Identifying a conducting sphere via exponential sums

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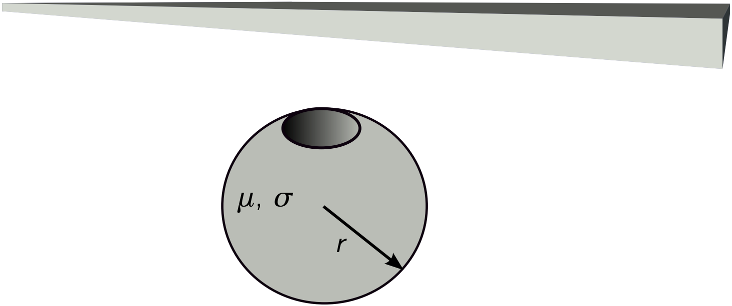
The scope of the research

Identifying a sphere in the underground
of radius r, conductivity σ and permeability μ



Application: discover of unexploded ordnance.

Our starting point

J. R. Wait
A conducting sphere in a time varying magnetic field
Geophysics, 16:666-672, 1951.

J.R.Wait, K. P. Spies.
Quasi static transient response of a conducting permeable sphere.
Geophysics, 34(5):789-792, 1969.

Our starting point

A mathematical model describing the step-function response is proposed.

The step-function response is the time behaviour of the outputs of a general system when its inputs change from zero to one in a very short time.

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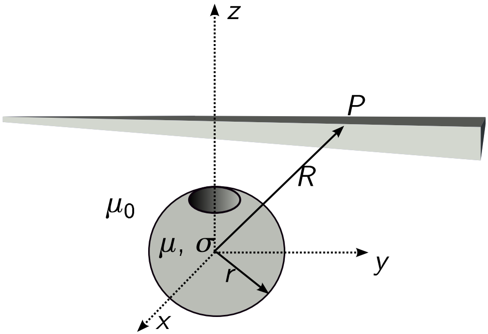
The setting of the problem

- Assume that in a the neighborhood of the sphere a uniform magnetic field $H = H_0 e^{\mathbf{i}\omega t}$ is applied with $\mu = \mu_0 = 4\pi 10^{-7}$, $\sigma = 0$ and is parallel to the polar axis.



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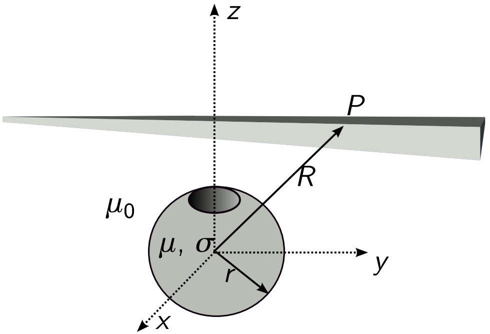
- Once the external field is applied eddy current is induced and produce a two secondary magnetic field inside and outside the sphere.



The step-function response

The mathematical model

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$$\tan x = \frac{(\mu_r - 1)x}{(\mu_r - 1) + x^2}, \qquad \mu_r = \frac{\mu}{\mu_0}$$

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- $c_n = \frac{12 \pi r}{\mu_0 \sigma} \frac{\delta_n^2}{(\mu_r + 2)(\mu_r - 1) + \delta_n^2}$.

A nonlinear approximation problem

Let us assume that we know the value of the function

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in 2N equispaced points $t_j = jh$, $j = 1, 2, 3, \ldots$.

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The method consists of three steps

1. identifying the coefficients d_n;
2. identifying the parameters c_n;
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Remark: We assume that we know the integer N such that

$$h(t) = \sum_{n=1}^{\infty} c_n \; e^{-d_n t} \simeq \sum_{n=1}^{N} c_n \; e^{-d_n t}$$

A look at the literature

There exists two families of numerical methods:

- Prony-like methods
- Matrix-pencil method

B. de Prony.
Essai expérimental et analytique sur les lois de la Dilatabilité des fluides élastiques et sur celles de la Force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures.
J. l'École Polytech., 1:24–76, 1795.

D. Potts and M. Tasche.
Parameter estimation for nonincreasing exponential sums by Prony-like methods.
Linear Algebra and Its Applications, 439(4):1024–1039, 2013.

L. Fermo, C. van der Mee and S. Seatzu.
Parameter estimation of monomial-exponential sums
Electronic Transactions on Numerical Analysis (ETNA), **41**: 249-261, 2014.

