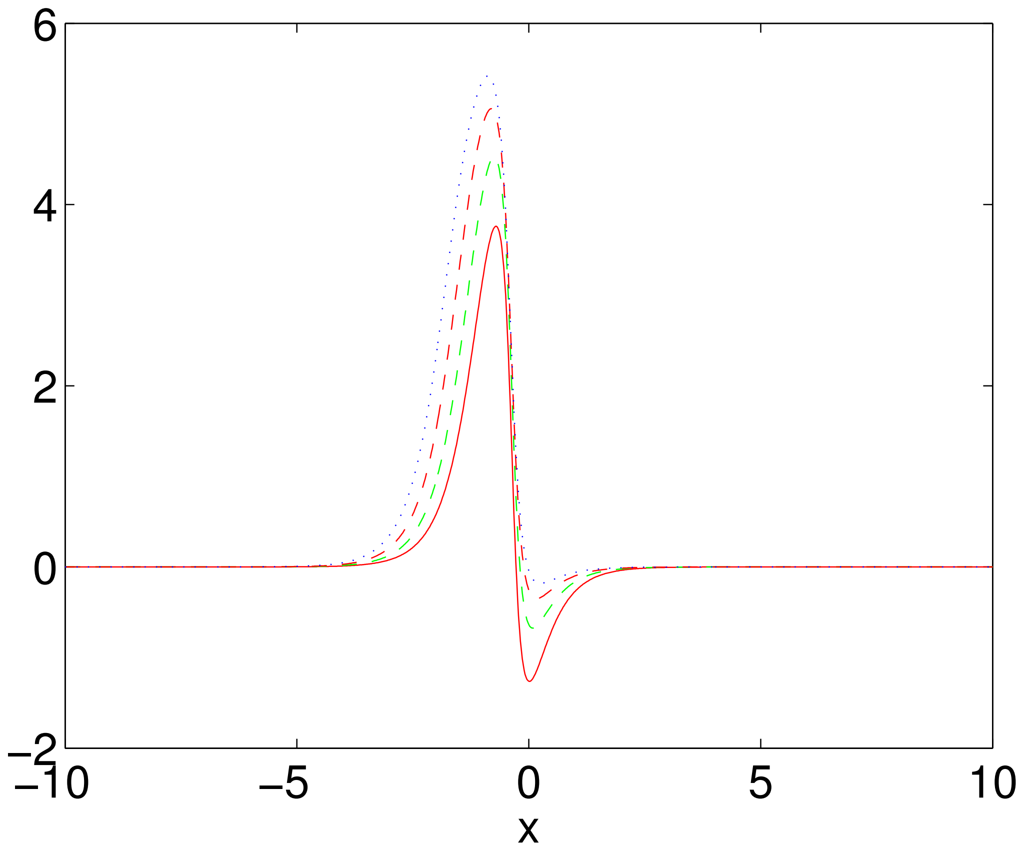



Fig. 7: antikink-antikink collision

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

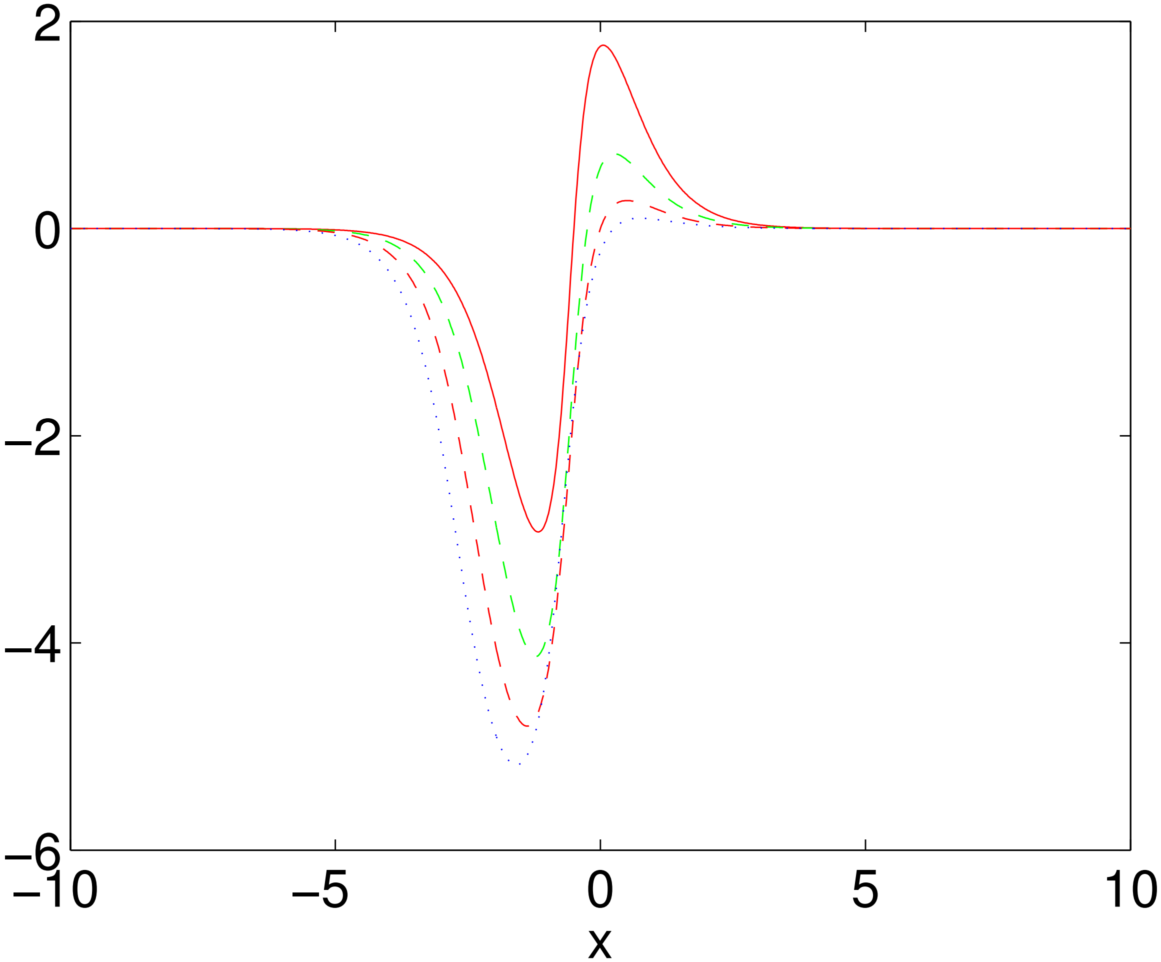



Fig. 8: Double pole: $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$

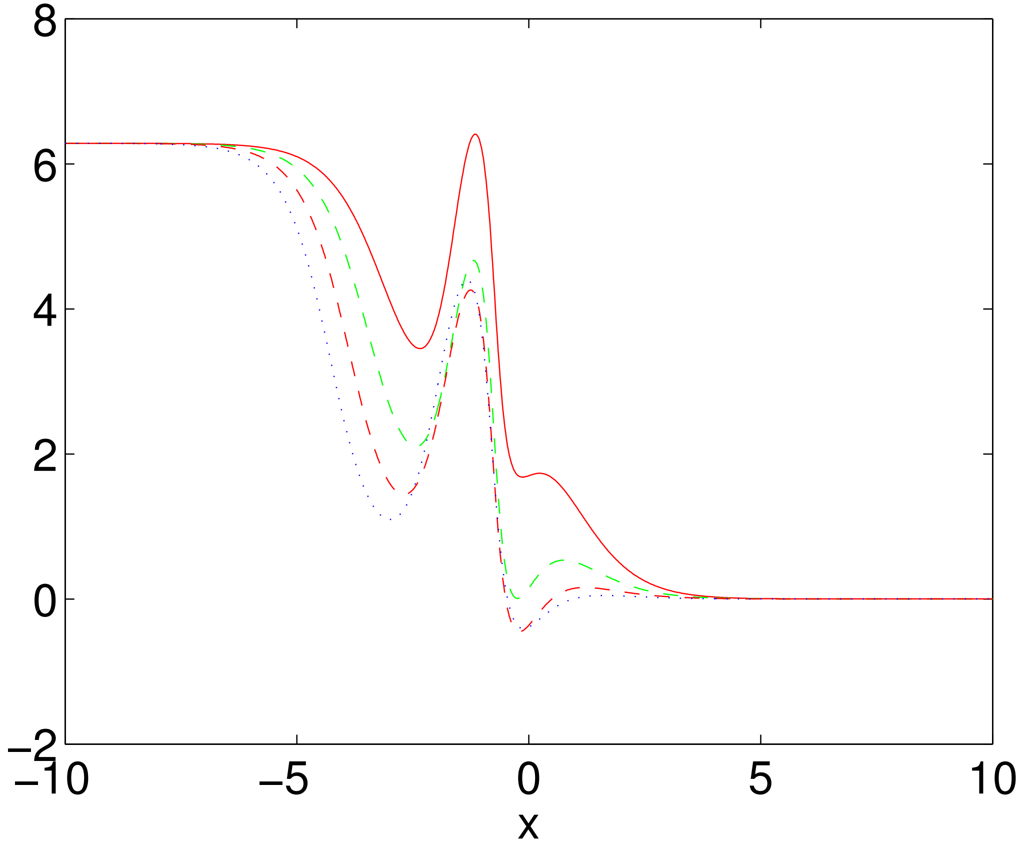



Fig. 9: Triple pole: $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

Heisenberg ferromagnetic equation: Find a real vector $\boldsymbol{m}(x,t)$ of unit length satisfying the initial-value problem

$$\begin{cases} \boldsymbol{m}_t = \boldsymbol{m} \times \boldsymbol{m}_{xx}, \\ \boldsymbol{m}(x,t) \to \boldsymbol{e}_3, \qquad x \to \pm\infty, \\ \boldsymbol{m}(x,0) \text{ known.} \end{cases}$$

Here $\boldsymbol{m}(x,t)$ is the magnetization vector as a function of position-time $(x,t) \in \mathbb{R}^2$ and $x \in \mathbb{R}$ runs along $\boldsymbol{e}_1$, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ being the canonical basis of $\mathbb{R}^3$.

The above equation is the continuous limit of the (quantum) ferromagnetic Heisenberg equation chain in a constant field (in the direction $e_3$) when the wavelength of the excited modes is larger than the lattice distance.

$$V_x = [i\lambda(\boldsymbol{m}\cdot\boldsymbol{\sigma})]V,$$
$$V_t = [-2i\lambda^2(\boldsymbol{m}\cdot\boldsymbol{\sigma}) - i\lambda(\boldsymbol{m}\times\boldsymbol{m}_x\cdot\boldsymbol{\sigma})]V,$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices [Zakharov and Takhtajan, 1979].

