Cornelis VAN DER MEE, Spring 2008, Math 3330, Sample Final Exam

Name:..............................**Grade**:..........**Rank**:...........

1. Bring the following matrix to reduced row echelon form:

$$A = \begin{pmatrix} 0 & 0 & 1 & 3 & -7 \\ 1 & -5 & 0 & 2 & 4 \\ 0 & 0 & 2 & 6 & -2 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix},$$

and determine its rank and nullity. Solution: Interchange the first two rows and then subtract twice the second row from the third row:

$$\begin{pmatrix} 1 & -5 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & -7 \\ 0 & 0 & 2 & 6 & -2 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & -5 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}.$$

Divide the third row by 12 and the fourth row by 9 and subtract the third row from the fourth row, thus creating a final row of zeros. Then add seven times the third row to the second row, subtract three times the third row from the first row:

$$\begin{pmatrix} 1 & -5 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{1} & -5 & 0 & 2 & 0 \\ 0 & 0 & \mathbf{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is the row reduced echelon form with the leading one's written in boldface. Thus the rank is 3, the number of leading one's. The nullity is $5 - 3 = 2$, the number of columns not containing a leading 1.

2. Find **all** solutions of the linear system $A\vec{\boldsymbol{x}} = \mathbf{0}$, where

$$A = \begin{pmatrix} 3 & 6 & 7 \\ 0 & 2 & -1 \end{pmatrix}.$$

Solution: Divide the first row by 3 and the second row by 2, and then subtract twice the second row from the first row:

$$\begin{pmatrix} 1 & 2 & 7/3 \\ 0 & 1 & -1/2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 10/3 \\ 0 & 1 & -1/2 \end{pmatrix},$$

which is the row reduced echelon form. The variable x_3 will be the unique parameter in the solution and the solution is given by

$$x_1 = -\frac{10}{3}x_3, \qquad x_2 = \frac{1}{2}x_3.$$

3. Consider the following 4×4 matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 2 & 6 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

   a. Find a basis of the image of A and show that it really is a basis.
   b. Find a basis of the kernel of A and show that it really is a basis.

Solution: Interchange the first two rows and the last two rows, then interchange the second and third rows, and then divide the second and last rows by 2,

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Next subtract the second row from the third row and then the third row from the last row, creating a last row of zeros:

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is the row reduced echelon forms with the leading one's written in boldface. Thus the first three columns of the original matrix A form a basis of the image of A:

$$\operatorname{Im} A = \operatorname{span}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}\right].$$

Using the above row reduced echelon form we now determine the kernel of A by solving the corresponding linear system with zero right-hand sides. This system $[x_1 - 2x_4 = 0,\ x_2 = 0,\ x_3 + 3x_4,\ 0 = 0]$ has the solution $x_1 = 2x_4$, $x_2 = 0$, and $x_3 = -3x_4$, where x_4 is a parameter. Thus

$$\operatorname{Ker} A = \operatorname{span}\left[\begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix}\right].$$

4. Argue why or why not the set of polynomials

$$1, \qquad x, \qquad 2x^2 - 1, \qquad 4x^3 - 3x, \qquad 8x^4 - 8x^2 + 1,$$

is a basis of the vector space of polynomials of degree ≤ 4. Solution: The vector space of all (real) polynomials of degree ≤ 4 has as a basis $\{1, x, x^2, x^3, x^4\}$ and hence has dimension 5. With respect to this basis, the five given polynomials can be represented by the five respective column vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -3 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -8 \\ 0 \\ 8 \end{pmatrix}.$$

The 5×5 matrix having these five vectors as columns,

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix},$$

is upper triangular with nonzero diagonal entries and hence invertible. Thus the five given polynomials form a basis of the vector space of polynomials of degree ≤ 4.

