Secondo PARZIALE

LUNEDI 7 GENNAIO

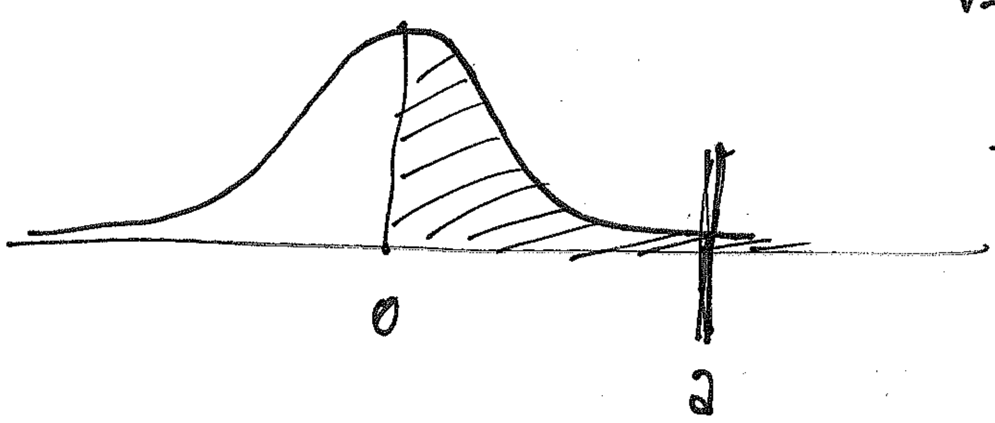
15:30 – 18:30

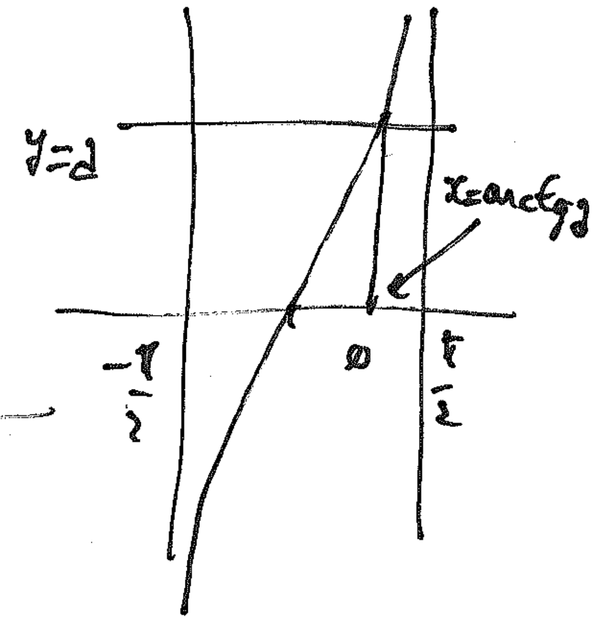
Aula Magna di MATEMATICA

<u>orali</u>: 9 gennaio 9-13, magari 10 gennaio

INTEGRALI generalizzati/impropri

$$\int_0^\infty \frac{dx}{1+x^2}$$





$$f(x) = \frac{1}{1+x^2}$$

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

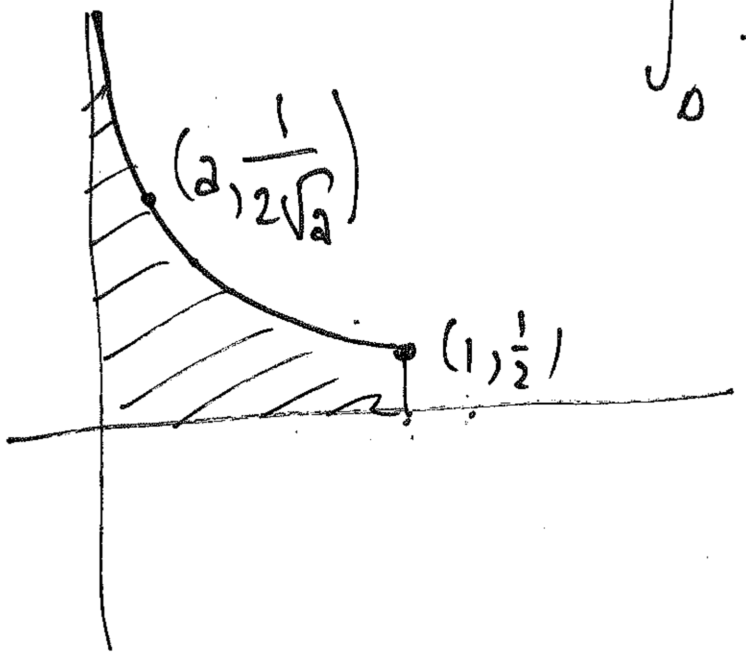
