Regularization preconditioners for frame-based deconvolution

Marco Donatelli

Dept. of Science and High Tecnology – U. Insubria (Italy)

Joint work with
M. Hanke (U. Mainz), *D. Bianchi* (U. Insubria),
Y. Cai, T. Z. Huang (UESTC, P. R. China)

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Outline

Ill-posed problems and iterative regularization

A nonstationary preconditioned iteration

Image deblurring

Combination with frame-based methods

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The model problem

Consider the solution of ill-posed equations

$$Tx = y\,, \tag{1}$$

where $T : \mathcal{X} \to \mathcal{Y}$ is a linear operator between Hilbert spaces.

- T is a compact operator, the singular values of T decay gradually to zero without a significant gap.
- Assume that problem (1) has a solution $x^\dagger$ of minimal norm.

Goal

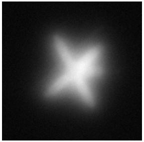
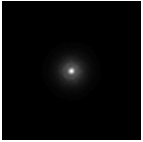
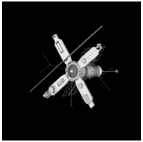
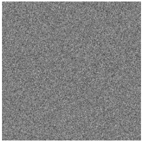
Compute an approximation of $x^\dagger$ starting from approximate data $y^\delta \in \mathcal{Y}$, instead of the exact data $y \in \mathcal{Y}$, with

$$\|y^\delta - y\| \leq \delta\,, \tag{2}$$

where $\delta \geq 0$ is the corresponding noise level.

Image deblurring problems

$$y^\delta \quad = \quad T \quad * \quad x \quad + \quad \xi$$

- T is doubly Toeplitz, large and severely ill-conditioned (discretizzation of an integral equations of the first kind)
- y^δ are known measured data (blurred and noisy image)
- ξ is noise; $\|\xi\| = \delta$

$\longrightarrow$ discrete ill-posed problems (Hansen, 90's)

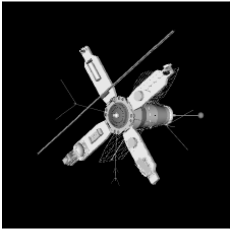
Regularization

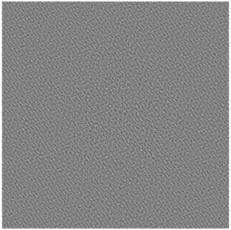
- The singular values of T are large in the low frequencies, decays rapidly to zero and are small in the high frequencies.
- The solution of $Tx = y^\delta$ requires some sort of regularization:

$$x = T^\dagger y^\delta = x^\dagger + T^\dagger \xi,$$

where $\|T^\dagger \xi\|$ is large.

$x^\dagger \quad \Longrightarrow \quad x = T^\dagger y^\delta$





Tikhonov regularization

Balance the the data fitting and the "explosion" of the solution

$$\min_x \{\|Tx - y^\delta\|^2 + \alpha\|x\|^2\}$$

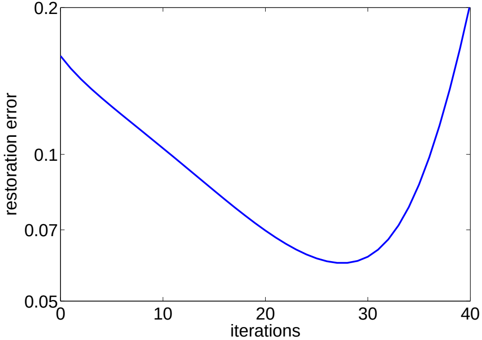
which is equivalent to

$$x = (T^*T + \alpha I)^{-1} T^* y^\delta,$$

where $\alpha > 0$ is a regularization parameter.

Iterative regularization methods (semi-convergence)

- Classical iterative methods firstly reduce the algebraic error into the low frequencies (well-conditioned subspace), when they arrive to reduce the algebraic error into the high frequencies then the restoration error increases because of the noise.
- The regularization parameter is the stopping iteration.



Preconditioned regularization

Replace the original problem $Tx = y^\delta$ with

$$P^{-1}Tx = P^{-1}y^\delta$$

such that

1. inversion of P is cheap
2. $P \approx T$ but not too much ($T^\dagger$ unbounded while P^{-1} must be bounded!)