Assuming that $\boldsymbol{m}(\cdot,t) - \boldsymbol{e}_3$ and $\boldsymbol{m}_x(\cdot,t)$ have only $L^1(\mathbb{R})$ components and $m_3(x) > -1$, we have the gauge transformation

$$\boldsymbol{m}(x,t)\cdot\boldsymbol{\sigma} = \Psi_{\text{ZS}}(x,0;t)^{-1}\sigma_3\Psi_{\text{ZS}}(x,0;t),$$

where $\Psi_{\text{ZS}}(x,\lambda;t)$ is the focusing Zakharov-Shabat Jost matrix from the right.

One-soliton solution: $A = (a)$, $B = (1)$, $C = (c)$, $a = p + iq$, $p > 0$

$$\begin{pmatrix} m_1(x,t) \\ m_2(x,t) \end{pmatrix} = \frac{1 - m_3(x,t)}{p} \begin{pmatrix} \cos\beta(x,t) & -\sin\beta(x,t) \\ \sin\beta(x,t) & \cos\beta(x,t) \end{pmatrix} \begin{pmatrix} q \cosh\kappa(x,t) \\ p \sinh\kappa(x,t) \end{pmatrix},$$

$$m_3(x,t) = 1 - \frac{2p}{p^2+q^2}\operatorname{sech}^2\kappa(x,t),$$

$$\kappa(x,t) = 2p(x - x_0 - vt) = \sqrt{\omega - \tfrac{v^2}{4}}\,(x - x_0 - vt),$$

$$\beta(x,t) = \omega t + \tfrac{v}{2}(x - x_0 - vt) + \varphi_0,$$

where $\omega = 4(p^2+q^2)$, $p = \frac{1}{2}\sqrt{\omega - \frac{v^2}{4}}$, and $q = \frac{v}{4}$.

$v$ speed, $\omega$ precession frequency, $x_0 = \frac{1}{2p}\ln(\frac{|c|}{2p})$ initial position, and $\varphi_0 = -\arg(c)$ initial phase in $m_1$-$m_2$-plane.

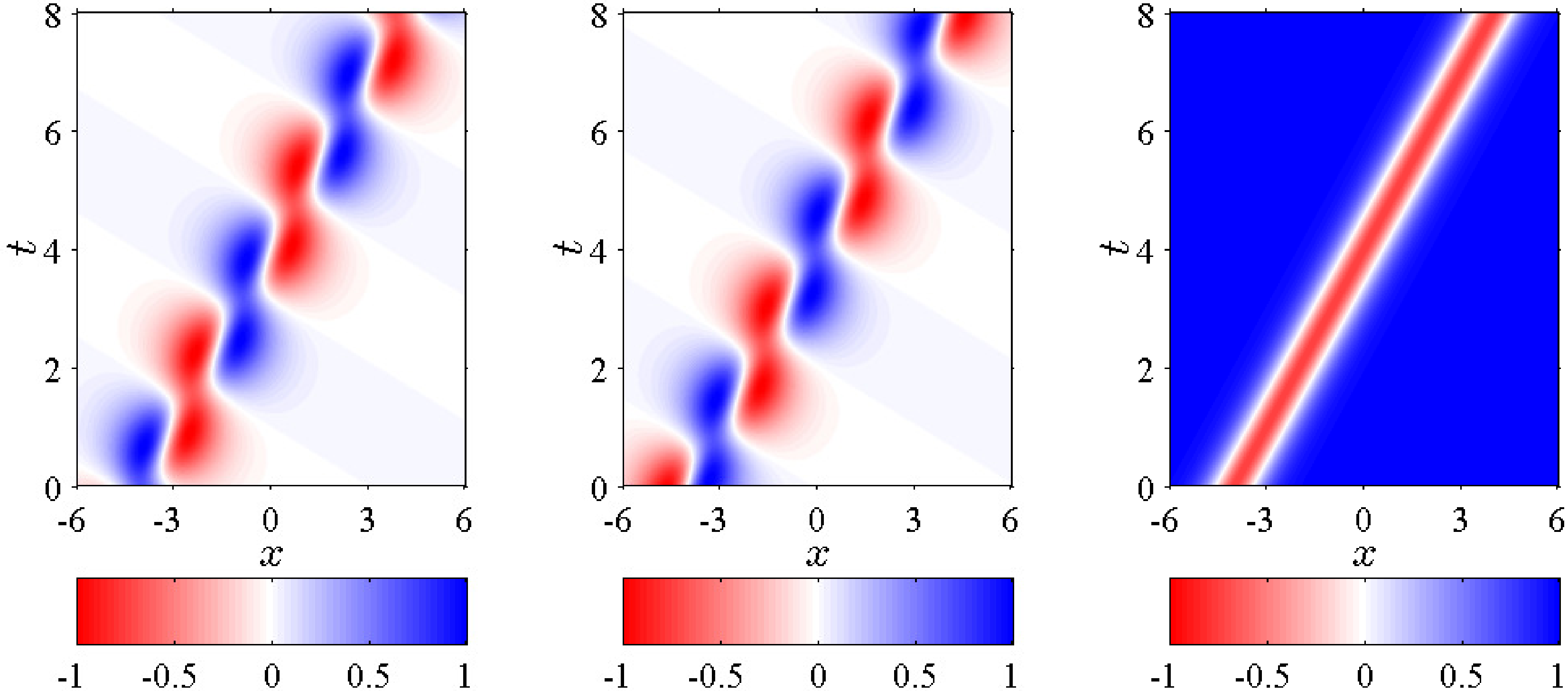



Fig. 1: Propagating, one-soliton solution.