5. Find the matrix of the orthogonal projection of $\mathbb{R}^4$ onto the hyperplane

$$x_1 - 2x_2 + 3x_3 - 4x_4 = 0.$$

Solution: The orthogonal projection is given by

$$\vec{\boldsymbol{u}}_1(\vec{\boldsymbol{u}}_1)^T + \vec{\boldsymbol{u}}_2(\vec{\boldsymbol{u}}_2)^T + \vec{\boldsymbol{u}}_3(\vec{\boldsymbol{u}}_3)^T,$$

where $\{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \vec{\boldsymbol{u}}_3\}$ is an orthonormal basis of this hyperplane. However, the hyperplane consists of all vectors which are perpendicular to the column vector $(1, -2, 3, -4)^T$. Dividing this vector by its length $\sqrt{30}$, we see that the orthogonal projection onto the line containing the column vector $(1, -2, 3, -4)^T$ equals

$$\frac{1}{30}\begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 & -4 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix}.$$

The orthogonal projection onto the hyperplane is obtained by subtraction the previous projection from the identity matrix, i.e.,

$$P = \frac{1}{30}\begin{pmatrix} 29 & 2 & -3 & 4 \\ 2 & 26 & 6 & -8 \\ -3 & 6 & 21 & 12 \\ 4 & -8 & 12 & 14 \end{pmatrix}.$$

6. Apply the Gram-Schmidt process to the given basis vectors of

$$V = \operatorname{span}\left[\begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 3 \end{pmatrix}\right]$$

to obtain an orthonormal basis of V. Solution: Let us call the three basis vectors $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$, respectively. The orthonormal basis of V,

$\{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \vec{\boldsymbol{u}}_3\}$, is computed as follows:

$$\vec{\boldsymbol{u}}_1 = \frac{\vec{\boldsymbol{v}}_1}{\|\vec{\boldsymbol{v}}_1\|} = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix},$$

$$\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{v}}_2 - (\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} -12/5 \\ 0 \\ 9/5 \\ 0 \end{pmatrix},$$

$$\vec{\boldsymbol{u}}_2 = \frac{\vec{\boldsymbol{w}}_2}{\|\vec{\boldsymbol{w}}_2\|} = \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix},$$

$$\begin{aligned} \vec{\boldsymbol{w}}_3 &= \vec{\boldsymbol{v}}_3 - (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 - (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_2)\vec{\boldsymbol{u}}_2 \\ &= \begin{pmatrix} 4 \\ 0 \\ 0 \\ 3 \end{pmatrix} + \frac{16}{5} \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix} - \frac{12}{5} \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \end{aligned}$$

$$\vec{\boldsymbol{u}}_3 = \frac{\vec{\boldsymbol{w}}_3}{\|\vec{\boldsymbol{w}}_3\|} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

7. Find a least-squares solution to the system

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Solution: Writing the system as $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, a least squares solution is given by

$$(A^T A)^{-1} A^T \vec{\boldsymbol{b}},$$

where

$$A^T A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 10 \end{pmatrix}.$$

Thus a least squares solution is given by

$$\begin{pmatrix}5 & 2\\ 2 & 10\end{pmatrix}^{-1}\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 3\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix} = \frac{1}{46}\begin{pmatrix}10 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 3\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix}$$
$$= \frac{1}{46}\begin{pmatrix}10 & 18 & -6\\ -2 & 1 & 15\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix} = \frac{1}{46}\begin{pmatrix}14\\11\end{pmatrix}.$$

8. Find the factors Q and R in the QR factorization of the matrix

$$M = \begin{pmatrix}1 & 0\\ 2 & 4\\ 1 & -2\end{pmatrix}$$

by using the Gram-Schmidt process. Solution: Use the Gram-Schmidt process to find an orthonormal basis of the linear span $\operatorname{Im} M$ of the columns $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ of M. We get successively

$$\vec{\boldsymbol{u}}_1 = \frac{\vec{\boldsymbol{v}}_1}{\|\vec{\boldsymbol{v}}_1\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix},$$

$$\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{v}}_2 - (\vec{\boldsymbol{v}}_2\cdot\vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 = \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1 = \begin{pmatrix}-1\\2\\-3\end{pmatrix}, \quad \vec{\boldsymbol{u}}_2 = \frac{\vec{\boldsymbol{w}}_2}{\|\vec{\boldsymbol{w}}_2\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}-1\\2\\-3\end{pmatrix}.$$

