Cornelis VAN DER MEE, Spring 2008, Math 3330, Exam 1

Name:................................**Grade**:..........**Rank**:...........

To receive full credit, show all of your work. Neither calculators nor computers are allowed.

1. Bring the following matrix to reduced row echelon form:

$$A = \begin{pmatrix} 1 & 3 & 0 & 5 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

and determine its rank. Answer: Subtract the second row from the third row, multiply the third row by -1, and divide the fourth row by 5 to get

$$\begin{pmatrix} \mathbf{1} & 3 & 0 & 5 & 3 \\ 0 & 0 & \mathbf{1} & 4 & -1 \\ 0 & 0 & 0 & \mathbf{1} & -3 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix},$$

where the leading one's have been written in boldface. We now clean out the last column above the leading 1 by subtracting three times the fourth row from the first, add three times the fourth row to the third row, and adding the fourth row to the second row, operations that do not change the first four columns. We get

$$\begin{pmatrix} \mathbf{1} & 3 & 0 & 5 & 0 \\ 0 & 0 & \mathbf{1} & 4 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}.$$

Finally, we clean out the fourth column above the leading 1 by subtracting five times the third row from the first row and four times the third row from the second row. As a result, we get the row reduced echelon form

$$\text{rref}(A) = \begin{pmatrix} \mathbf{1} & 3 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}.$$

Consequently, $\text{rank}(A) = 4$, the number of leading one's in $\text{rref}(A)$.

2. If the augmented matrix for a nonhomogeneous system of linear equations has been reduced by row operations to the matrix

$$\left(\begin{array}{ccccc:c} 1 & 7 & 0 & 0 & -3 & 11 \\ 0 & 0 & 2 & 0 & 8 & -10 \\ 0 & 0 & 0 & 1 & -6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

what is the solution to this linear system? Answer: Let us divide the second row by 2 and arrive at the row reduced echelon form

$$\left(\begin{array}{ccccc:c} \mathbf{1} & 7 & 0 & 0 & -3 & 11 \\ 0 & 0 & \mathbf{1} & 0 & 4 & -5 \\ 0 & 0 & 0 & \mathbf{1} & -6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where the leading one's have been written in boldface. Hence the last equation is the irrelevant tautology $0 = 0$, the second and fifth variables end up as parameters in the solution, and

$$x_1 = 11 - 7x_2 + 3x_5, \qquad x_3 = -5 - 4x_5, \qquad x_4 = 7 + 6x_5.$$

3. Find **all** solutions of the linear system $A\vec{\boldsymbol{x}} = \mathbf{0}$, where

$$A = \begin{pmatrix} 2 & 3 & 6 \\ 0 & 1 & -1 \end{pmatrix}.$$

Answer: Construct the augmented matrix by adding a zero column:

$$\left(\begin{array}{ccc:c} 2 & 3 & 6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right).$$

Next, divide the first row by 2 to arrive at

$$\left(\begin{array}{ccc:c} 1 & \frac{3}{2} & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right).$$

Next we subtract 3/2 times the second row from the first row to arrive at the row reduced echelon form

$$\left(\begin{array}{ccc:c} \mathbf{1} & 0 & \frac{9}{2} & 0 \\ 0 & \mathbf{1} & -1 & 0 \end{array}\right),$$

where the leading one's have been written in boldface. Thus the rank of the coefficient matrix and that of the augmented matrix both equal 2, while the third variable ends up as the parameter in the solution which is given by

$$x_1 = -\frac{9}{2}x_3, \qquad x_2 = x_3.$$

4. Evaluate the inverse of the matrix

$$A = \begin{pmatrix} 11 & 7 \\ -4 & -3 \end{pmatrix}.$$

Answer: The determinant equals $\det A = 11 \times (-3) - 7 \times (-4) = -5$. Thus

$$A^{-1} = \frac{1}{-5}\begin{pmatrix} -3 & -7 \\ 4 & 11 \end{pmatrix} = \begin{pmatrix} 3/5 & 7/5 \\ -4/5 & -11/5 \end{pmatrix}.$$

