$$(1+x)^n \geq 1+nx \qquad x \geq -1 \quad n = 1, 2, 3, \ldots$$

BERNOUILLI

$$(1+x)^{n+1} = (1+x)(1+x)^n$$

$$\geq (1+x)(1+nx)$$

$$= 1+(n+1)x+nx^2$$

$$\geq 1+(n+1)x$$

$$a_n = \left(1+\frac{1}{n}\right)^n \qquad b_n = \left(1+\frac{1}{n}\right)^{n+1}$$

$$a_{n-1} \leqslant a_n \Longleftrightarrow \left(1+\frac{1}{n}\right)^n \geqslant \left(1+\frac{1}{n-1}\right)^{n-1} \longleftrightarrow \left(\frac{n+1}{n}\right)^n \geqslant \left(\frac{n}{n-1}\right)^{n-1}$$

$$\longleftrightarrow \left(\frac{n+1}{n}\right)^n \geqslant \left(\frac{n}{n-1}\right)^n \frac{n-1}{n} \longleftrightarrow \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^n \geqslant \frac{n-1}{n}$$

$$\longleftrightarrow \left(\frac{n^2-1}{n^2}\right)^n \geqslant \frac{n-1}{n} \longleftrightarrow \left(1-\frac{1}{n^2}\right)^n \geqslant 1 - n\cdot\frac{1}{n^2}$$

$$x = -\frac{1}{n^2} \geqslant -1$$

$$b_{n-1} \geq b_n \iff \left(1+\frac{1}{n}\right)^{n+1} \leq \left(1+\frac{1}{n-1}\right)^{n} \iff \left(\frac{n+1}{n}\right)^{n+1} \leq \left(\frac{n}{n-1}\right)^{n}$$

$$\iff \frac{n+1}{n}\left(\frac{n+1}{n}\right)^{n} \leq \left(\frac{n}{n-1}\right)^{n} \iff \frac{n+1}{n} \leq \left(\frac{n}{n-1}\right)^{n}\left(\frac{n}{n+1}\right)^{n}$$

$$\iff \frac{n+1}{n} \leq \left(\frac{n^2}{n^2-1}\right)^{n} \iff \left(1+\frac{1}{n^2-1}\right)^{n} \geq 1+\frac{1}{n}$$

$$x = \frac{1}{n^2-1} \geq -1, \; n \geq 2 \qquad \left(1+\frac{1}{n^2-1}\right)^{n} \geq 1+\frac{n}{n^2-1} \geq 1+\frac{n}{n^2} = 1+\frac{1}{n}$$

$$2 = a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_2 \leq b_1 = 4$$

$$a_n = \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n}\right)^{n+1} = b_n$$

$$\alpha = \sup\{a_n : n \in \mathbb{N}\} \leq 4$$

$$\beta = \inf\{b_n : n \in \mathbb{N}\} \geq 2$$

$$\frac{b_n}{a_n} = 1 + \frac{1}{n} \longrightarrow 1$$

$$2 \leq \alpha \leq \beta \leq 4$$

$$\alpha = \beta \underset{=}{\overset{\text{def}}{=}} e$$

$$e \simeq 2{,}718284\ldots$$

$(a_n)_{n=1}^{\infty}$ crescente, $\sup\{a_n : n \in \mathbb{N}\} = a < +\infty$

$$\lim_{n\to\infty} a_n = a$$

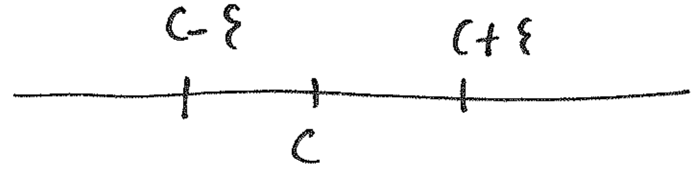
$(b_n)_{n=1}^{\infty}$ decrescente, $\inf\{b_n : n \in \mathbb{N}\} = b > -\infty$

$$\lim_{n\to\infty} b_n = b$$

$\lim\limits_{n\to\infty} c_n = c$ Per $\varepsilon > 0$, abbiamo $|c_n - c| < \varepsilon$ per $n > N$

$c \to$ NUMERO REALE

$\{c_n : n \in \mathbb{N}\}$ è limitata



$a_n = (-1)^n \qquad -1 \leq a_n \leq +1$

$-1, 1, -1, 1, -1, 1, \ldots$

$\to -1, -1, -1, -1, \ldots \longrightarrow -1$

$\to 1, 1, 1, 1, 1, \ldots \longrightarrow +1$

Esiste una sottosuccessione $(a_{n_k})_{k=1}^{\infty}$ convergente

~~TEOR~~

$n_1 < n_2 < n_3 < n_4 < \cdots \longrightarrow +\infty$

TEOREMA di Bolzano-Weierstrass:

Ogni successione limitata ha una sottosuccessione convergente.