Alert!

Preconditioners can be used to accelerate the convergence, but an imprudent choice of preconditioner may spoil the achievable quality of computed restorations.

Classical preconditioner

- Historically, the first attempt of this sort was by Hanke, Nagy, and Plemmons (1993): In that work

$$P = C_\varepsilon,$$

  where C_ε is the optimal doubly circulant approximation of T, with eigenvalues set to be one for frequencies above $1/\varepsilon$. Very fast, but the choice of ε is delicate and not robust.
- Subsequently, other regularizing preconditioners have been suggested: Bertero and Piana (1997), Kilmer and O'Leary (1999), Estatico (2002), Egger and Neubauer (2005), Brianzi, Di Benedetto, and Estatico (2008).

Hybrid regularization

- Combine iterative and direct regularization (Björck, O'Leary, Simmons, Nagy, Reichel, Novati, ...).
- Main idea:
  1. Compute iteratively a Krylov subspace by Lanczos or Arnoldi.
  2. At every iteration solve the projected Tikhonov problem in the small size Krylov subspace.
- Usually few iterations, and so a small Krylov subspace, are enough to compute a good approximation.

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Nonstationary iterated Tikhonov regularization

Given x_0 compute for $n = 0, 1, 2, \ldots$

$$z_n = (T^*T + \alpha_n I)^{-1} T^* r_n \,, \qquad r_n = y^\delta - Tx_n \,, \tag{3a}$$

$$x_{n+1} = x_n + z_n \,. \tag{3b}$$

This is some sort of regularized iterative refinement.

Choices of α_n:

- $\alpha_n = \alpha > 0$, $\forall n$, stationary.
- $\alpha_n = \alpha q^n$ where $\alpha > 0$ and $0 < q \leq 1$, geometric sequence (fastest convergence), [Groetsch and Hanke, 1998].

$T^*T + \alpha I$ and $TT^* + \alpha I$ could be expensive to invert!

The starting idea

The iterative refinement applied to the error equation $Te_n \approx r_n$ is correct up to noise, hence consider instead

$$Ce_n \approx r_n \,, \tag{4}$$

possibly tolerating a slightly larger misfit.

$$\Downarrow$$

Approximate T by C and iterate

$$h_n = (C^*C + \alpha_n I)^{-1} C^* r_n \,, \qquad r_n = y^\delta - Tx_n \,, \tag{5}$$

$$x_{n+1} = x_n + h_n \,. \tag{6}$$

Preconditioner $\Rightarrow$ $P = (C^*C + \alpha_n I)^{-1} C^*$

Nonstationary preconditioning

Differences to previous preconditioners:

- gradual approximation of the optimal regularization parameter
- nonstationary scheme, not to be used in combination with CGLS
- essentially as fast as nonstationary iterated Tikhonov regularization

An hybrid regularization

Instead of projecting into a small size Krylov subspace, project the error equation in a nearby space of the same size but where the operator is diagonal (for image deblurring). The projected linear system (the rhs r_n) changes at every iteration.

Estimation of α_n

Assumption:

$$\|(C - T)z\| \leq \rho \, \| Tz \| \, , \qquad z \in \mathcal{X} \, , \tag{7}$$

for some $0 < \rho < 1/2$.

Adaptive choice of α_n

Choose α_n s.t. the (4) is solved up to a certain relative amount:

$$\|r_n - Ch_n\| = q_n \|r_n\| \, , \tag{8}$$

where $q_n < 1$, but not too small ($q_n > \rho + (1 + \rho)\delta/\|r_n\|$).

The Algorithm (AIT)

Choose $\tau = (1+2\rho)/(1-2\rho)$ and fix $q \in (2\rho, 1)$.
While $\|r_n\| > \tau\delta$, let $\tau_n = \|r_n\|/\delta$, and compute α_n s.t.

$$\|r_n - Ch_n\| = q_n\|r_n\|\,, \qquad q_n = \max\{\, q,\, 2\rho + (1+\rho)/\tau_n \,\}\,. \quad (9a)$$

Then, update

$$h_n = (C^*C + \alpha_n I)^{-1} C^* r_n\,, \quad (9b)$$

$$x_{n+1} = x_n + h_n\,, \qquad r_{n+1} = y^\delta - Tx_{n+1}\,. \quad (9c)$$

Details

- The parameter q prevents that r_n decreases too rapidly.
- The unique α_n can be computed by Newton iteration.