$a = \frac{1}{4}\sqrt{7} + \frac{1}{4}i$, $c = \frac{-7+3i\sqrt{7}}{8}e^{-2(i+\sqrt{7})}$, $v = 1$, $\omega = 2$, $x_0 = -4$, $\varphi_0 = 0$

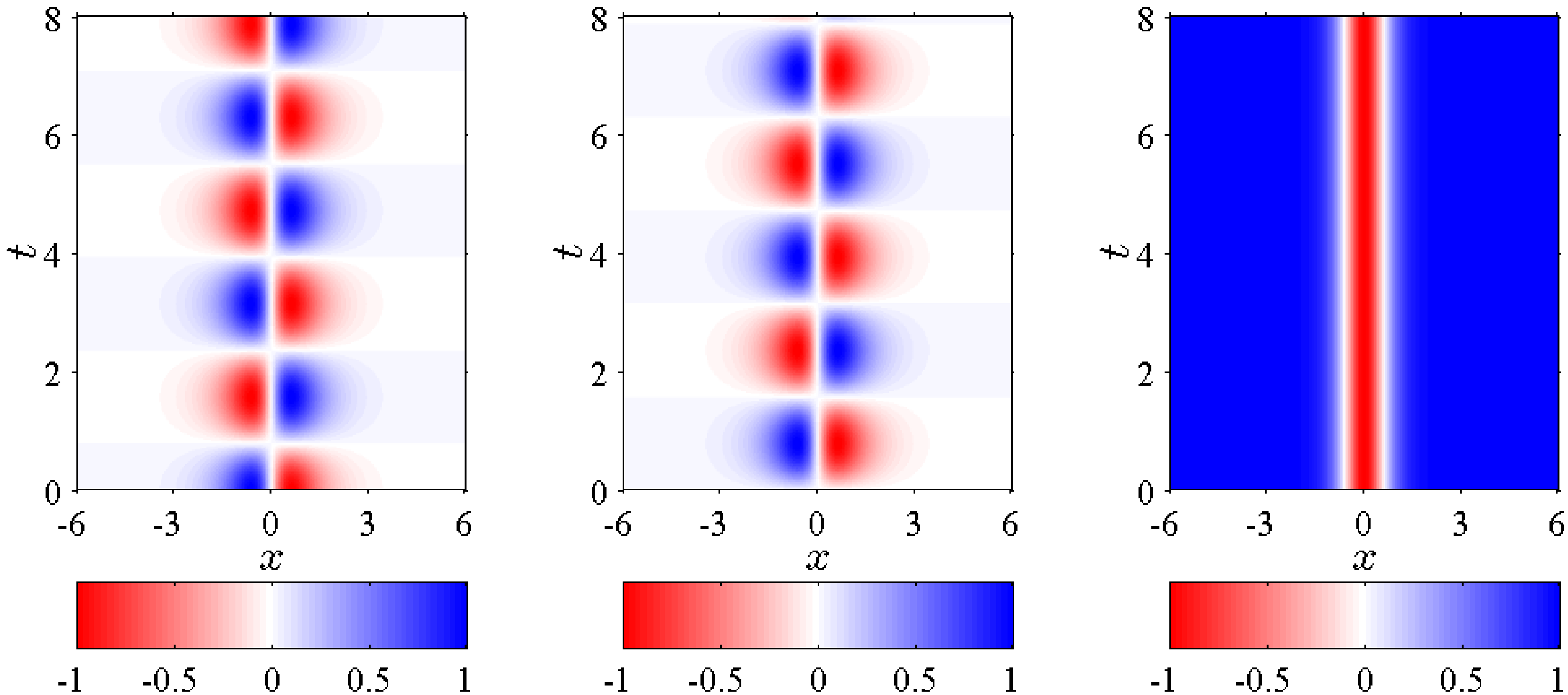



Fig. 2: Stationary, one-soliton solution.

$a = \frac{1}{2}\sqrt{2}$, $c = \sqrt{2}$, $\omega = 2$, $v = x_0 = \varphi_0 = 0$

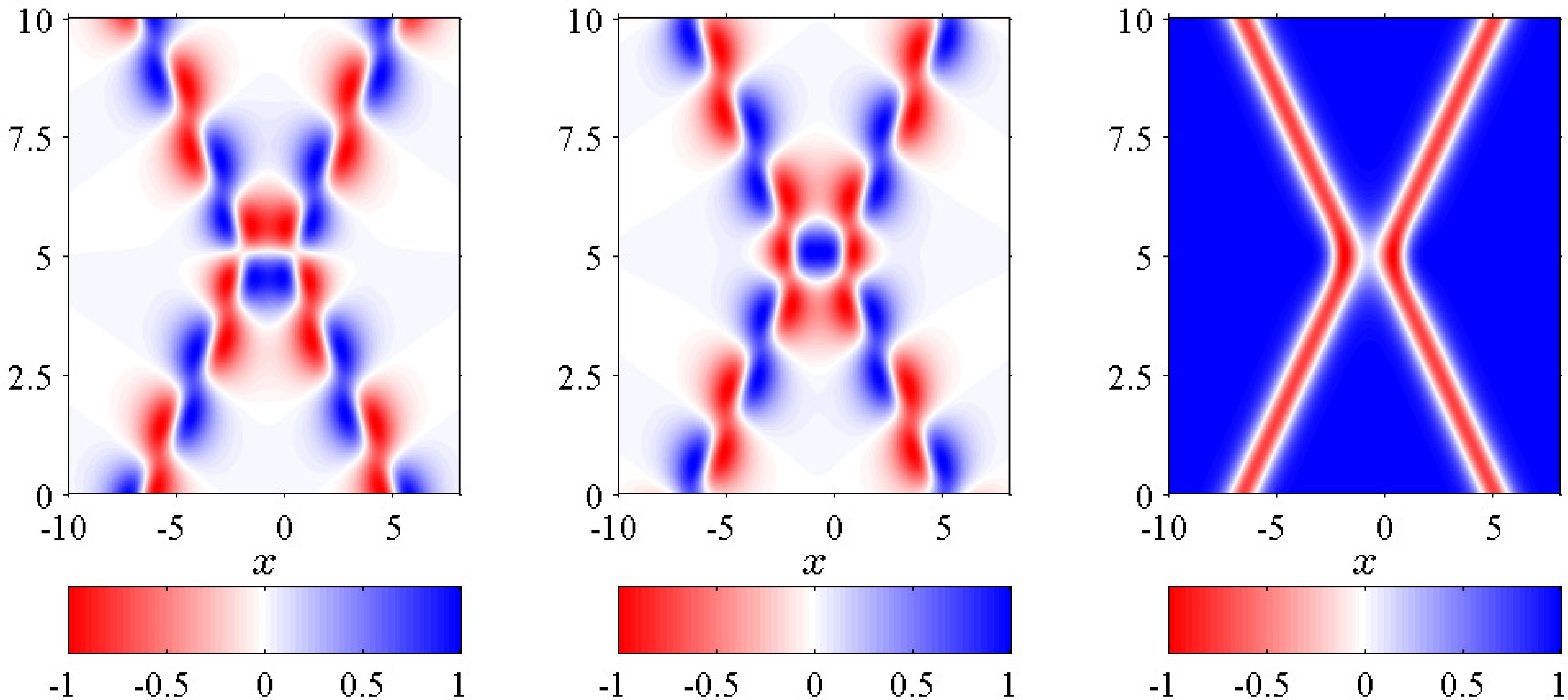



Fig. 3: Head-on collision between two solitons propagating in opposite directions.

$$(v, \omega, x_0, \varphi_0) \mapsto (1, 2, -5, 0) \& (-1, 2, 5, \tfrac{\pi}{2})$$