Da ogni successione (limitata) si può estrarre una sottosuccessione convergente.

$$a_n = \left(1+\frac{1}{n}\right)^n \qquad b_n = \left(1+\frac{1}{n}\right)^{n+1}$$

$$2 = a_1 < a_2 < a_3 < \dots < a_n < a_{n+1} < \dots < b_{n+1} < b_n < \dots < b_2 < b_1 = 4$$

$$e = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

SE $\lim_{n\to\infty} a_n = a$ (finito), allora:

Per ogni $\varepsilon > 0$ esiste $N(\varepsilon)$ tale che $|a_n - a_m| < \varepsilon$

per $n, m > N$ SUCCESSIONE DI CAUCHY

$$\lim_{n \to \infty} \frac{n^b}{a^n} = \emptyset$$

dove $b > 0$

$a > 1$

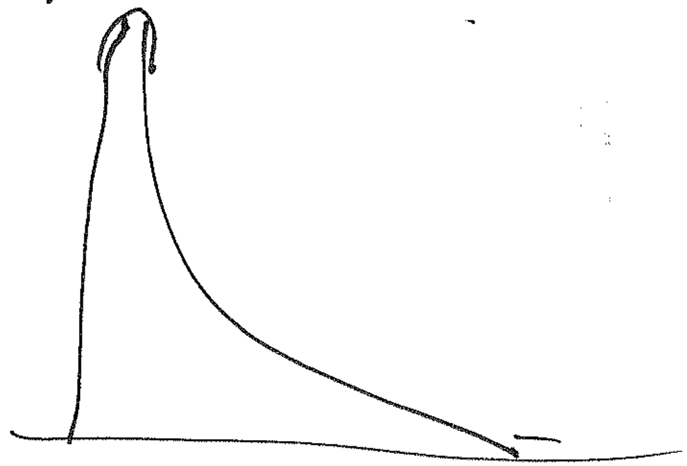
$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^b \frac{1}{a}$$

$$a_n = \frac{n^2}{3^n} \longrightarrow \emptyset$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \frac{1}{3} \longrightarrow \frac{1}{3}$$

$a = 1{,}000001 \qquad b = 1000000$

$$a_n = \frac{n^{1000000}}{(1{,}000001)^n} \longrightarrow 0$$



$$\frac{a_{n+1}}{a_n} = \underbrace{\left(\frac{n+1}{n}\right)^b}_{\leq 2^b \text{ costante}} \frac{1}{a} \qquad a > 1$$

$$\lim_{n \to \infty} \frac{n^b}{a^n} = 0 \quad \text{se } b > 0, \; a > 1$$

$$0 < a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_3}{a_2} \frac{a_2}{a_1} a_1$$

$$\leq \left(2^b\right)^n \frac{1}{a^n} a_1 = \frac{\text{costante}}{a^n} = \text{costante} \underbrace{\left(\frac{1}{a}\right)^n}_{\to 0}$$

$$\to 0$$

$$\boxed{\begin{gathered} c, b > 0 \\ \lim_{n\to\infty} \frac{(\log_y n)^k}{n^b} = 0 \end{gathered}}$$

Base del logaritmo $y > 1$

$$\log_{10} 1000000 = 6$$

$$\lim_{n\to\infty} \frac{\log n}{n} = 0$$

$$n = 10^k, \quad k \to +\infty$$

$$\lim_{k\to\infty} \frac{k}{10^k} = 0$$

$$\frac{n^b}{2^n} \to 0$$

$a > 1$

$$\lim_{n\to\infty} \frac{a^n}{n!} = \emptyset$$

~~Le~~ $\cancel{b > a > 1}$,

$$a_n = \frac{a^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n}$$

$$= \frac{a^{n+1}}{a^n} \frac{n!}{(n+1)!} = a \frac{1 \times 2 \times 3 \times \cdots \times n}{1 \times 2 \times 3 \times \cdots \times n \times (n+1)}$$

$$= \frac{a}{n+1} \longrightarrow \emptyset$$

$(\log n)^c \qquad n^b \qquad a^n \qquad n! \qquad n^n$

$c > 0 \qquad b > 0 \qquad a > 1$

$$\frac{n^n}{n!} \to +\infty$$

$$\lim_{n\to\infty} \frac{n!}{n^n} \qquad \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!}$$

