Sia $f : [a,b] \to \mathbb{R}$ continua. Sia

$$F(x) = \int_a^x f(t)\,dt$$

allora $F(a) = 0$. Inoltre, F è derivabile sull'intervallo $[a,b]$ e

$$F'(x) = f(x), \qquad a \le x \le b$$

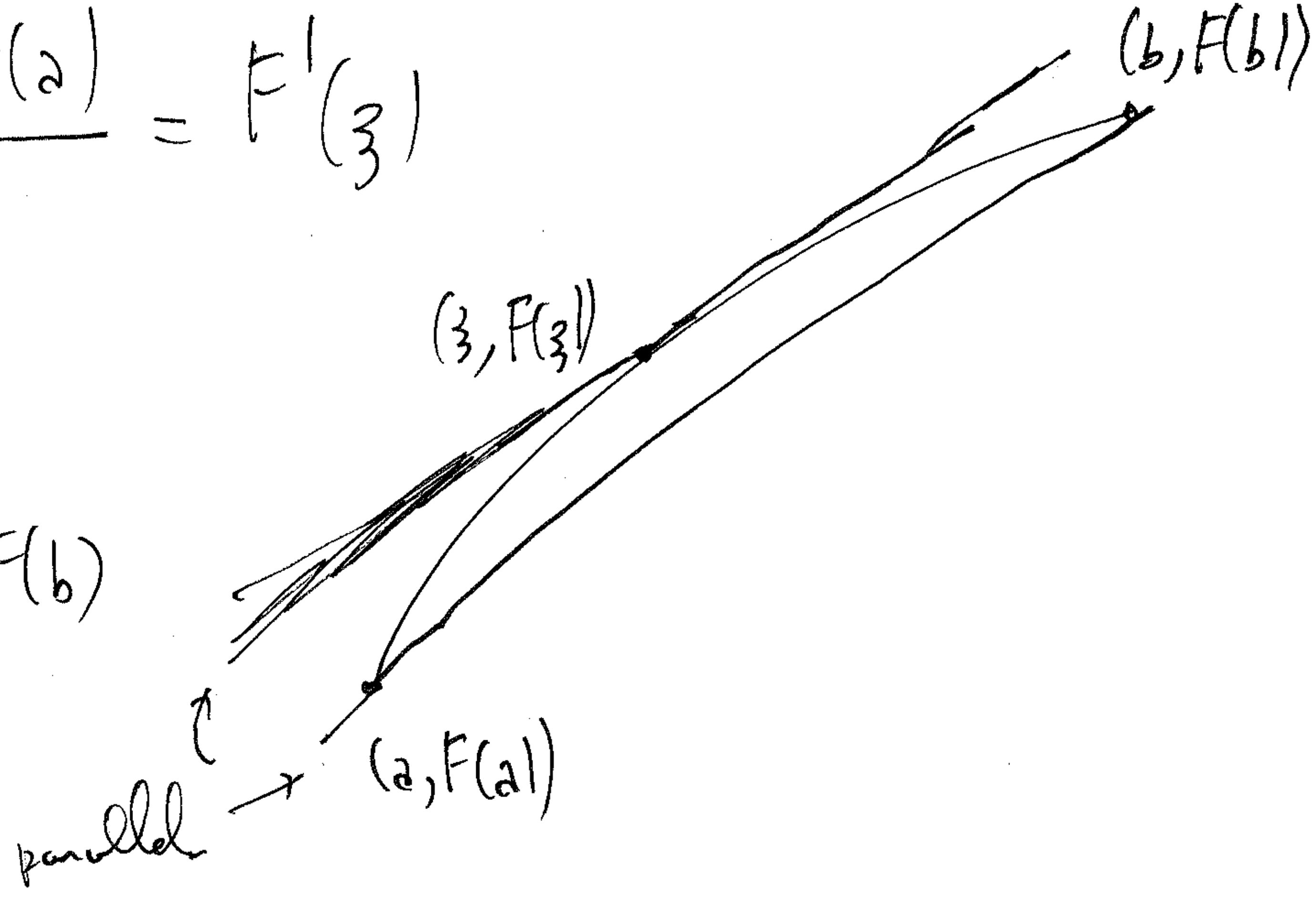
TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