The Prony method

We rewrite

$$h(t_j) = \sum_{n=1}^{N} c_n \ e^{-d_n jh} \qquad \Longrightarrow \qquad h(j) = \sum_{n=1}^{N} c_n \ z_n^j, \quad z_n = e^{-d_n h}.$$

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We see it as the general solution of a homogeneous linear difference equation of order N of the type

$$\sum_{k=0}^{N-1} p_k h(k+m) = -h(m+n), \qquad m = 0, 1, \dots, N-1$$

where p_k are the coefficients of the Prony polynomial

$$P(z) = \prod_{j=1}^{n} (z - z_j)^{m_j} = \sum_{k=0}^{N-1} p_k z^k, \quad p_N \equiv 1$$

- Compute the coefficients of such a polynomial by solving the following system whose matrix of coefficients is a Hankel matrix

$$\begin{pmatrix} h(1) & h(2) & h(3) & \dots & h(n-1) \\ h(2) & h(3) & h(4) & \dots & h(n) \\ h(3) & h(4) & h(5) & \dots & h(n+1) \\ \dots & \dots & \dots & \ddots & \vdots \\ h(n-2) & h(n-1) & h(n) & \dots & h(2n-3) \\ h(n-1) & h(n) & h(n+1) & \dots & h(2n-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-2} \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} -h(n) \\ -h(n+1) \\ -h(n+2) \\ \vdots \\ -h(2n-2) \\ -h(2n-1) \end{pmatrix}$$

- Once we solve this system, by computing the eigenvalues of the following Companion matrix,

$$C = \begin{pmatrix} 0 & 0 & 0 & \dots & -p_0 \\ 1 & 0 & 0 & \dots & -p_1 \\ 0 & 1 & 0 & \dots & -p_2 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -p_{n-1} \end{pmatrix}$$

we can recover the z_n and then $d_n = -\frac{\log z_n}{h}$

II STEP Computation of $\{c_n\}$

The coefficients $\{c_n\}$ are the solution of the following system having a Vandermonde matrix as matrix of coefficients.

$$\begin{pmatrix} z_1^0 & z_2^0 & z_3^0 & \dots & z_n^0 \\ z_1^1 & z_2^1 & z_3^1 & \dots & z_n^1 \\ z_1^2 & z_2^2 & z_3^2 & \dots & z_n^2 \\ \dots & \dots & \dots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & z_3^{n-1} & \dots & z_n^{n-1} \\ z_1^n & z_2^n & z_3^n & \dots & z_n^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix} = \begin{pmatrix} -h(n) \\ -h(n+1) \\ -h(n+2) \\ \vdots \\ -h(2n-2) \\ -h(2n-1) \end{pmatrix}$$

Our first experiments

We have assumed that the sphere is made from a certain material and we evaluate the step-function

$$h(t) = \sum_{n=1}^{N} c_n \; e^{-d_n t}, \qquad t > 0$$

where

- $d_n = \frac{\delta_n^2}{r^2 \mu \sigma}$ with δ_n the nth zero of the following equation

$$\tan x = \frac{(\mu_r - 1)x}{(\mu_r - 1) + x^2}, \qquad \mu_r = \frac{\mu}{\mu_0}$$

- $c_n = \frac{12\pi r}{\mu_0 \sigma} \frac{\delta_n^2}{(\mu_r+2)(\mu_r-1)+\delta_n^2}.$

The numerical solution of the equation $\tan x = \dfrac{(\mu_r - 1)x}{(\mu_r - 1) + x^2}$

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However the presence of the asymptotes make unstable the method.

In order to overcome these difficulties we use the following numerical trick: if $\sqrt{\mu_r - 1} > n\pi + \frac{\pi}{2}$ we solve equation

$$\cot x = \frac{1}{x} + \frac{x}{\mu_r - 1}$$

if $\sqrt{\mu_r - 1} \leq n\pi + \frac{\pi}{2}$ we solve

$$\tan x = \frac{(\mu_r - 1)x}{(\mu_r - 1) + x^2}.$$

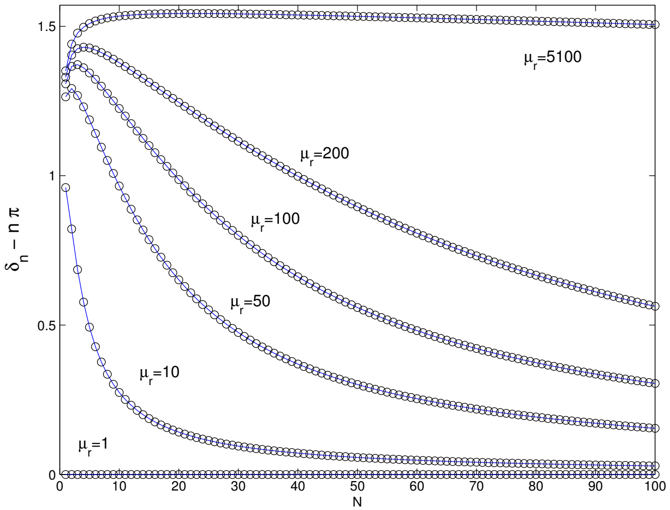


Figure: δ_n as a function of series terms and parametrically as a function of the relative permeability μ_r