$$= \underbrace{\frac{(n+1)!}{n!} \frac{1}{n+1}}_{=1} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e} < 1$$

$$a_n \sim \text{cost} \cdot \left(\frac{1}{e}\right)^n \to 0$$

$$\lim_{n\to\infty} \frac{n! + 2^n}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} + \lim_{n\to\infty} \frac{2^n}{(n+1)!} = \emptyset + \emptyset \cdot \emptyset = \emptyset$$

$$\frac{1\times 2\times \cdots \times n}{1\times 2\times 3\times \cdots \times n\times (n+1)} = \frac{1}{n+1}$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = \emptyset$$

$a > 1$

$$\frac{1}{n+1} \; \boxed{\frac{2^n}{n!}}$$

$$\lim_{n\to\infty} \frac{n}{1+\log^3 n} = \lim_{n\to\infty} \frac{n}{1+(\log n)^3} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}+\left(\frac{\log n}{n^{1/3}}\right)^3} = +\infty$$

$$\boxed{\text{SE } b_n > 0 \text{ e } \lim_{n\to\infty} b_n = 0 \text{ allora } \lim_{n\to\infty} \frac{1}{b_n} = +\infty}$$

$$\lim_{n\to\infty} \frac{(\log n)^2}{n} = \lim_{n\to\infty} \left(\frac{\log n}{n^{1/2}}\right)^2 = 0$$

$$\lim_{n\to\infty} \frac{(-1)^n}{\underbrace{n^2}_{b_n}} = 0$$

$$\lim_{n\to\infty} \frac{\log n}{n^b} = 0, \quad b > 0$$

$$\frac{1}{b_n} = (-1)^n n^2$$

NON HA LIMITE

$$\lim_{n\to\infty} \frac{3^n}{n^3} = +\infty$$

$$\cancel{\frac{3^n}{n^3}}$$

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \quad \text{per } b>0,\ a>1$$

$$\lim_{n\to\infty} \frac{n^3}{a^3} = 0$$

positivo

$$\boxed{\begin{array}{c} \log n \\ n^b \\ a^n \\ n! \\ n^n \end{array}}$$

$$\lim_{n\to\infty} \frac{5^n - n^5}{4^n + n^6} = \lim_{n\to\infty} \underbrace{\frac{5^n}{4^n+n^6}}_{\to +\infty} - \lim_{n\to\infty} \underbrace{\frac{n^5}{4^n+n^6}}_{\to 0} = +\infty$$

$$\cancel{\frac{5^n - n^5}{4^n + n^6}}$$

$$0 < \frac{n^5}{4^n+n^6} < \frac{n^5}{4^n}$$

$$\underbrace{0 \qquad\qquad 0}_{\to 0}$$

$$\frac{5^n}{4^n+n^6} = \frac{1}{\left(\frac{4}{5}\right)^n + \frac{n^6}{5^n}} \to +\infty$$

$$\lim_{n\to\infty} \frac{n^3}{n!} = 0$$

$$\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^n$$

$k = 2n$

$$\lim_{k\to\infty} \left\{\left(1+\frac{1}{k}\right)^k\right\}^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$$

$\log n$

n^b

2^n

$n!$

n^n

$$\lim_{n\to\infty}\left(\sqrt{n^2+6n+10}-\sqrt{n^2+2n}\right)$$

$$\frac{\left(\sqrt{n^2+6n+10}-\sqrt{n^2+2n}\right)\left(\sqrt{n^2+6n+10}+\sqrt{n^2+2n}\right)}{\sqrt{n^2+6n+10}+\sqrt{n^2+2n}}$$

$$=\frac{(n^2+6n+10)-(n^2+2n)}{\sqrt{n^2+6n+10}+\sqrt{n^2+2n}}=\frac{4n+10}{\sqrt{n^2+6n+10}+\sqrt{n^2+2n}}$$

$$= \frac{4 + \frac{10}{n}}{\sqrt{1 + \frac{6}{n} + \frac{10}{n^2}} + \sqrt{1 + \frac{2}{n}}} \longrightarrow \frac{4 + 0}{\sqrt{1+0+0} + \sqrt{1+0}} = 2$$

$$\lim_{n\to\infty} \frac{2n^2+5}{3n^2+n} = \lim_{n\to\infty} \frac{2 + \frac{5}{n^2}}{3 + \frac{1}{n}} = \frac{2+0}{3+0} = \frac{2}{3}$$