Consequently,

$$\underbrace{\left(\vec{\boldsymbol{v}}_1 \,\vdots\, \vec{\boldsymbol{v}}_2\right)}_{=M} = \underbrace{\left(\vec{\boldsymbol{u}}_1 \,\vdots\, \vec{\boldsymbol{u}}_2\right)}_{=Q}\underbrace{\begin{pmatrix}1/\sqrt{6} & -1/\sqrt{14}\\ 0 & 1/\sqrt{14}\end{pmatrix}}_{=R} = \underbrace{\left(\vec{\boldsymbol{u}}_1 \,\vdots\, \vec{\boldsymbol{u}}_2\right)}_{=Q}\underbrace{\begin{pmatrix}\sqrt{6} & \sqrt{6}\\ 0 & \sqrt{14}\end{pmatrix}}_{=R}$$

or in other words [by using $R = Q^T M$]

$$\begin{pmatrix}1 & 0\\ 2 & 4\\ 1 & -2\end{pmatrix} = \begin{pmatrix}1/\sqrt{6} & -1/\sqrt{14}\\ 2/\sqrt{6} & 2/\sqrt{14}\\ 1/\sqrt{6} & -3/\sqrt{14}\end{pmatrix}\begin{pmatrix}\sqrt{6} & \sqrt{6}\\ 0 & \sqrt{14}\end{pmatrix}.$$

9. Find the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 5 \\ 2 & 6 & 4 \end{pmatrix}.$$

Solution: By Sarrus's rule,

$$\begin{aligned} \det A &= 1.0.4 + 2.5.2 + 2.6.(-3) - 2.0.(-3) - 2.2.4 - 6.5.1 \\ &= 0 + 20 - 36 - 0 - 16 - 30 = -62. \end{aligned}$$

10. Compute the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 3 \\ 1 & 7 \end{pmatrix}.$$

Use this information to diagonalize the matrix A if possible. Otherwise indicate why diagonalization is not possible. Solution: The eigenvalues of A are the solutions of the quadratic equation $\det(\lambda I - A) = 0$. In fact,

$$\begin{aligned} \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 5 & -3 \\ -1 & \lambda - 7 \end{pmatrix} &= (\lambda - 5)(\lambda - 7) - 3 \\ = \lambda^2 - 12\lambda + 32 &= (\lambda - 4)(\lambda - 8), \end{aligned}$$

hence the eigenvalues are 4 and 8. Having distinct eigenvalues, the matrix A is diagonalizable. Let us compute the eigenvectors:

$$\lambda = 4 : \begin{pmatrix} -1 & -3 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathrm{Ker}(4I - A) = \mathrm{span} \left[\begin{pmatrix} 3 \\ -1 \end{pmatrix} \right],$$

$$\lambda = 8 : \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathrm{Ker}(8I - A) = \mathrm{span} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

Thus the diagonalizing transformation S is given as follows:

$$A \underbrace{\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}}_{S} \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}.$$

11. Find all eigenvalues (real and complex) of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -9 & -9 & -1 \end{pmatrix}.$$

Why or why not is it possible to diagonalize the matrix A? Solution: The eigenvalues of A are the zeros of the cubic equation $\det(\lambda I - A) = 0$. In fact,

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & 0 & -1 \\ -1 & \lambda & 0 \\ 9 & 9 & \lambda + 1 \end{pmatrix} = \lambda \begin{vmatrix} \lambda & 0 \\ 9 & \lambda + 1 \end{vmatrix} - \begin{vmatrix} -1 & \lambda \\ 9 & 9 \end{vmatrix}$$
$$= \lambda^2(\lambda + 1) + (9 + 9\lambda) = (\lambda + 1)(\lambda^2 + 9).$$

Thus the eigenvalues of A are -1 and $\pm 3i$. Since the eigenvalues of A are distinct, the matrix A is diagonalizable.

12. Find the solution of the discrete dynamical system

$$x(n + 1) = Ax(n), \qquad n = 0, 1, 2, 3, \ldots,$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}, \qquad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution: The eigenvalues of A are the zeros of the quadratic equation

$$0 = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 3 & -4 \\ -4 & \lambda + 3 \end{pmatrix} = (\lambda - 3)(\lambda + 3) - 16 = \lambda^2 - 25,$$

hence $\lambda = \pm 5$. The eigenvectors are to be found as follows:

$$\lambda = -5 : \begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{Ker}(-5I - A) = \text{span} \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} \right],$$
$$\lambda = 5 : \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{Ker}(5I - A) = \text{span} \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right].$$

Now let us $x(0)$ as a linear combination of the eigenvectors:

$$x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{2}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Consequently,

$$x(n) = \frac{1}{5} A^n \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{2}{5} A^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{5} (-5)^n \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{2}{5} 5^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$