LAGRANGE:

Sia $F:[a,b] \to \mathbb{R}$ continua, e derivabile in (a,b).

allora esiste $\xi \in (a,b)$ tale che

$$\frac{F(b)-F(a)}{b-a} = F'(\xi)$$



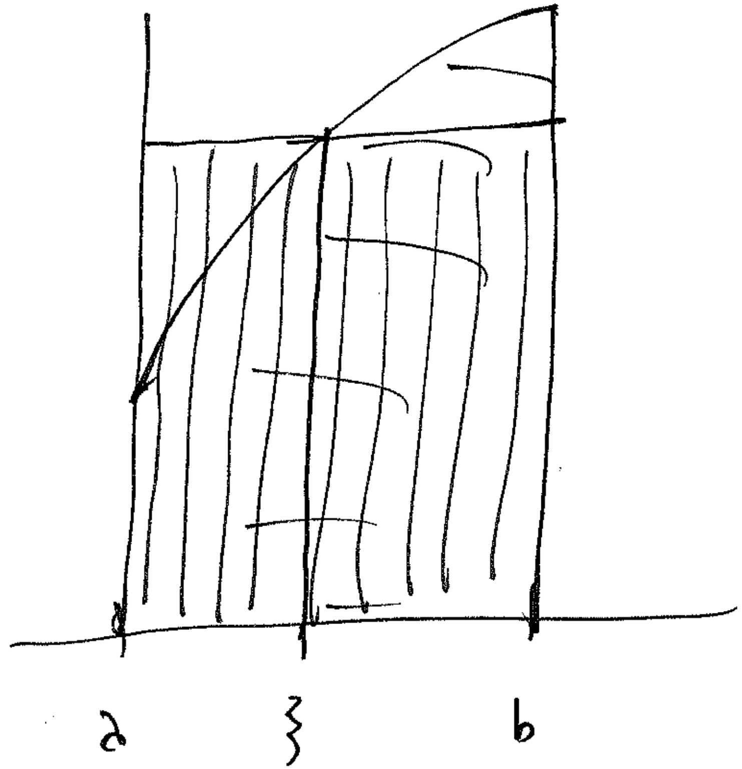
ROLLE: Caso particolare per $F(a)=F(b)$

$f : [a,b] \to \mathbb{R}$ continua

$$F(x) = \int_a^x f(t)\,dt$$

$$\frac{F(b) - F(a)}{b-a} = \frac{\int_a^b f(t)\,dt}{b-a}$$

$\exists\, \xi \in (a,b) : \quad f(\xi)\cdot(b-a) = \int_a^b f(t)\,dt$



Sia $F:[a,b] \to \mathbb{R}$ derivabile con derivata uguale a zero

$$\exists \xi \in (a,b) : \frac{F(b)-F(a)}{b-a} = F'(\xi) = 0 \longrightarrow F(a) = F(b)$$

Prendiamo ora a_1, b_1 tali che $a \leq a_1 < b_1 \leq b$

$$\exists \eta \in (a_1,b_1) : \frac{F(b_1)-F(a_1)}{b_1-a_1} = F'(\eta) = 0 \to F(a_1) = F(b_1)$$

QUINDI: F è costante

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad P(0) = a_0$$

$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} \qquad P'(0) = a_1$$

$$P''(x) = 2a_2 + 6a_3 x + \cdots + n(n-1) a_n x^{n-2} \qquad P''(0) = 2a_2$$

$$P'''(x) = 6a_3 + \cdots + n(n-1)(n-2) a_n x^{n-3} \qquad P'''(0) = 6a_3$$

$$P^{(4)}(x) = 24a_4 + \cdots + n(n-1)(n-2)(n-3) a_n x^{n-4} \qquad P^{(4)}(0) = 24a_4$$

$$\vdots$$

$$P^{(n)}(x) = n!\, a_n \qquad P^{(n)}(0) = n!\, a_n$$

TAYLOR:

$$p(x) = p(\emptyset) + \frac{p'(\emptyset)}{1!}x + \frac{p''(\emptyset)}{2!}x^2 + \frac{p'''(\emptyset)}{3!}x^3 + \dots$$

