Cornelis VAN DER MEE, Spring 2008, Math 3330, Exam 3

Name:................................**Grade**:..........**Rank**:...........

To receive full credit, show all of your work. Neither calculators nor computers are allowed.

1. Consider the two vectors

$$\vec{\boldsymbol{u}} = \begin{pmatrix} -15 \\ 20 \end{pmatrix}, \qquad \vec{\boldsymbol{v}} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}.$$

   a. Compute the lengths of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

   b. Compute the cosine of the angle between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

   c. Construct an orthogonal 2×2 matrix A such that $A\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}}$.

   d. Is it possible to choose the matrix A in part c in such a way that $\det(A) = 1$? If it is possible, compute such an orthogonal matrix A and explain its geometrical meaning. If it is not possible, argue why not.

Solution: The lengths of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are given by $\|\vec{\boldsymbol{u}}\| = \sqrt{(-15)^2 + 20^2} = 25$ and $\|\vec{\boldsymbol{v}}\| = \sqrt{7^2 + 24^2} = 25$. Hence, the cosine of the angle α between them is given by $(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})/(\|\vec{\boldsymbol{u}}\|\,\|\vec{\boldsymbol{v}}\|) = 375/25^2 = \frac{3}{5}$. Since $\|\vec{\boldsymbol{u}}\| = \|\vec{\boldsymbol{v}}\|$, there exists a rotation matrix A such that $A\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}}$. Using that $\sin\alpha = -\frac{4}{5}$ (because $\vec{\boldsymbol{u}}$ is in the second quadrant and $\vec{\boldsymbol{v}}$ in the first quadrant, we need to rotate in the clockwise direction), we get

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix},$$

which satisfies $\det(A) = 1$.

2. Find an orthonormal basis for

$$V = \text{span}\left[\begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}\right]$$

and use this information to write down the orthogonal projection of $\mathbb{R}^4$ onto V. Solution: Write $V = \text{span}[\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3]$. Since $\|\vec{\boldsymbol{v}}_1\| = \sqrt{5}$, we get for the first orthonormal basis vector

$$\vec{\boldsymbol{u}}_1 = \vec{\boldsymbol{v}}_1/\sqrt{5} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{pmatrix}.$$

Next, compute

$$\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{v}}_2 - \underbrace{(\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{u}}_1)}_{=8/\sqrt{5}} \vec{\boldsymbol{u}}_1 = \begin{pmatrix} 8/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\|\vec{\boldsymbol{w}}_2\| = \frac{4}{5}\sqrt{5}$, we get for the second orthonormal basis vector

$$\vec{\boldsymbol{u}}_2 = \vec{\boldsymbol{w}}_2/\frac{4}{5}\sqrt{5} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \\ 0 \end{pmatrix}.$$

Next, compute

$$\vec{\boldsymbol{w}}_3 = \vec{\boldsymbol{v}}_3 - \underbrace{(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_1)}_{=-1/\sqrt{5}} \vec{\boldsymbol{u}}_1 - \underbrace{(\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_2)}_{=2/\sqrt{5}} \vec{\boldsymbol{u}}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix},$$

so that $\|\vec{\boldsymbol{w}}_3\| = 2$. Hence the third orthonormal basis vector is given by

$$\vec{\boldsymbol{u}}_3 = \frac{1}{2}\vec{\boldsymbol{w}}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The orthogonal projection P of $\mathbb{R}^4$ onto V is given by

$$\begin{aligned}
P &= \vec{\boldsymbol{u}}_1\vec{\boldsymbol{u}}_1^T + \vec{\boldsymbol{u}}_2\vec{\boldsymbol{u}}_2^T + \vec{\boldsymbol{u}}_3\vec{\boldsymbol{u}}_3^T \\
&= \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ -\frac{2}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{4}{5} & \frac{2}{5} & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Second Solution: Since the matrix having $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$ as its columns has rank 3 and has a third row of zeros, it is clear that $V = \{\vec{\boldsymbol{x}} \in \mathbb{R}^4 : x_3 = 0\}$. This subspace has the orthonormal basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus the orthogonal projection P of $\mathbb{R}^4$ onto V is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Find a least-squares solution to the system

