Cornelis VAN DER MEE, Spring 2008, Math 3330, Exam 4

Name:................................**Grade**:..........**Rank**:...........
To receive full credit, show all of your work. Neither calculators nor computers are allowed.

1. Compute the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 6 & 1 \\ 3 & 8 \end{pmatrix}.$$

Use this information to diagonalize the matrix A if possible. Otherwise indicate why diagonalization is not possible. Solution: The eigenvalues of A are the zeros of $\det(\lambda I - A)$. In fact,

$$\begin{aligned}\det(\lambda I - A) &= \begin{vmatrix} \lambda-6 & -1 \\ -3 & \lambda-8 \end{vmatrix} = (\lambda-6)(\lambda-8)-3 \\ &= \lambda^2 - 14\lambda + 45 = (\lambda-5)(\lambda-9).\end{aligned}$$

Thus the eigenvalues, 5 and 9, are distinct and hence A is diagonalizable. The eigenvectors are to be computed as follows:

$$\lambda = 5: \begin{pmatrix} -1 & -1 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{Ker}(5I - A) = \text{span}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right],$$

$$\lambda = 9: \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{Ker}(9I - A) = \text{span}\left[\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right].$$

Moreover,

$$A\underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}}_{S} \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix},$$

where S is the diagonalizing transformation.

2. Find a 2×2 matrix A such that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are eigenvectors of A, with eigenvalues -2 and 1, respectively. Solution: The matrix A must satisfy

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \qquad A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

In other words, by combining columns in one matrix we get

$$A\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} = \begin{pmatrix}-2 & 2\\ -4 & -1\end{pmatrix}.$$

Hence,

$$\begin{aligned} A &= \begin{pmatrix}-2 & 2\\ -4 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}^{-1} \\ &= \frac{1}{-5}\begin{pmatrix}-2 & 2\\ -4 & -1\end{pmatrix}\begin{pmatrix}-1 & -2\\ -2 & 1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2 & -6\\ -6 & -7\end{pmatrix}. \end{aligned}$$

3. Consider the discrete dynamical system

$$x(n+1) = Ax(n), \qquad n = 0, 1, 2, 3, \ldots,$$

where

$$A = \begin{pmatrix}3 & -3\\ 1 & -1\end{pmatrix}, \qquad x(0) = \begin{pmatrix}2\\ 0\end{pmatrix}.$$

   a. Write $x(0)$ as a linear combination of eigenvectors of A.
   b. Compute $x(n)$ for $n = 1, 2, 3, \ldots$.

Solution: The matrix A has two proportional and hence is not invertible. Thus one of its eigenvalues is $\lambda = 0$. Since $\operatorname{Tr} A = 3 - 1 = 2$, the other eigenvalue is $\lambda = 2$. Let us now find the eigenvectors:

$$\lambda = 0: \begin{pmatrix}-3 & 3\\ -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \operatorname{Ker} A = \operatorname{span}\left[\begin{pmatrix}1\\ 1\end{pmatrix}\right],$$

$$\lambda = 2: \begin{pmatrix}-1 & 3\\ -1 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \operatorname{Ker}(2I - A) = \left[\begin{pmatrix}3\\ 1\end{pmatrix}\right].$$

It is now easily seen that

$$x(0) = \begin{pmatrix}2\\ 0\end{pmatrix} = \begin{pmatrix}3\\ 1\end{pmatrix} - \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Hence, for $n = 1, 2, 3, \ldots$ we obtain

$$x(n) = A^n x(0) = A^n\begin{pmatrix}3\\ 1\end{pmatrix} - A^n\begin{pmatrix}1\\ 1\end{pmatrix} = 2^n\begin{pmatrix}3\\ 1\end{pmatrix} - 0^n\begin{pmatrix}1\\ 1\end{pmatrix} = 2^n\begin{pmatrix}3\\ 1\end{pmatrix}.$$

4. Find all eigenvalues (real and complex) of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{pmatrix}.$$

Explain why or why not the matrix A is diagonalizable. Solution: The eigenvalues of A are the zeros of the cubic polynomial $\det(\lambda I - A)$. In fact,

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda - 3 \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -1 \\ 2 & \lambda - 3 \end{vmatrix}$$
$$= \lambda\{\lambda(\lambda - 3) + 2\} = \lambda(\lambda - 1)(\lambda - 2).$$

Thus the eigenvalues, 0, 1, and 2, are distinct and A is diagonalizable.

5. Compute the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Use this information to diagonalize the matrix A if possible. Otherwise indicate why diagonalization is not possible. Solution: The matrix A has the following block structure:

$$A = \begin{pmatrix} A^{\text{up}} & 0_{2\times 1} & 0_{2\times 1} \\ 0_{1\times 2} & 3 & 0 \\ 0_{1\times 2} & 0 & 4 \end{pmatrix},$$

where

$$A^{\text{up}} = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$$

is an upper triangular matrix. Thus the eigenvalues of A^{up} are 1 and 2 and those of A are 1, 2, 3, and 4. Thus A has distinct eigenvalues and hence is diagonalizable. Let us now diagonalize A^{up}. Indeed,

$$\lambda = 1: \begin{pmatrix} 0 & -5 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \operatorname{Ker}(I - A^{\text{up}}) = \operatorname{span}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right],$$

$$\lambda = 2: \begin{pmatrix} 1 & -5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{Ker}(2I - A^{\text{up}}) = \text{span}\left[\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right].$$

Now the eigenbasis of A corresponding to the respective eigenvalues 1, 2, 3, and 4 is given by

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right],$$

while by lining up these column vectors into one 4×4 matrix one gets the diagonalizing transformation S. Consequently,

$$A \underbrace{\begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=S} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}}_{=\text{diag}(1,2,3,4)}.$$

6. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

a. Compute the eigenvalues (real and complex) of the matrix A.

b. Compute the **algebraic** multiplicities of these eigenvalues.

c. Explain why your result is in full agreement with the values of $\text{Tr}(A)$ and $\det(A)$.

Solution: The eigenvalues of A are the zeros of the quadratic polynomial $\det(\lambda I - A)$. In fact,

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 & 1 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 2 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 2 \\ 0 & -1 & \lambda \end{vmatrix} - \begin{vmatrix} -1 & \lambda & 0 \\ 0 & -1 & \lambda \\ 0 & 0 & -1 \end{vmatrix}$$

$$= \lambda^2(\lambda^2 + 2) + 1 = (\lambda^2 + 1)^2.$$

Thus $\lambda = \pm i$ are both eigenvalues of algebraic multiplicity two.[1] Thus the sum of the eigenvalues is $2[i+(-i)] = 0$ (where the multiplicities are to be taken into account), which coincides with the sum of the diagonal elements, $\mathrm{Tr}(A)$, of A. The product of the eigenvalues is $[i.(-i)]^2 = 1$, which coincides with the determinant of A.

[1]It can be shown that either eigenvalue has geometric multiplicity one, thus A is not diagonalizable.