Cornelis VAN DER MEE, Spring 2008, Math 3330, Sample Exam 3

Name:...............................**Grade**:..........**Rank**:...........
To receive full credit, show all of your work. Neither calculators nor computers are allowed.

1. Consider the two vectors

$$\vec{\boldsymbol{u}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \vec{\boldsymbol{v}} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}.$$

   a. Compute the cosine of the angle between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.
   b. Compute the distance between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.
   c. Does there exist an orthogonal 3×3 matrix A such that $A\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}}$? If it exists, construct one. If it does not exist, explain why not.

Answer: a) We first compute

$$\|\vec{\boldsymbol{u}}\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}, \qquad \|\vec{\boldsymbol{v}}\| = \sqrt{3^2 + 0^2 + 4^2} = 5,$$
$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = (1.3) + ((-1).0) + (0.4) = 3.$$

Thus $\angle(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})/(\|\vec{\boldsymbol{u}}\| \, \|\vec{\boldsymbol{v}}\|) = (3/5\sqrt{2})$. b) The distance between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ equals $\|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}\| = \sqrt{(1-3)^2 + ((-1)-0)^2 + (0-4)^2} = \sqrt{21}$. c) Orthogonal matrices A preserve length in the sense that $\|A\vec{\boldsymbol{u}}\| = \|\vec{\boldsymbol{u}}\|$ for any vector $\vec{\boldsymbol{u}} \in \mathbb{R}^3$. Since $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ do not have the same length, such an orthogonal matrix A does not exist.

2. Find an orthonormal basis for

$$V = \text{span}\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

and use this information to write down the orthogonal projection of $\mathbb{R}^4$ onto V. Answer: Denote the three above vectors spanning V

by $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$, respectively. Now apply the Gram-Schmidt process. Compute $\|\vec{\boldsymbol{v}}_1\| = \sqrt{2}$ and define the unit vector $\vec{\boldsymbol{u}}_1 = \vec{\boldsymbol{v}}_1/\sqrt{2} = (1/\sqrt{2}, 0, 1/\sqrt{2}, 0)^T$. Then put

$$\vec{\boldsymbol{u}}_2 = \frac{\vec{\boldsymbol{v}}_2 - (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1}{\|\vec{\boldsymbol{v}}_2 - (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1\|} = \frac{1}{\|(-\frac{3}{2}, 0, \frac{3}{2}, 0)^T\|}\begin{pmatrix} -\frac{3}{2} \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since $\vec{\boldsymbol{v}}_3$ is orthogonal to $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ and hence to all of their linear combinations (and hence in particular to $\vec{\boldsymbol{u}}_1$ and $\vec{\boldsymbol{u}}_2$) and $\|\vec{\boldsymbol{v}}_3\| = 1$, the orthonormal basis of V is given by the three vectors in the right-hand side of

$$V = \operatorname{span}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right].$$

3. Find a least-squares solution to the system

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Answer: Writing the above system as $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, we have to find the vector $\vec{\boldsymbol{x}}^*$ which minimizes the distance $\|A\vec{\boldsymbol{x}}^* - \vec{\boldsymbol{b}}\|$. We first compute

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}.$$

Then the least-squares solution vector $\vec{\boldsymbol{x}}^*$ is given by

$$\begin{aligned} \vec{\boldsymbol{x}}^* = (A^T A)^{-1} A^T \vec{\boldsymbol{b}} &= \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{2}{9}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}. \end{aligned}$$

3.* (**TO BE REPLACED**) Find the factors Q and R in the QR factorization of the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 1 & -2 \end{pmatrix}$$

by using the Gram-Schmidt process. Answer: Apply the Gram-Schmidt process to find an orthonormal basis of the linear span $\operatorname{Im} M$ of the columns $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ of M. We get successively

$$\vec{\boldsymbol{u}}_1 = \frac{\vec{\boldsymbol{v}}_1}{\|\vec{\boldsymbol{v}}_1\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

$$\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{v}}_2 - (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 = \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \quad \vec{\boldsymbol{u}}_2 = \frac{\vec{\boldsymbol{w}}_2}{\|\vec{\boldsymbol{w}}_2\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}.$$

Consequently,

$$\underbrace{\left(\vec{\boldsymbol{v}}_1 \,\vdots\, \vec{\boldsymbol{v}}_2 \right)}_{=M} = \underbrace{\left(\vec{\boldsymbol{u}}_1 \,\vdots\, \vec{\boldsymbol{u}}_2 \right)}_{=Q} \underbrace{\begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{14} \\ 0 & 1/\sqrt{14} \end{pmatrix}^{-1}}_{=R} = \underbrace{\left(\vec{\boldsymbol{u}}_1 \,\vdots\, \vec{\boldsymbol{u}}_2 \right)}_{=Q} \underbrace{\begin{pmatrix} \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{14} \end{pmatrix}}_{=R}$$

or in other words

$$\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{14} \\ 2/\sqrt{6} & 2/\sqrt{14} \\ 1/\sqrt{6} & -3/\sqrt{14} \end{pmatrix} \begin{pmatrix} \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{14} \end{pmatrix}.$$

4. Find the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & -5 \\ 0 & 2 & 4 \\ 3 & 6 & 9 \end{pmatrix}.$$

Describe the parallelepiped whose volume is given by this determinant. Answer: By Sarrus's rule, $\det(A) = (1.2.9) + (1.4.3) + ((-5).0.6) - (3.2.(-5)) - (0.1.9) - (1.6.4) = 36$. The vectors pointing from the origin to the points with Cartesian coordinates $(1,0,3)$, $(1,2,6)$, and $(-5,4,9)$ span the parallelepiped whose volume is given by this determinant. Instead of these three points, we may also take the parallelepiped spanned by the vectors pointing from the origin to the points with Cartesian coordinates $(1,1,-5)$, $(0,2,4)$, and $(3,6,9)$.

5. Find the determinants of the 4×4 matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 2 & 5 & -1 \\ 4 & 2 & 8 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Answer: In A we interchange the first and fourth columns and then the second and third columns, which does not change the determinant. Thus

$$\det(A) = \begin{vmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 5 & 2 & 0 \\ -2 & 8 & 2 & 4 \end{vmatrix} = 3.1.2.4 = 24,$$

because the resulting matrix is lower triangular. To compute $\det(B)$, we apply Laplace expansion down the fourth column and get

$$\det(B) = 4 \times \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\begin{aligned} &= 4 \times (1 \times 1 \times 1) - \{(1.1.3) + (1.1.1) + (0.0.2) \\ &- (1.1.0) - (0.1.3) - (1.2.1)\} = 2. \end{aligned}$$

Here we used the fact that the first 3×3 determinant involved an upper triangular matrix. The second 3×3 determinant was computed by applying Sarrus's rule.

6. Let A be a 7×7 matrix with $\det(A) = -3$.

   a. Compute $\det(-2A)$.
   b. Compute $\det(AA^T)$.
   c. Compute $\det(A^T A^{-1})$.
   d. Compute the determinant of the matrix obtained from A by first interchanging the last two columns and then interchanging the first two rows.

   Answer:

a. $\det(-2A) = (-2)^7 \det(A) = 384$.

b. $\det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = 9$.

c. $\det(A^T A^{-1}) = \det(A^T)\det(A^{-1}) = \det(A)/\det(A) = 1$.

d. Each row or column interchange multiplies the determinant by -1. Since two such interchanges are applies, the determinant does not change and remains -3.