Theoretical results [D., Hanke, IP 2013]

Theorem

The norm of the iteration error $e_n = x^\dagger - x_n$ decreases monotonically as long as

$$\|r_n\| \le \tau\delta \le \|r_{n-1}\|, \qquad \tau > 1 \text{ fixed.}$$

Theorem

For exact data ($\delta = 0$) the iterates x_n converges to the solution of $Tx = y$ that is closest to x_0 in the norm of $\mathcal{X}$.

Theorem

For noisy data ($\delta > 0$), as $\delta \to 0$, the approximation x^δ converges to the solution of $Tx = y$ that is closest to x_0 in the norm of $\mathcal{X}$.

Extensions [Buccini, manuscript 2015]

- Projection into convex set Ω:

$$x_{n+1} = P_\Omega(x_n + h_n).$$

- In the computation of h_n by Tikhonov, replace I with L, where L is a regularization operator (e.g., first derivative):

$$h_n = (C^*C + \alpha_n L^*L)^{-1} C^* r_n \,,$$

  under the assumption that L and C have the same basis of eigenvectors.

- In both cases the previous convergence analysis can be extended even if it is not straightforward (take care of $\mathcal{N}(L) \ldots$)

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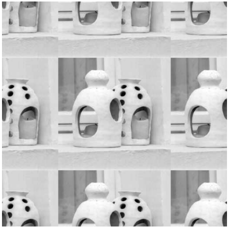
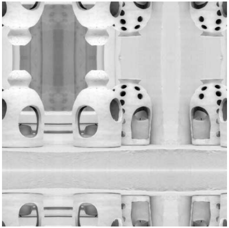
Boundary Conditions (BCs)

zero Dirichlet



Periodic



Reflective

Antireflective

The matrix C

Space invariant point spread function (PSF)

$$\Downarrow$$

T has a doubly Toeplitz-like structure that carries the "correct" boundary conditions.

- doubly circulant matrix C diagonalizable by FFT, that corresponds to periodic BCs.
- The boundary conditions have a very local effect

$$T - C = E + R, \tag{10}$$

where E is of small norm and R of small rank.

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Synthesis approach

- Images have a sparse representation in the wavelet domain.
- Let W^* be a wavelet or tight-frame synthesis operator ($W^*W = I$) and v the frame coefficients such that

$$x = W^* v.$$

- The deblurring problem can be reformulated in terms of the frame coefficients v as

$$\min_{v \in \mathbb{R}^s} \left\{ \mu \|v\|_1 + \frac{1}{2\lambda} \|v\|^2 : v = \arg\min_{v \in \mathbb{R}^s} \| TW^* v - y^\delta \|_P^2 \right\}. \tag{11}$$

Modified Linearized Bregman algorithm (MLBA)

- Denote by S_μ the soft-thresholding function

$$[S_\mu(v)]_i = \mathrm{sgn}(v_i) \max\{|v_i| - \mu, \, 0\}\,. \tag{12}$$

- The MLBA proposed in [Cai, Osher, and Shen, SIIMS 2009]

$$\begin{cases} z^{n+1} = z^n + WT^*P(y^\delta - TW^*v^n), \\ v^{n+1} = \lambda S_\mu(z^{n+1}), \end{cases} \tag{13}$$

  where $z^0 = v^0 = 0$.

- Choosing $P = (TT^* + \alpha I)^{-1} \Rightarrow \lambda = 1$ the iteration (13) converges to the unique minimizer of (11).
- The authors of MLBA proposed to use $P = (CC^* + \alpha I)^{-1}$.
- If $v^n = z^n$ the first equation (inner iteration) of MLBA is preconditioned Landweber.