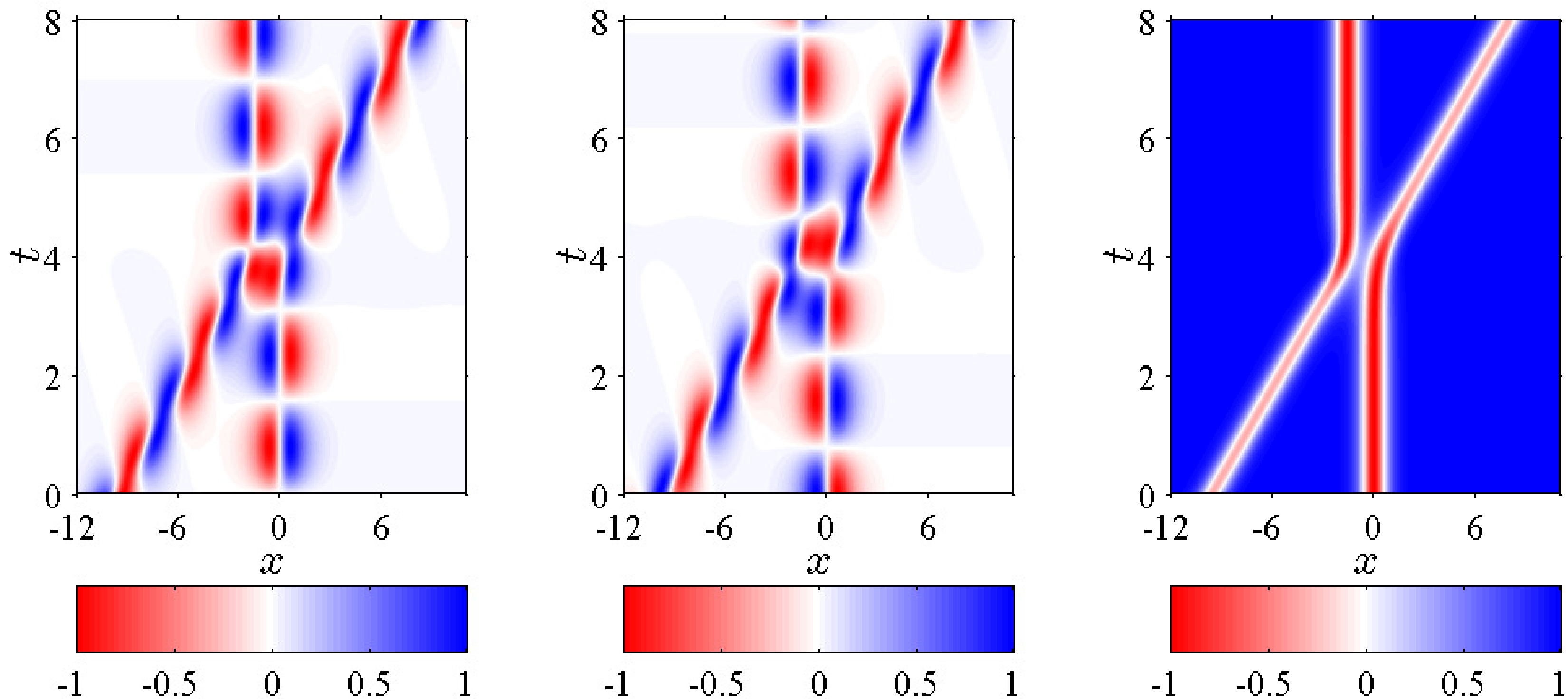



Fig. 4: Scattering between a propagating and a stationary soliton.
In $m_3$, observe the spatial shift experienced by the stationary soliton
(in the opposite direction with respect to the propagating one) after the interaction.

$$(v, \omega, x_0, \varphi) \mapsto (2, 3, -8, 0) \& (0, 2, 0, \tfrac{\pi}{2})$$

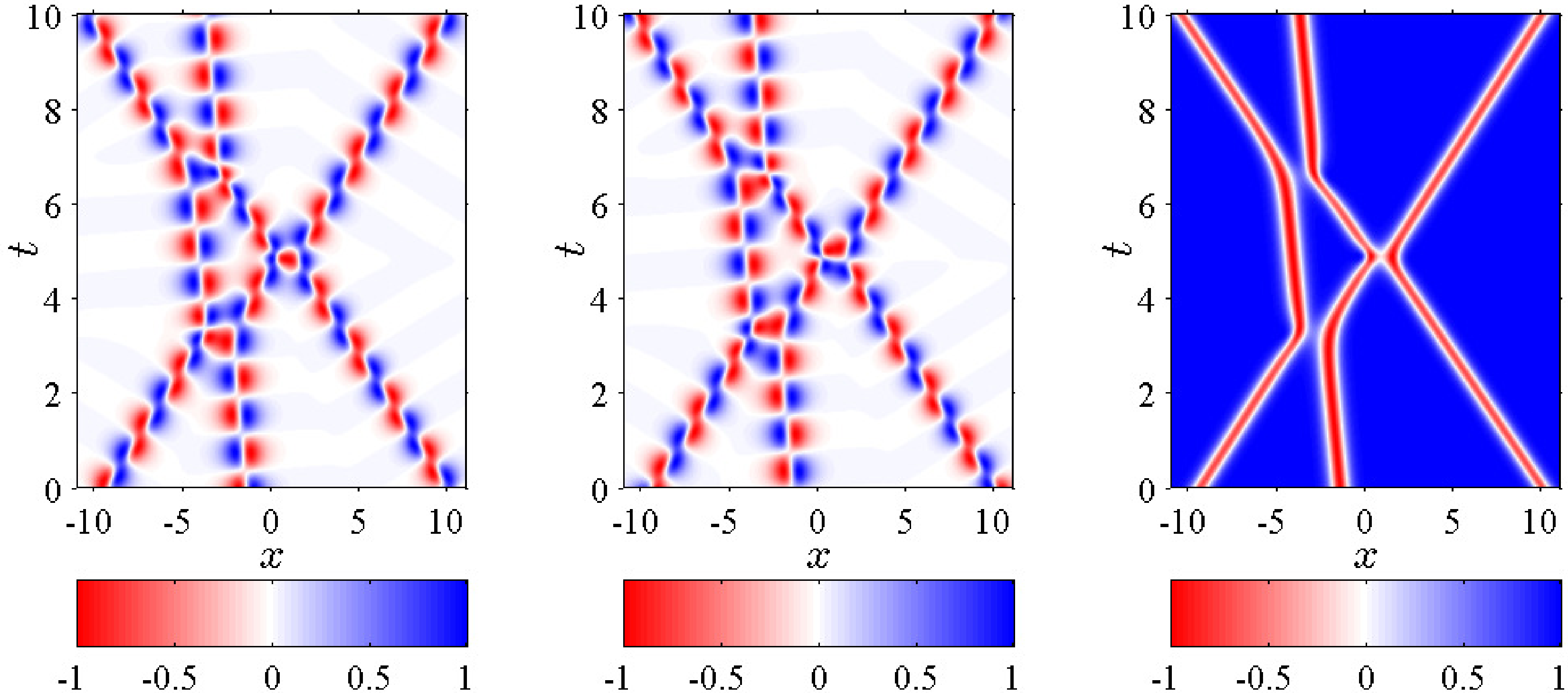



Fig. 5: Interaction of three propagating solitons with different velocities.

The solitons emerged unchanged from the interaction.