$$\int_0^a \frac{dx}{1+x^2} = \Big[\operatorname{arctg} x\Big]_0^a = \operatorname{arctg} a \longrightarrow \frac{\pi}{2}$$

se $a \to +\infty$

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{a \to +\infty} \int_0^a \frac{dx}{1+x^2} = \frac{\pi}{2}$$

$$\int_0^1 \frac{dx}{2\sqrt{x}}$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

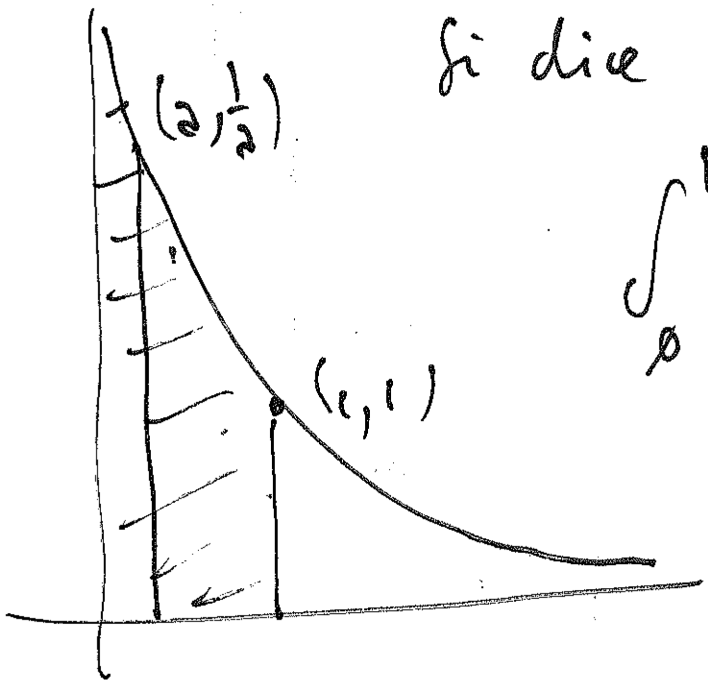


For $a \in (0,1)$,

$$\int_a^1 \frac{dx}{2\sqrt{x}} = \left[\sqrt{x}\right]_a^1 = 1 - \sqrt{a} \longrightarrow 1 \text{ as } a \to 0^+$$