Metallic material	σ (S/m)	μ (H/m)
Aluminium	3.50×10^{7}	1.256665×10^{-6}
Carbon Steel	6.99×10^{6}	1.26×10^{-4}
Ferritic stainless steel	1.45×10^{6}	2×10^{-3}
Iron	1.00×10^{7}	6.3×10^{-3}

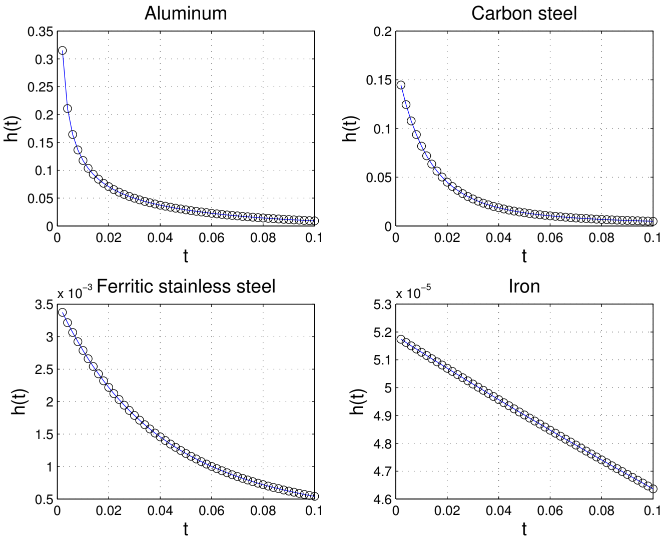


Figure: The behavior of the sum $h(t)$ for different materials for $n = 10$

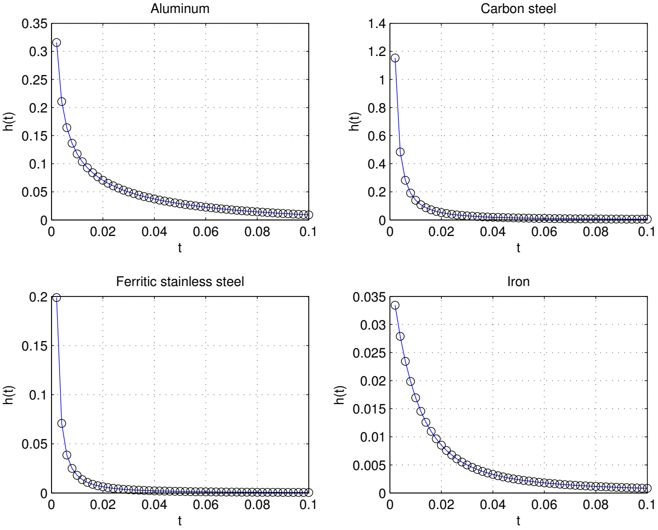


Figure: The behavior of the sum $h(t)$ for different materials for $n = 100$

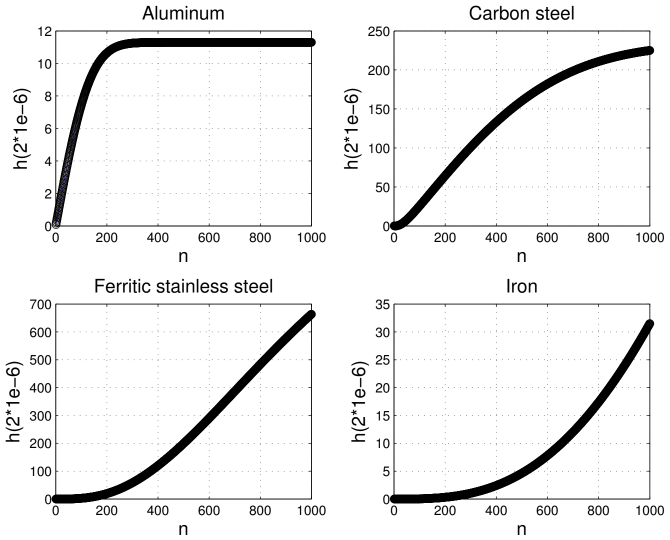


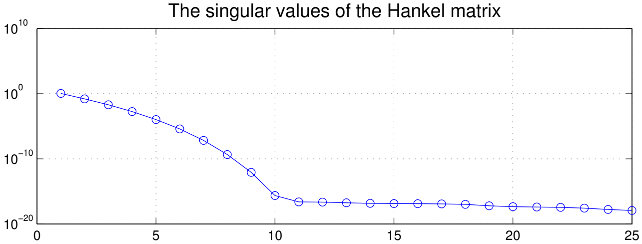
Figure: The behavior of the sum as function of n for different materials at $t = 2 \times 10^{-6}$