$(v, \omega, x_0, \varphi_0) \mapsto (1.7, 5, -7, 0) \& (-\frac{1}{4}, 4, \frac{1}{4}, 0) \& (-1.8, 5.5, 10, 0)$

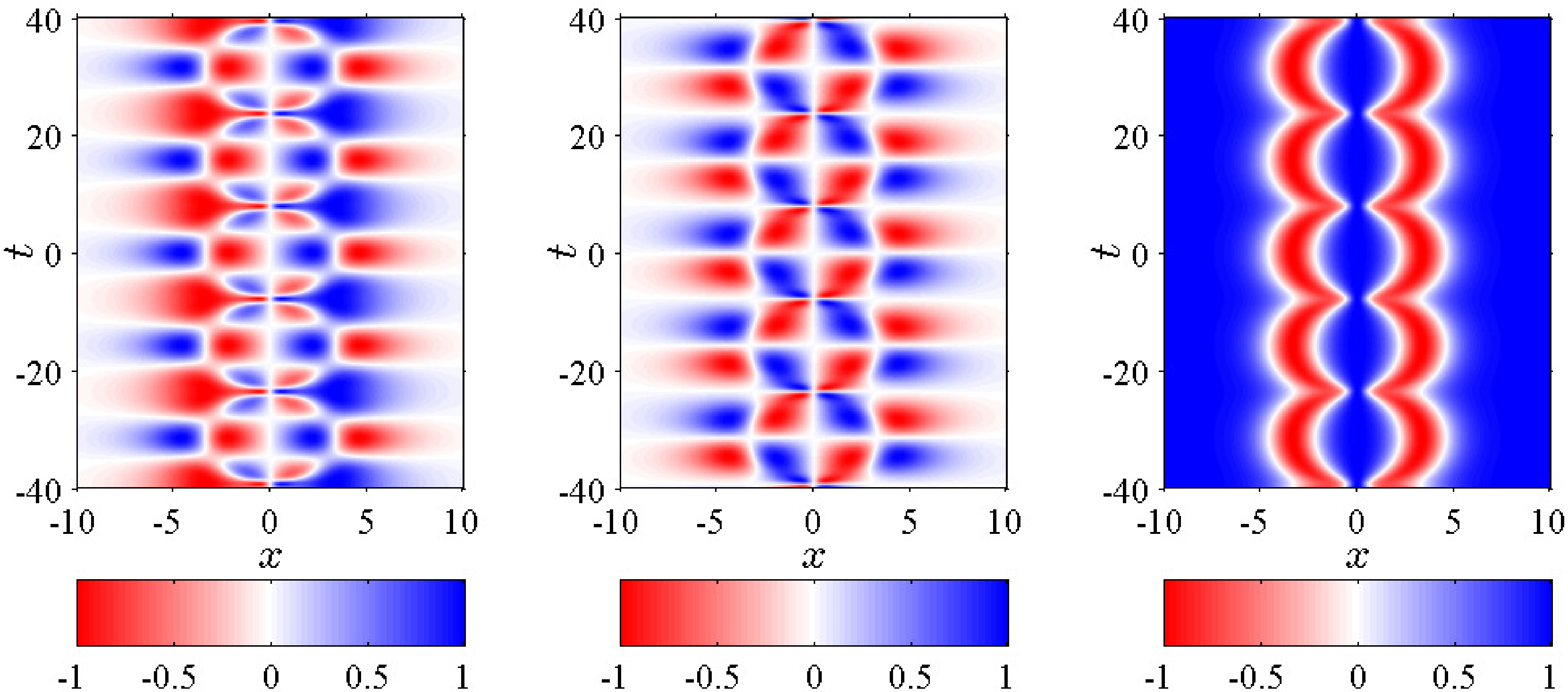



Fig. 6: Stationary, breather-like soliton.

$$(v, \omega) \mapsto (0, 0.8) \& (0, 0.4)$$

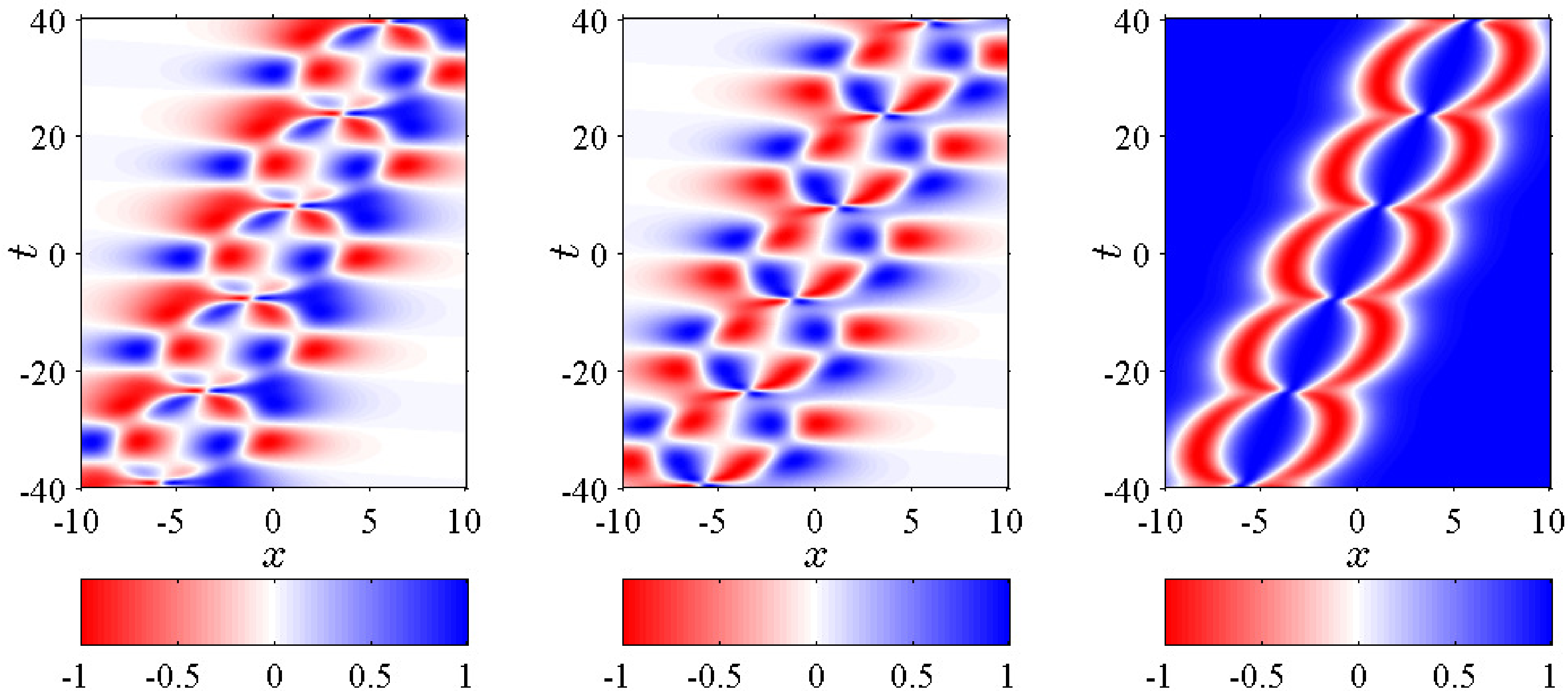
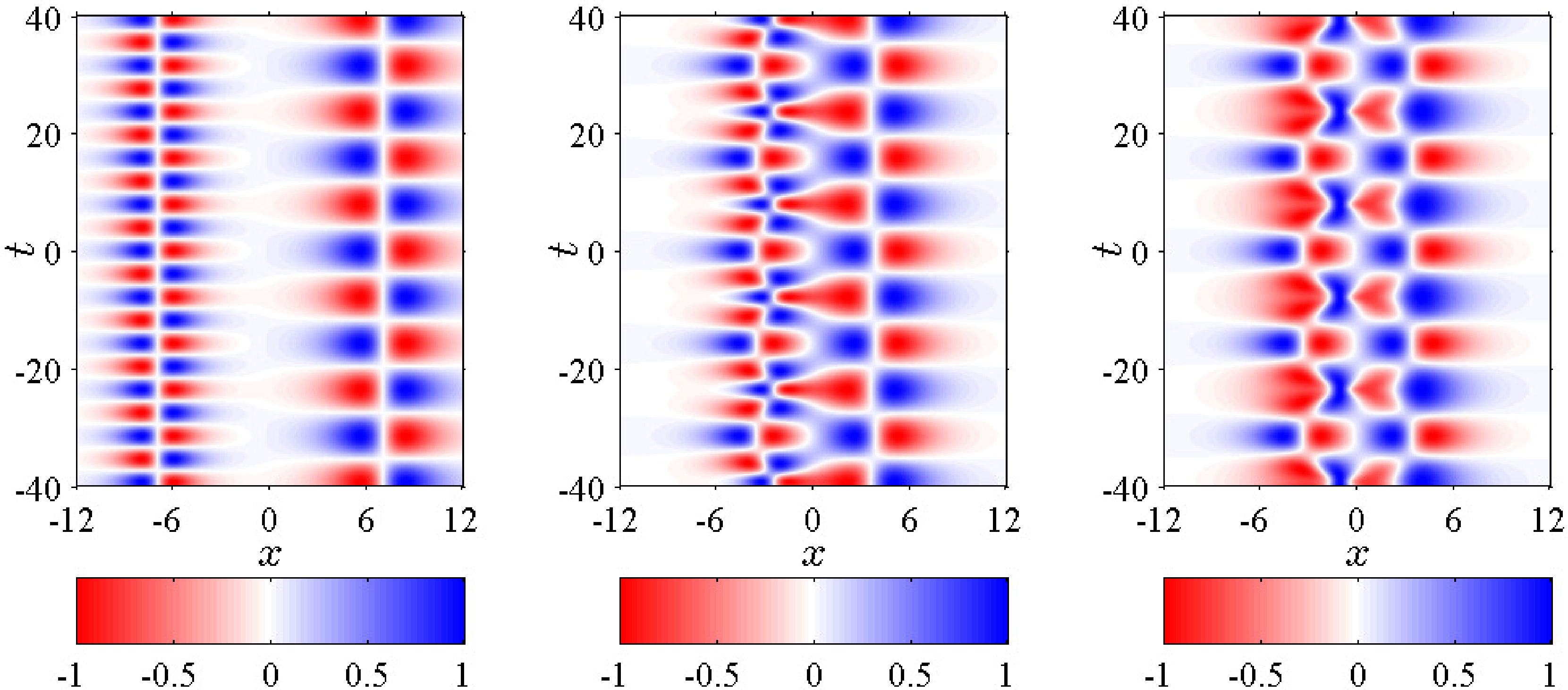