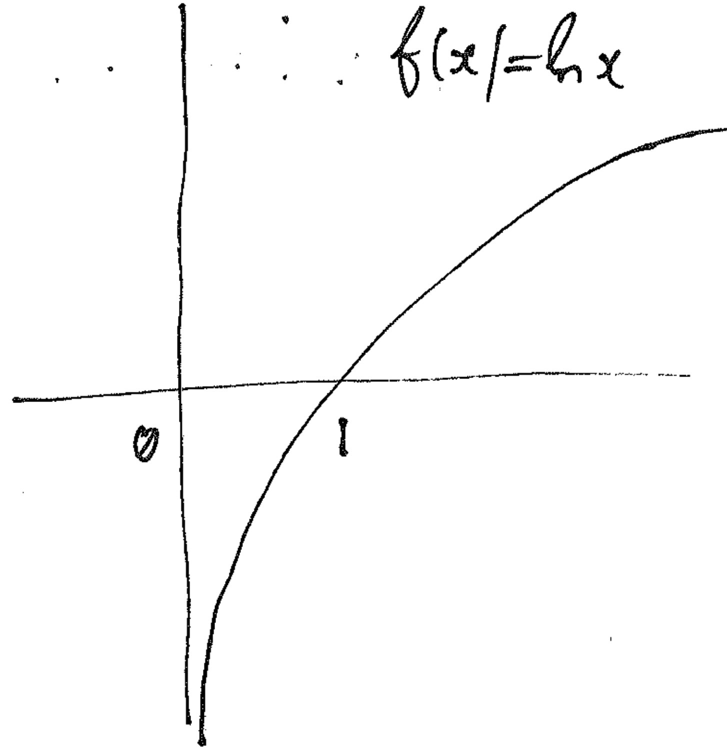
$$\int_0^1 \frac{dx}{2\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{2\sqrt{x}}$$

$$\int_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} \left[\ln x\right]_a^1 = -\ln a \longrightarrow +\infty \text{ se } a\to 0^+$$



si dice che è divergente

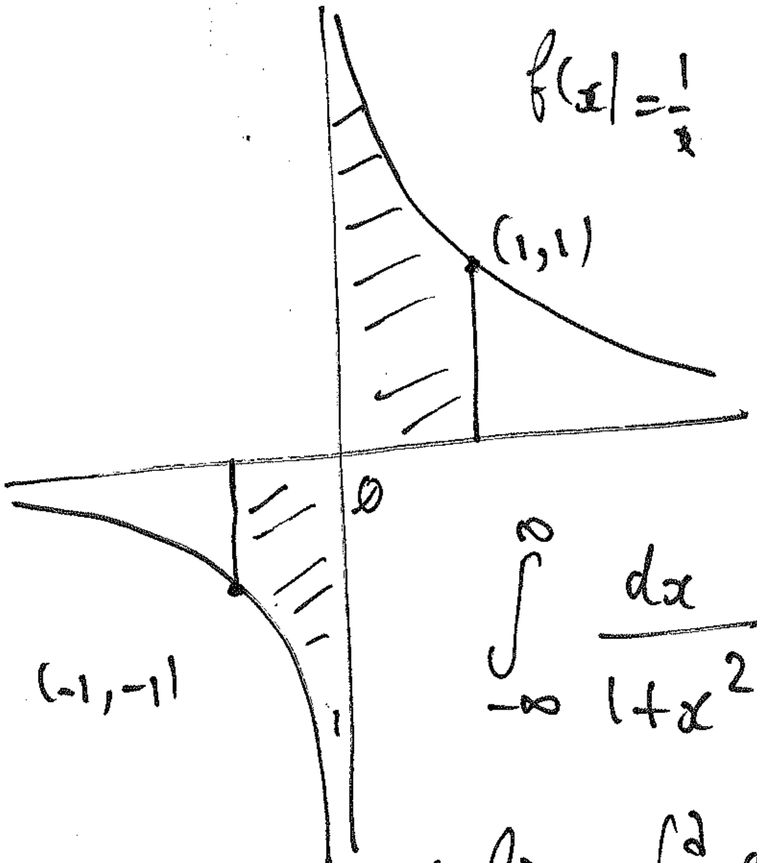
$$\int_0^1 \frac{dx}{x}$$



$$\int_{-1}^{+1} \frac{dx}{x}$$

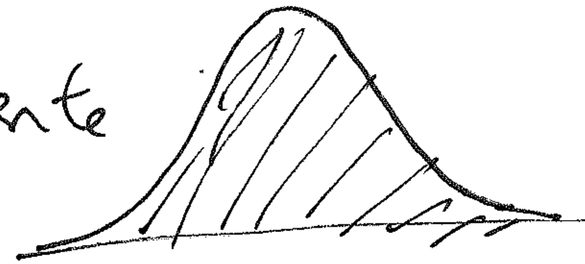
$$\int_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} \{-\ln a\} = +\infty$$