$$\begin{pmatrix} 3 & 4 \\ -4 & 3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Solution: Writing the system of equations as $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ and a least squares solution as $\vec{\boldsymbol{x}}^*$, we get

$$\begin{aligned}
\vec{\boldsymbol{x}}^* &= (A^TA)^{-1}A^T\vec{\boldsymbol{b}} = \begin{pmatrix} 25 & 0 \\ 0 & 50 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -4 & 0 \\ 4 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{50} \end{pmatrix} \begin{pmatrix} 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 3/25 \\ -3/25 \end{pmatrix}.
\end{aligned}$$

This is the only least squares solution, because $\operatorname{Ker} A = \{0\}$.

4. Find the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & -5 \\ -1 & 1 & 8 \\ 3 & 3 & 7 \end{pmatrix}.$$

Describe the parallelepiped whose volume is given by this determinant.
Solution: By Sarrus's rule,

$$\begin{aligned}\det(A) &= (1.1.7) + (2.8.3) + ((-1).3.(-5)) - (3.1.(-5)) - ((-1).2.7) \\ &\quad - (1.8.3) = 7 + 48 + 15 + 15 + 14 - 24 = 75.\end{aligned}$$

The vectors pointing from the origin to the points with Cartesian coordinates $(1, -1, 3)$, $(2, 1, 3)$, and $(-5, 8, 7)$ span the parallelepiped whose volume is given by this determinant.

5. Find the determinants of the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 5 \\ 4 & 3 & 9 & -7 \\ 0 & 3 & 2 & -2 \\ 0 & 0 & 2 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 9 & -8 & 0 & 0 & 5 \end{pmatrix}.$$

Solution: To compute $\det(A)$, we interchange, in A the first two rows, then the second and third rows, and finally the last two rows, resulting in an upper triangular matrix. Therefore,

$$\det(A) = -\begin{vmatrix} 4 & 3 & 9 & -7 \\ 0 & 3 & 2 & -2 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 5 \end{vmatrix} = -(4.3.2.5) = -120.$$

To compute $\det(B)$, we subtract $9/2$ times the first row from the last row and add $25/4$ times the second row to the last row. We then expand the resulting determinant with respect to the first column twice, and finally compute a 3×3 determinant by Sarrus's rule. In other words,

$$\det(B) = \det \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & -\frac{25}{2} & 0 & 0 & 5 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & \frac{25}{4} & 0 & 5 \end{pmatrix}$$

$$= 2 \times \det \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & \frac{25}{4} & 0 & 5 \end{pmatrix} = 2 \times 2 \times \det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ \frac{25}{4} & 0 & 5 \end{pmatrix}$$

$$= 2 \times 2 \times \left(20 + \frac{25}{4} + 0 - 0 - 0 - 0 \right) = 105.$$

6. Let A be an 8×8 matrix with $\det(A) = -2$.

   a. Compute $\det(-\sqrt{2}\,A)$.

   b. Compute $\det(A^T A^3)$.

   c. Compute $\det(SA^2S^{-1})$, where S is an 8×8 matrix satisfying $\det(S) = 7$.

   d. Compute the determinant of the matrix obtained from A by first interchanging the first two columns, then interchanging the last two columns, and then dividing the second row by 2.

Solution: $\det(-\sqrt{2}\,A) = (-\sqrt{2})^8 \det(A) = 16.(-2) = -32$. Next, $\det(A^T A^3) = \det(A^T)\det(A)^3 = \det(A)^4 = (-2)^4 = 16$. Next,

$$\det(SA^2S^{-1}) = \det(A^2) = \det(A)^2 = (-2)^2 = 4.$$

Finally, since the number of row/column interchanges is two (hence even), the only change in the determinant is caused by dividing a row by 2. Thus the final matrix in Part d) has determinant $(-2)/2 = -1$.