Cornelis VAN DER MEE, Spring 2008, Math 3330, Sample Exam 4

Name:. . . . . **Grade**:. . . . . **Rank**:. . . . .
To receive full credit, show all of your work. Neither calculators nor computers are allowed.

1. Compute the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 15 & -10 \end{pmatrix}.$$

Use this information to diagonalize the matrix A if possible. Otherwise indicate why diagonalization is not possible. Solution: First we compute the characteristic polynomial

$$\det(\lambda I_2 - A) = \det \begin{pmatrix} \lambda - 3 & 2 \\ -15 & \lambda + 10 \end{pmatrix} = (\lambda-3)(\lambda+10)+30 = \lambda(\lambda+7).$$

The eigenvalues of A are its zeros 0 and -7. The eigenspace corresponding to the eigenvalue $\lambda = 0$ (which is $\operatorname{Ker} A$) consists of the multiples of the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. The eigenspace corresponding to the eigenvalue -7 (which is $\operatorname{Ker}(A + 7I_2) = \operatorname{Ker} \begin{pmatrix} 10 & -2 \\ 15 & -3 \end{pmatrix}$) consists of the multiples of the vector $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Since the eigenvalues of A are distinct, we can diagonalize A. In fact,

$$A \underbrace{\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}}_{=S} \begin{pmatrix} 0 & 0 \\ 0 & -7 \end{pmatrix},$$

where S is the (invertible) diagonalizing transformation.

2. Find a 2×2 matrix A such that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are eigenvectors of A, with eigenvalues -2 and 3, respectively. Solution: We obviously have

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \qquad A \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix},$$

or in other words

$$A\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix} = \begin{pmatrix}-2 & 6\\ -4 & 9\end{pmatrix}.$$

Hence,

$$A = \begin{pmatrix}-2 & 6\\ -4 & 9\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}^{-1} = \begin{pmatrix}-2 & 6\\ -4 & 9\end{pmatrix}\begin{pmatrix}-3 & 2\\ 2 & -1\end{pmatrix} = \begin{pmatrix}18 & -10\\ 30 & -17\end{pmatrix}.$$

3. Find the solution of the discrete dynamical system

$$x(n+1) = Ax(n), \qquad n = 0, 1, 2, 3, \ldots,$$

where

$$A = \begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}, \qquad x(0) = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

Solution: The solution is, in abstract form, $x(n) = A^n x(0)$, where $n = 0, 1, 2, \ldots$. To find a more amenable solution we diagonalize A, $AS = S\mathrm{diag}(\lambda_1, \lambda_2)$, put $y(n) = Sx(n)$, and write the dynamical system in the simplified form

$$y(n+1) = Sx(n+1) = SAx(n) = \mathrm{diag}(\lambda_1, \lambda_2)Sx(n) = \mathrm{diag}(\lambda_1, \lambda_2)y(n),$$

which is easy to solve. Since A is upper triangular, its eigenvalues are its diagonal elements $\lambda_1 = 2$ and $\lambda_2 = 1$, while it is easily seen that $A\begin{pmatrix}1\\ 0\end{pmatrix} = 2\begin{pmatrix}1\\ 0\end{pmatrix}$. To find the other eigenvector (which must be linearly independent and hence must have a nonzero second entry), we try

$$A\begin{pmatrix}x\\ 1\end{pmatrix} = \begin{pmatrix}x\\ 1\end{pmatrix},$$

which implies $2x + 3 = x$ and $1 = 1$. Thus $x = -3$ and $\begin{pmatrix}-3\\ 1\end{pmatrix}$ is the eigenvector corresponding to the eigenvalue 1. We now line up the two eigenvectors as column of a 2×2 matrix to get

$$S = \begin{pmatrix}1 & -3\\ 0 & 1\end{pmatrix}.$$

Since $y(0) = Sx(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, we get

$$y(n) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \cdot 2^n \\ 1 \end{pmatrix}.$$

Hence,

$$x(n) = S^{-1}y(n) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \cdot 2^n \\ 1 \end{pmatrix} = \begin{pmatrix} 3[1 - 2^n] \\ 1 \end{pmatrix}.$$

4. Find all eigenvalues (real and complex) of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{pmatrix}.$$

Why or why not is it possible to diagonalize the matrix A? Solution: Let us compute the characteristic polynomial of A:

$$\begin{aligned} \det(\lambda I_3 - A) &= \det \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 4 & \lambda - 1 \end{pmatrix} \\ &= \lambda \det \begin{pmatrix} \lambda & -1 \\ 4 & \lambda - 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -4 & \lambda - 1 \end{pmatrix} \\ &= \lambda\{\lambda(\lambda - 1) + 4\} - 4 = \lambda^3 - \lambda^2 + 4\lambda - 4 \\ &= (\lambda^2 + 4)(\lambda - 1). \end{aligned}$$

Since the eigenvalues [1, $2i$, and $-2i$] are distinct, it is possible to diagonalize A.

5. Compute the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Use this information to diagonalize the matrix A if possible. Otherwise indicate why diagonalization is not possible. Solution: Let us compute the characteristic polynomial

$$\det(\lambda I_4 - A) = \det\begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda-3 & 3 & -1 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{pmatrix} = \lambda \det\begin{pmatrix} \lambda-3 & 3 & -1 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{pmatrix}$$
$$= \lambda\{\lambda^2(\lambda-3) - 1 + 3\lambda\} = \lambda(\lambda-1)^3.$$

Thus the eigenvalues are 0 and 1, where 1 has algebraic multiplicity 3. Since the first column of A has only zeros, the eigenvector corresponding to the zero eigenvalue is $(1,0,0,0)^T$. Now observe that

$$(I_4 - A)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_4 \\ -2x_2 + 3x_3 - x_4 \\ -x_2 + x_3 \\ -x_3 + x_4 \end{pmatrix},$$

which can only be the zero vector if $x_1 = x_2 = x_3 = x_4 = 1$. Thus the eigenspace corresponding to the eigenvalue 1 is one-dimensional and hence 1 is an eigenvalue of geometric multiplicity 1. Since algebraic and geometric multiplicities differ and hence there is no basis of $\mathbb{R}^4$ consisting of eigenvectors of A, the matrix A cannot be diagonalizable.

6. Compute the eigenvalues (real and complex) and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Explain why your result is in full agreement with the values of $\mathrm{Tr}(A)$

and $\det(A)$. Solution: Let us compute the characteristic polynomial

$$\begin{aligned}
\det(\lambda I_4 - A) &= \det \begin{pmatrix} \lambda & 0 & 0 & -1 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{pmatrix} \\
&= \lambda \det \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{pmatrix}}_{\text{lower triangular}} + \begin{pmatrix} 0 & 0 & -1 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{pmatrix} \\
&= \lambda^4 - 1 = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i).
\end{aligned}$$

Thus the eigenvalues are 1, -1, i, and $-i$. Their sum is $\operatorname{Tr} A = 0$. Their product is $\det(A) = -1$.

7. Why is $45 = 1+2+3+4+5+6+7+8+9$ an eigenvalue corresponding to any 9×9 sudoku matrix? What is the corresponding eigenvector?