$f(x) = \frac{1}{x}$ $\quad \int_{-1}^{0} \frac{dx}{x} = \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x} = \lim_{b\to 0^-} \ln|b| = -\infty$



$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

Divergente

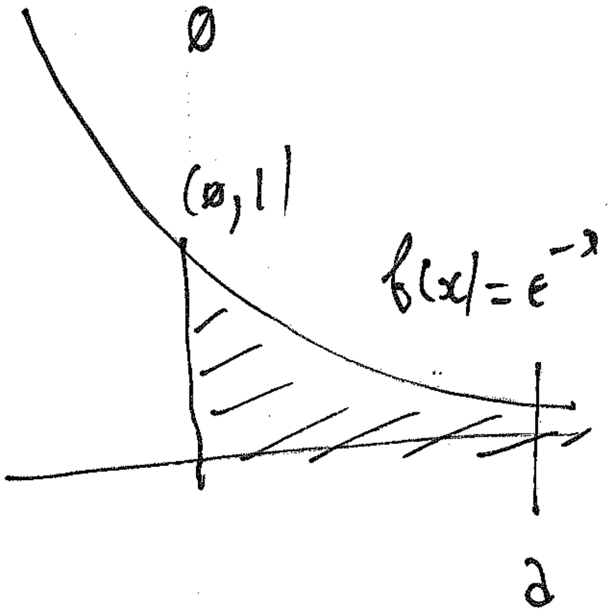


$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to -\infty} \int_b^0 \frac{dx}{1+x^2}$$

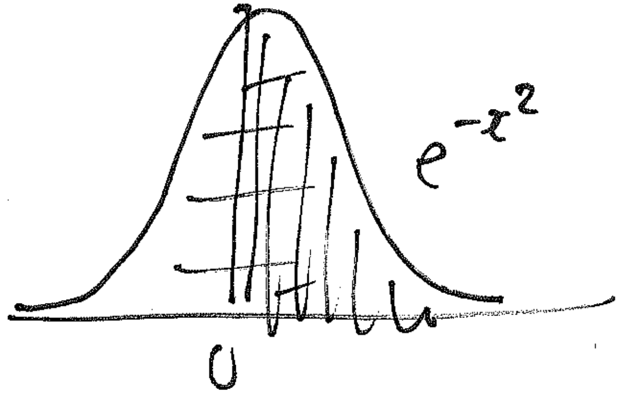
$$+ \lim_{a\to +\infty} \int_0^a \frac{dx}{1+x^2} = \lim_{b\to -\infty} (-\operatorname{arctg} b) + \lim_{a\to +\infty} (\operatorname{arctg} a) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\int_0^{\infty} e^{-x}\,dx = \lim_{a\to+\infty}\int_0^a e^{-x}\,dx = \lim_{a\to+\infty}\left[-e^{-x}\right]_0^a$$