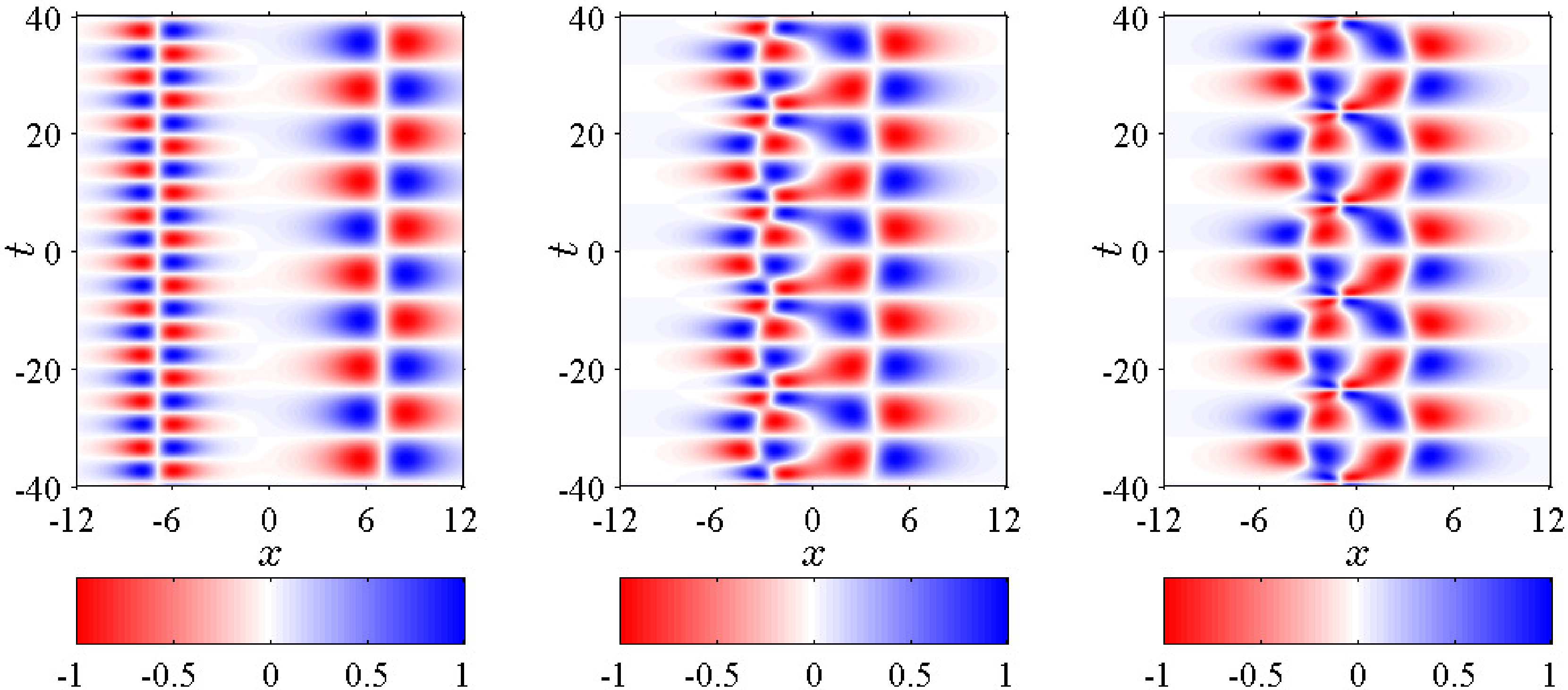

Fig. 7: Propagating, breather-like soliton.

$$(v, \omega) \mapsto (0.15, 0.8) \& (0.15, 0.4)$$




Transition from two stationary solitons to a pair of solitons forming a single stationary breather-like soliton (only $m_1(x,t)$ is shown).

$$(v, \omega, x_0, \varphi_0) \mapsto (0, 0.8, -3, 0) \& (0, 0.4, 7, 0)$$




Transition from two stationary solitons to a pair of solitons forming a single stationary breather-like soliton (only $m_2(x,t)$ is shown).

$$(v, \omega, x_0, \varphi_0) \mapsto (0, 0.8, -3, 0) \& (0, 0.4, 7, 0)$$

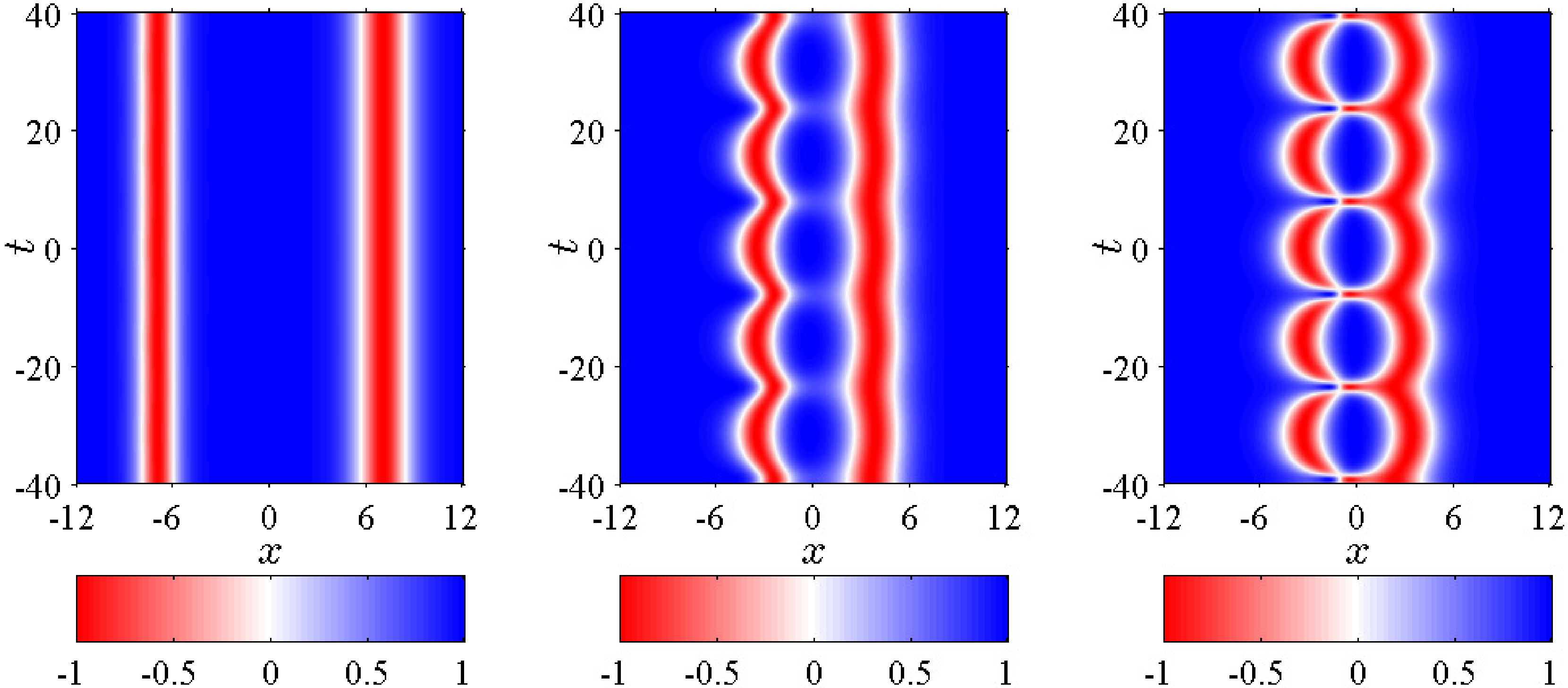



Fig. 8: Transition from two stationary solitons to a pair of solitons forming a single stationary breather-like soliton (only $m_3(x,t)$ is shown).