AIT + Bregman splitting

- Replace preconditioned Landweber with AIT.
- Usual assumption

$$\|(C - T)u\| \le \rho \|Tu\|, \qquad u \in \mathcal{X}.$$

- Further assumption

$$\|CW^*(v - S_\mu(v))\| \le \rho\delta, \qquad \forall\, v \in \mathbb{R}^s, \tag{14}$$

which is equivalent to consider the soft-threshold parameter $\mu = \mu(\delta)$ and such that $\mu(\delta) \to 0$ as $\delta \to 0$.

AIT + Bregman splitting – 2

Algorithm [Cai, D., Bianchi, Huang, 2016]

$$\begin{cases} z^{n+1} = z^n + WC^*(CC^* + \alpha_n I)^{-1}(y^\delta - TW^*v^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{15}$$

where the parameter (α_n, stopping iteration, etc.) are fixed as in AIT.

Theorem

For noisy data ($\delta > 0$), as $\delta \to 0$, the approximation x^δ converges to the solution of $Tx = y$ that is closest to x_0 in the norm of $\mathcal{X}$.

If an estimation of the best α is available, we can fix $\alpha_n = \alpha_{\text{opt}}$.

Numerical Results

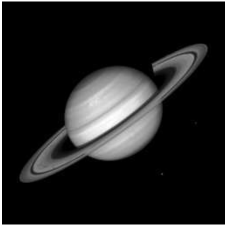
- W by linear B-spline.
- $\mathrm{PSNR} = 20 \log_{10} \frac{255 \cdot n}{\|x - \tilde{x}\|}$, with $\tilde{x}$ the computed approximation.
- Best regularization parameter by hand for every method.

Compared methods

- MLBA: iteration (13) by Cai, Osher, and Shen.
- AIT–Breg: our nonstationary iteration (15).
- AIT–Breg–opt: our iteration (15) with a stationary $\alpha_n = \alpha_{\mathrm{opt}}$ chosen by hand like in MLBA.
- FA–MD, TV–MD: ADMM [Almeida, Figueiredo, IEEE 2013] for Frame-based Analysis and Total Variation, respectively.
- FTVd: extension of FTVd in [Wang et al. SIIMS 2008] to deal with boundary artifacts [Bai et al., 2014].

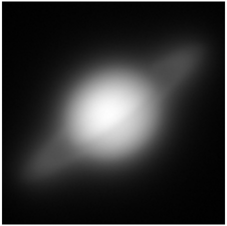
Example 3 (Saturn)

$\nu = 1\,\%$, Zero BCs.



True image

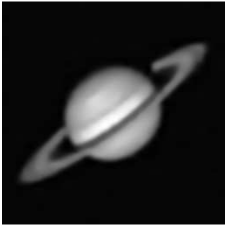
PSF



Observed image

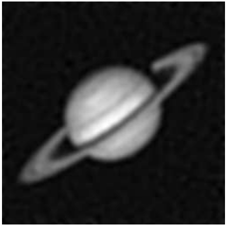
Restorations

Method	PSNR	CPU time
AIT–Breg	31.25	10.32
AIT–Breg–opt	31.49	16.56
MLBA	30.97	200.99
FA–MD	30.87	90.85
TV–MD	31.17	47.61
FTVd:	30.50	1.75



MLBA

AIT–Breg

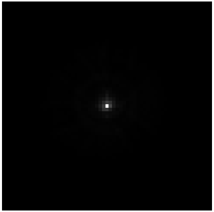


FTVd

Example 4 (Boat)

$\nu = 1\,\%$, Antireflective BCs.

True image



PSF

Observed image

Restorations

Method	PSNR	CPU time
AIT–Breg	29.77	19.57
AIT–Breg–opt	30.17	3.67
MLBA	29.43	34.26
FA–MD	29.61	15.95
TV–MD	29.87	16.74
FTVd:	28.95	0.73

MLBA

AIT–Breg–opt

TV–MD

Conclusions

- Under the assumption that an approximation C of T is available, our new scheme turns out to be fast and stable.
- The choice of ρ reflect how much we trust in the previous approximation (a too small ρ can be detected by α_n or $\|r_n\|$).
- Our scheme does not require T^*.
- Projection into a convex set can be added.
- It is possible to include a regularization matrix.
- It can be used as inner least-square iteration in nonlinear methods.

References

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