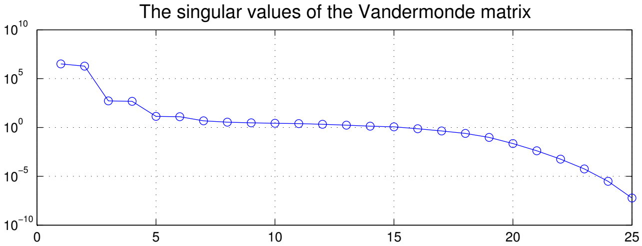
We have applied the Prony method to the sum

$$h(t) = \sum_{n=1}^{10} c_n \; e^{-d_n t}, \qquad t \in [0, 0.1]$$

in the aluminium case.

d_n	$\tilde{d}_n$	ϵ_n
2.243954973008462e+01	2.243954973008579e+01	5.20e-14
8.975789563175007e+01	8.975789563180739e+01	6.38e-13
2.019551388008130e+02	2.019551388187708e+02	8.89e-11
3.590312792374512e+02	3.590312827371441e+02	9.74e-09
5.609863169416867e+02	5.609866405940118e+02	5.76e-07
8.078202519135250e+02	8.078279136080553e+02	9.48e-06
1.099533084152968e+03	1.098811780829547e+03	6.56e-04
1.436124813660015e+03	1.385496497352069e+03	3.52e-02
1.817595440434669e+03	1.597756993664808e+03	1.20e-01
2.243944964476928e+03	2.107856078732385e+03	6.06e-02


The singular values of the Hankel matrix
10^10
10^0
10^−10
10^−20
0
5
10
15
20
25


The singular values of the Vandermonde matrix
10^10
10^5
10^0
10^−5
10^−10
0
5
10
15
20
25

Things to do...

1. "Modifying" the Prony method
2. Using the matrix-pencil methods
3. Develop a new method

A look at the matrix-pencil method - Computation of $\{z_j\}$

These methods are useful in the case when we have more data M then parameters N.

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$$\mathbf{H}_{MM} = \left(\begin{array}{c|ccc|} h(1) & h(2) & \dots & h(M) \\ h(2) & h(3) & \dots & h(M+1) \\ \vdots & \vdots & \vdots & \vdots \\ h(M) & h(M+1) & \dots & h(2M-1) \end{array} \right)$$

$$\mathbf{H}^1_{MM} = \left(\begin{array}{|ccc|c} h(2) & h(3) & \dots & h(M+1) \\ h(3) & h(4) & \dots & h(M+2) \\ \vdots & \vdots & \vdots & \vdots \\ h(M+1) & h(M+2) & \dots & h(2M) \end{array} \right)$$

The theory of finite difference equations allows us to prove the following lemma

Lemma

If the data are noiseless

- $\operatorname{rank} \mathbf{H}_{MM} = \operatorname{rank} \mathbf{H}^1_{MM} = N$

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Lemma

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- $\operatorname{rank} \mathbf{H}_{MM} = \operatorname{rank} \mathbf{H}^1_{MM} = N$
- $\mathbf{H}^1_{MN} = \mathbf{H}^0_{MN} \mathbf{C}_N(P)$ *where*
  - $\mathbf{C}_N(P)$ *is the companion matrix of the Prony polynomial, i.e.*

$$\mathbf{C}_N(P) = \begin{pmatrix} 0 & 0 & \dots & 0 & -p_0 \\ 1 & 0 & \dots & 0 & -p_1 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & -p_{N-1} \end{pmatrix}.$$

  - $\mathbf{H}_{MM} = [\mathbf{H}_{MN} \dots], \quad \mathbf{H}^1_{MM} = [\mathbf{H}^1_{MN} \dots]$

To the monomial-power sum $h(t)$ we associate the $N \times N$ matrix-pencil

$$\mathbf{H}_{NN}(z) = (\mathbf{H}_{MN})^*(\mathbf{H}^1_{MN} - z\mathbf{H}_{MN})$$

where the asterisk denotes the conjugate transpose.

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The difference equations properties allow us to state the following theorem

Theorem

The zeros z_j of the Prony polynomial, with their multiplicities, are exactly the generalized eigenvalues of the matrix-pencil $\mathbf{H}_{MM}(z)$.

THANKS FOR THE ATTENTION !