5. Find the matrix A such that

$$A\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \qquad A\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \qquad A\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}.$$

Answer: Dividing the first equality by 2, changing the signs in the second equality, and subtract the first equality from the third, we get

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}, \qquad A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Since we have in fact lined up the columns of A, we get

$$A = \begin{pmatrix} -2 & -5 & 0 \\ 1 & -3 & 3 \end{pmatrix}.$$

6. Determine the matrix of the projection of any point $\vec{\boldsymbol{x}} \in \mathbb{R}^3$ onto the line through the origin and the point $(3, 4, 0)$. Answer: The column vector with entries $(3, 4, 0)$ has length $\sqrt{3^2 + 4^2 + 0^2} = 5$. Let

$$\vec{\boldsymbol{u}} = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$$

be the proportional column vector of unit length. Then the projection matrix P is given by

$$P = \begin{pmatrix} (u_1)^2 & u_1u_2 & u_1u_3 \\ u_2u_1 & (u_2)^2 & u_2u_3 \\ u_3u_1 & u_3u_2 & (u_3)^2 \end{pmatrix} = \begin{pmatrix} 9/25 & 12/25 & 0 \\ 12/25 & 16/25 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $u_1 = \frac{3}{5}$, $u_2 = \frac{4}{5}$, and $u_3 = 0$.

7. Determine the matrix of the counterclockwise rotation in $\mathbb{R}^2$ through the angle $\theta = 45°$. Answer: This rotation moves the point $(1,0)$ to a point on $y = x$ in the first quadrant and $(0,1)$ to a point on $y = -x$ in the second quadrant, both of them at a distance of 1 from the origin. Thus

$$R = \begin{pmatrix} a & -a \\ a & a \end{pmatrix},$$

where $a > 0$ and $\sqrt{a^2 + a^2} = \sqrt{(-a)^2 + a^2} = 1$. Thus $a = \frac{1}{2}\sqrt{2}$. Another solution consists of using the well-known expression

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta = 45°$. We then use that $\cos 45° = \sin 45° = \frac{1}{2}\sqrt{2}$.

8. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 4 & 5 \end{pmatrix}.$$

Start from the augmented matrix

$$\left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 1 & 0 \\ 0 & 4 & 5 & 0 & 0 & 1 \end{array}\right)$$

Now switch the second and third rows to get

$$\left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 0 & 1 \\ 0 & 0 & 6 & 0 & 1 & 0 \end{array}\right)$$

Then divide the second row by 4 and the third row by 6 to arrive at

$$\left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 0 \end{array}\right)$$

Now subtract twice the second row from the first, resulting in

$$\left(\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 0 \end{array}\right)$$

Finally, subtract $\frac{1}{2}$ times the third row from the first row and $\frac{5}{4}$ times the third row from the second row. We thus obtain the reduced row echelon matrix

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -\frac{1}{12} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{5}{24} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 0 \end{array}\right)$$

Since

$$\operatorname{rref}\left(A \,\vdots\, I\right) = \operatorname{rref}\left(I \,\vdots\, A^{-1}\right),$$

we get

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{12} & -\frac{1}{2} \\ 0 & -\frac{5}{24} & \frac{1}{4} \\ 0 & \frac{1}{6} & 0 \end{pmatrix}.$$

9. Compute A^3, where $A = \begin{pmatrix} 1 & a \\ 0 & 3 \end{pmatrix}$ and a is a parameter. Answer:

$$A^3 = (A^2)A = \begin{pmatrix} 1 & 4a \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 13a \\ 0 & 27 \end{pmatrix}.$$

10. Compute the matrix product ABC, where

$$A = \begin{pmatrix} 5 & 7 & 0 \\ 2 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -5 & 3 \\ 0 & 3 & 2 \\ 0 & -2 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Answer: Compute

$$A(BC) = \begin{pmatrix} 5 & 7 & 0 \\ 2 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5 & 6 \\ 0 & -3 & 4 \\ 0 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 58 \\ 2 & 21 & 6 \\ 0 & 4 & 12 \end{pmatrix}.$$