$$(v, \omega, x_0, \varphi_0) \mapsto (0, 0.8, -3, 0) \& (0, 0.4, 7, 0)$$

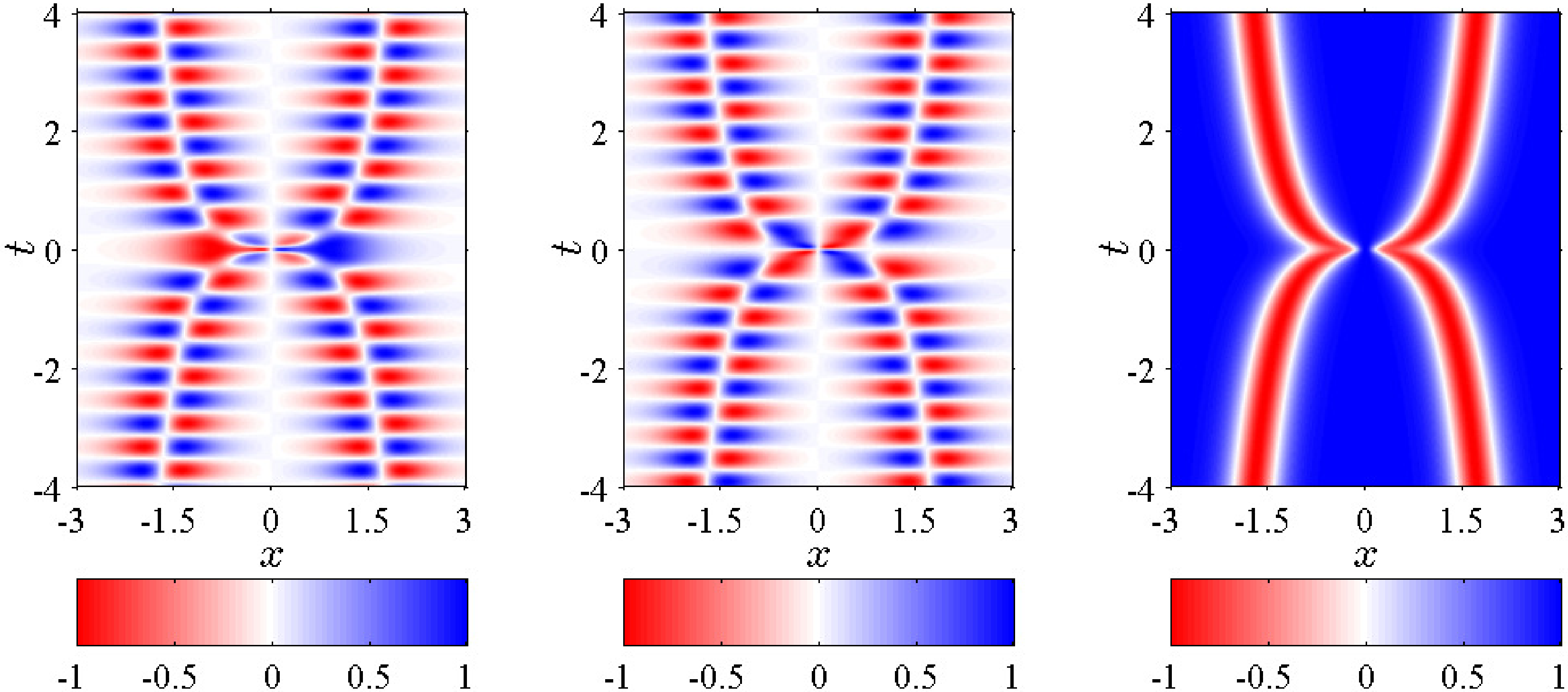



Fig. 9: A multipole soliton solution with algebraic multiplicity $n_1 = 2$.
$a = p + iq = \sqrt{2}$, $x_0 = \varphi_0 = 0$, Jordan block of order 2

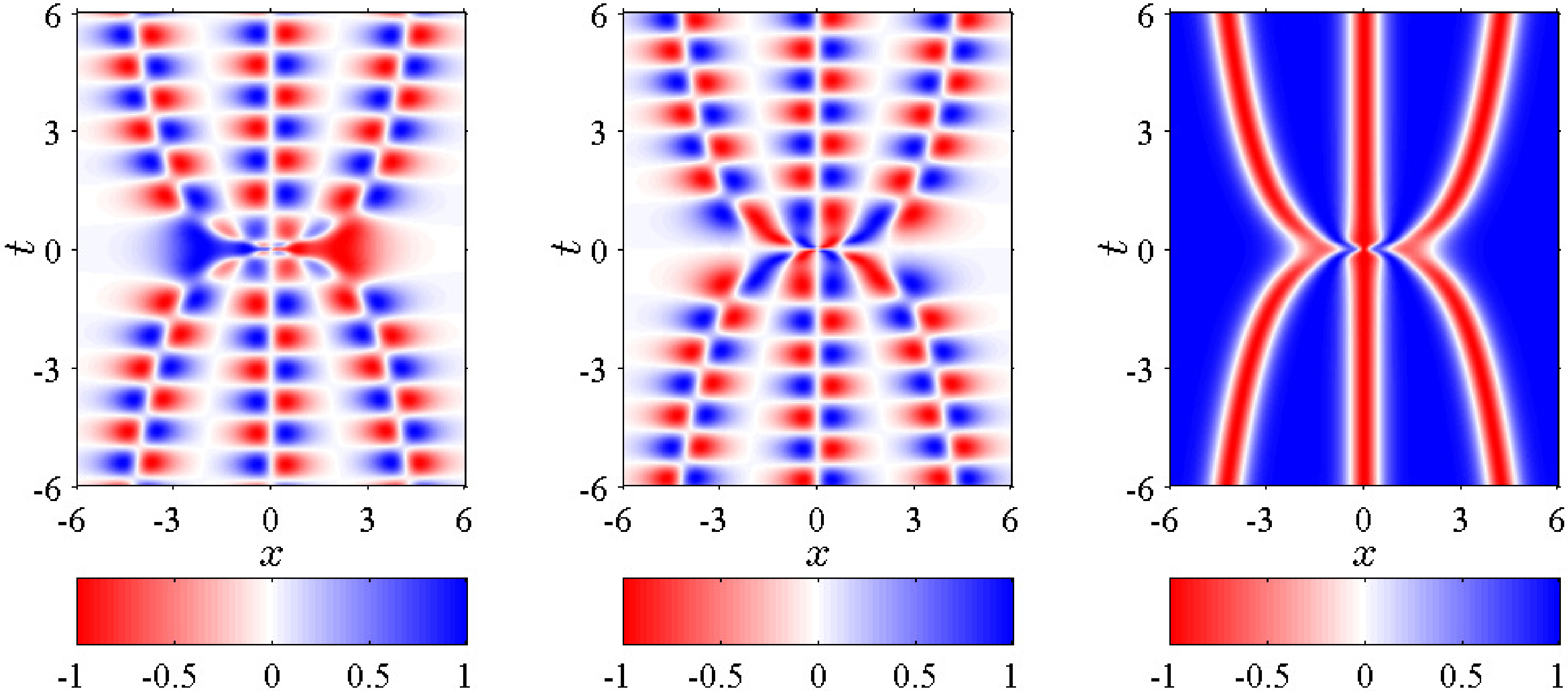



Fig. 10: A multipole soliton solution with algebraic multiplicity $n_1 = 3$.