$$= \lim_{a\to+\infty}\left(1-e^{-a}\right) = 1$$



$$\int_0^{\infty} e^{-x^2}\,dx \overset{?}{=} \frac{1}{2}\sqrt{\pi}$$

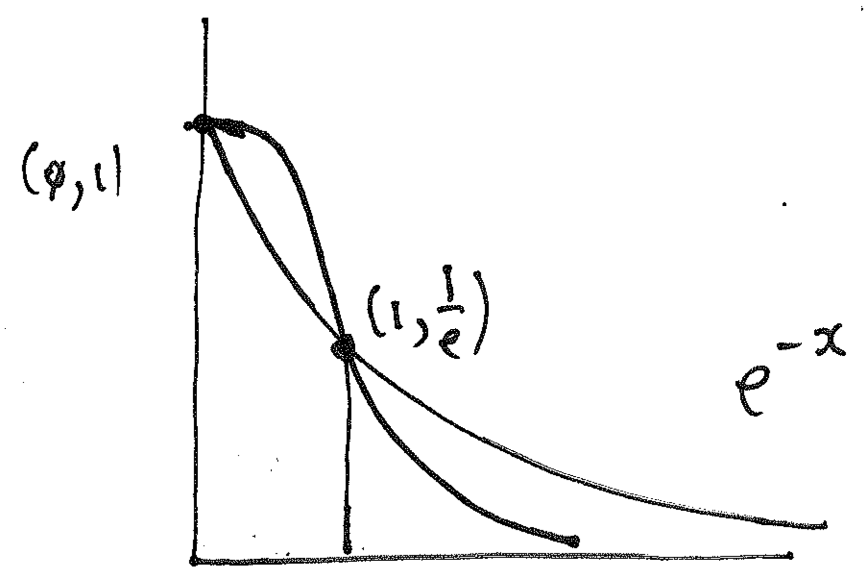


$$= \lim_{a\to+\infty}\int_0^a e^{-x^2}\,dx$$

$$F(a) = \int_0^a e^{-x^2}\,dx \longrightarrow F'(a) = e^{-a^2} > 0 \,,\quad F(0) = 0$$

F è crescente

~~Sia~~ Consideriamo $\quad \sup_{a>0} F(a) = \lim_{a \to +\infty} \int_0^a e^{-x^2}\,dx$



$$F(a) - F(1) = \int_1^a e^{-x^2}\,dx$$

$$< \int_1^a e^{-x}\,dx = \left[-e^{-x}\right]_1^a = \frac{1}{e} - e^{-a} \to \frac{1}{e}$$

se $a \to +\infty$

$$F(a) < F(1) + \frac{1}{e}$$

Sia convergente $\int_0^\infty g(x)\,dx$, dove $g \geq 0$

Sia $0 \leq f(x) \leq g(x)$, $x \geq 0$.

allora è convergente $\int_0^\infty f(x)\,dx$

CRITERIO del CONFRONTO

Sia divergente $\int_0^\infty f(x)\,dx$, dove $f \geq 0$.

Sia $0 \leq f(x) \leq g(x)$, $x \geq 0$

allora è divergente $\int_0^\infty g(x)\,dx$

$$\int_0^{\infty} \frac{1+\operatorname{sen}(x^{2007})}{1+x^2}\, dx$$

$$0 \leq \frac{1+\operatorname{sen}(x^{2007})}{1+x^2} \leq \frac{2}{1+x^2}$$

→ È convergente $\displaystyle\int_0^{\infty} \frac{1+\operatorname{sen}(x^{2007})}{1+x^2}\, dx$

$\displaystyle\int_0^{\infty} \frac{2}{1+x^2}\, dx$ è convergente $\left(= \lim_{a \to +\infty} 2 \operatorname{arctg} a = \pi \right)$

$$\int_0^{\infty} \frac{2+\operatorname{arctg}(x)}{\sqrt{1+x^2}}\, dx$$

$$\frac{d}{dx}\ln\left(x+\sqrt{1+x^2}\right)=\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$$

$$\int_0^a \frac{dx}{\sqrt{1+x^2}}=\left[\ln\left(x+\sqrt{1+x^2}\right)\right]_0^a=\ln\left(a+\sqrt{1+a^2}\right)\xrightarrow[\text{se } a\to+\infty]{}+\infty$$