$a = p + iq = 1$, $x_0 = \varphi_0 = 0$, Jordan block of order 3

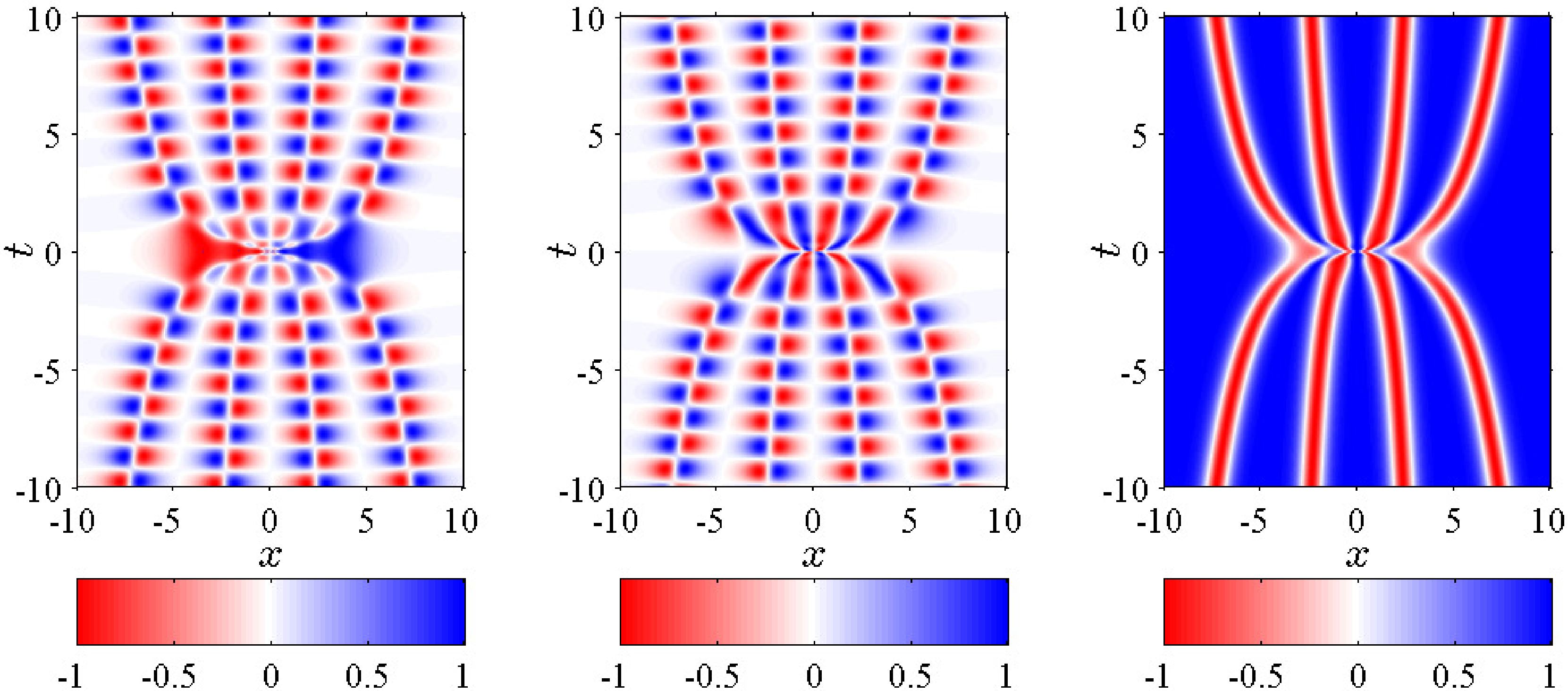



Fig. 11: A multipole soliton solution with algebraic multiplicity $n_1 = 4$.
$a = p + iq = \frac{1}{2}\sqrt{3}$, $x_0 = \varphi_0 = 0$, Jordan block of order 4

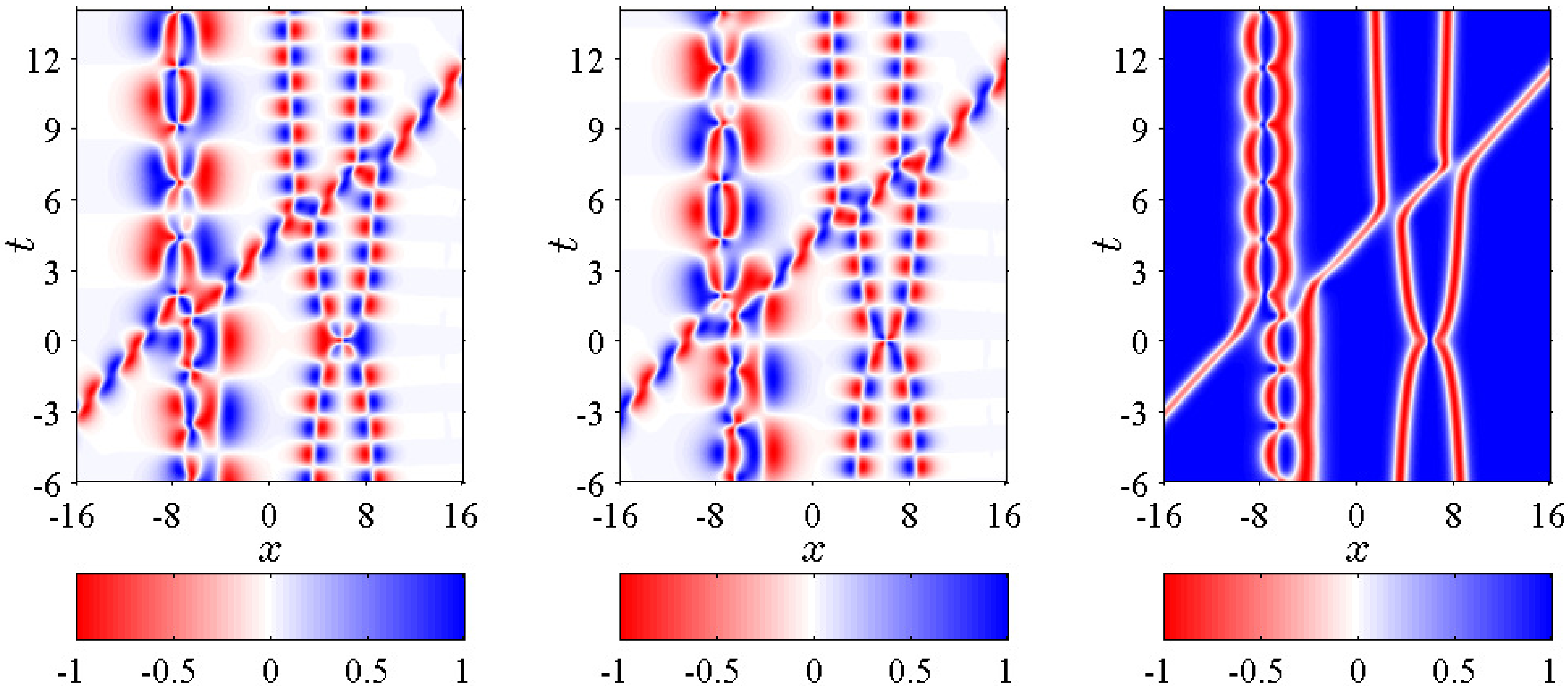



Fig. 12: Interaction of a propagating soliton with a breather-like soliton and a multipole soliton with algebraic multiplicity 2.

$(v, \omega, x_0, \varphi_0) \mapsto (0, 3.6, 0, 0) \& (0, 1, 0, 0) \& (1.75, 3, -4, \frac{3\pi}{4}) \& (0, 2.9, 6, 0)$

THANK YOU FOR YOUR ATTENTION