$\hookrightarrow$ è divergente $\int_0^a \frac{dx}{\sqrt{1+x^2}}$

$2+\operatorname{arctg}(x)\geq 2,\ x\geq 0$

$$\frac{2+\operatorname{arctg}(x)}{\sqrt{1+x^2}}\geq\frac{2}{\sqrt{1+x^2}}$$

$\rightarrow$ è divergente

$$\int_0^\infty \frac{2+\operatorname{arctg}(x)}{\sqrt{1+x^2}}\,dx$$

$$\int_e^{+\infty} \frac{dx}{x(\ln x)^p}, \quad p>0$$

$$\int_e^{a} \frac{dx}{x(\ln x)^p} = \int_1^{\ln a} \frac{dt}{t^p} \to \begin{cases} \dfrac{1}{p-1} & \text{se } p>1 \\ +\infty & \text{se } 0<p\leq 1 \end{cases}$$

$$\int_e^{\infty} \frac{dx}{x(\ln x)^p} \begin{cases} \text{convergente se } p>1 \\ \text{divergente se } 0<p\leq 1 \end{cases}$$

$$\int_e^{a} \frac{dx}{x(\ln x)^p}, \quad e<a, \quad p>0$$

$$t=\ln x \to dt = \frac{dx}{x}$$

$$= \int_1^{\ln a} \frac{dt}{t^p} = \begin{cases} \left[\dfrac{t^{1-p}}{1-p}\right]_1^{\ln a}, & p\neq 1 \\ \left[\ln t\right]_1^{\ln a}, & p=1 \end{cases}$$

$\underline{p>1}$: $$\frac{1}{p-1}\left(1-\frac{1}{(\ln a)^{p-1}}\right) \longrightarrow \frac{1}{p-1}, \quad a\to+\infty$$

$\underline{0<p<1}$: $$\frac{(\ln a)^{1-p}}{1-p} - \frac{1}{1-p} \longrightarrow +\infty \text{ se } a\to+\infty$$

$\underline{p=1}$: $$\ln \ln a \longrightarrow +\infty \text{ se } a\to+\infty$$

$$\int_{e^2}^{a} \frac{dx}{x(\ln x)(\ln\ln x)} = \int_{2}^{\ln a} \frac{dt}{t \ln t} = \int_{\ln 2}^{\ln\ln a} \frac{du}{u} = [\ln u]_{\ln 2}^{\ln\ln a}$$

$t = \ln x$

$dt = \frac{dx}{x}$

$u = \ln t$

$du = \frac{dt}{t}$

$$= \ln\ln\ln a - \ln\ln 2 \to +\infty$$

$$\int_{e^2}^{\infty} \frac{dx}{x(\ln x)(\ln\ln x)} \text{ è divergente}$$

$$\frac{d}{dx} \ln(\underbrace{\ln\ln x}) = \frac{1}{x(\ln x)(\ln\ln x)}$$

$$\frac{1}{\ln\ln x} \frac{d}{dx} \ln(\ln x) = \frac{1}{\ln\ln x} \frac{1}{\ln x} \frac{1}{x}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{a\to 1^-} \int_0^a \frac{dx}{\sqrt{1-x^2}} = \lim_{a\to 1^-} \Big[\operatorname{arcsen} x\Big]_0^a$$

$$= \lim_{a\to 1^-} \operatorname{arcsen} a = \frac{\pi}{2}$$

$$\sqrt[4]{\frac{1}{16}} = \frac{1}{2}, \quad \sqrt{\frac{1}{16}} = \frac{1}{4}$$

$$\int_0^1 \frac{dx}{\sqrt[4]{1-x^2}}$$

$$0 \le 1-x^2 \le 1$$

$$0 \le \sqrt{1-x^2} \le \sqrt[4]{1-x^2}$$

$$0 < \frac{1}{\sqrt[4]{1-x^2}} \le \frac{1}{\sqrt{1-x^2}}, \quad 0 \le x < 1$$

è convergente $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

è convergente $\int_0^1 \frac{dx}{\sqrt[4]{1-x^2}}$