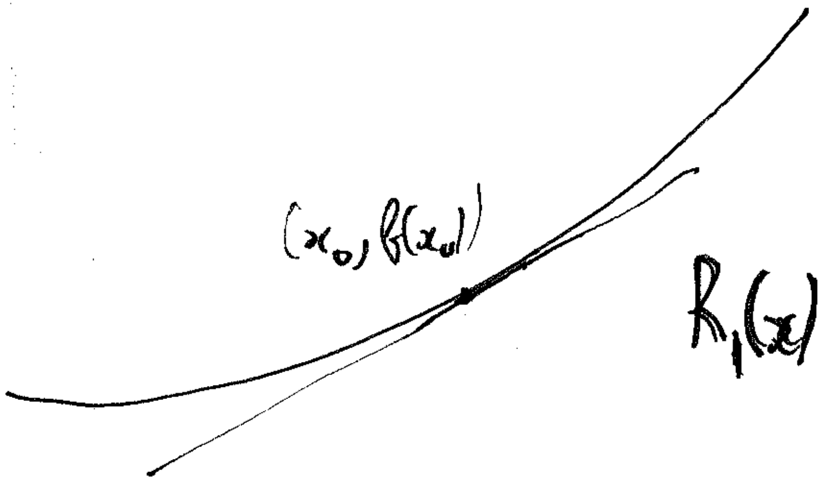
$$+ \frac{p^{(n)}(\emptyset)}{n!}x^n$$

SE gr $p > n$, allora $p(x) = a_0 + a_1 x + \dots + a_n x^n + a_{n+1} x^{n+1} + ???$

$$p(x) = p(\emptyset) + \frac{p'(\emptyset)}{1!}x + \frac{p''(\emptyset)}{2!}x^2 + \dots + \frac{p^{(n)}(\emptyset)}{n!}x^n + \underbrace{O\left(x^{n+1}\right)}_{R_n(x)}$$

<u>$n=1$</u>

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \underbrace{R_1(x)}_{\int_{x_0}^{x}(x-t)\,f''(t)\,dt}$$



$$R_1(x) = \int_{x_0}^{x} \underbrace{(x-t)}_{F(t)}\,\underbrace{f''(t)}_{G'(t)}\,dt$$

$$= \Big[(x-t)\,f'(t)\Big]_{t=x_0}^{x} - \int_{x_0}^{x} \underbrace{(-1)}_{F'(t)}\,f'(t)\,dt = -(x-x_0)\,f'(x_0) + \int_{x_0}^{x} f'(t)\,dt$$

$$= -(x-x_0)\,f'(x_0) + f(x) - f(x_0)$$

Sia f derivabile $n+1$ volte in $[a,b]$, con derivata $(n+1)$-esima continua. Sia $x_0 \in (a,b)$. Allora

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$
$$+ \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$

$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \overset{(?)}{=} -\frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_{n-1}(x)$$

$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

$$= \left[\frac{(x-t)^n}{n!} f^{(n)}(t) \right]_{t=x_0}^{x} - \int_{x_0}^{x} \frac{\cancel{-n}(x-t)^{n-1}}{\cancel{n!}\ (n-1)!} f^{(n)}(t)\,dt$$

$$= -\frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_{n-1}(x)$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$

$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt = (x-x_0)\frac{(x-\xi)^n}{n!} f^{(n+1)}(\xi)$$

dove ξ tra x_0 e x

$R_n(x) \leq |x-x_0|^{n+1}$ costante.

$$R_n(x) = O\left((x-x_0)^{n+1}\right)$$

$$f(x) = e^x \longrightarrow f^{(n)}(x) = e^x \longrightarrow f^{(n)}(\emptyset) =$$

$x_0 = \emptyset$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + O\left(x^{n+1}\right)$$

$$R_n(x) = \int_0^x \frac{(x-t)^n}{n!} e^t \, dt$$

$$\cos x \longrightarrow -\operatorname{sen} x \longrightarrow -\cos x \longrightarrow \operatorname{sen} x$$

$f(x) = \cos x$ $\quad 1 \longrightarrow \emptyset \longrightarrow -1 \longrightarrow \emptyset \longrightarrow 1 \longrightarrow \emptyset \longrightarrow -1 \longrightarrow \emptyset \longrightarrow \cdots$

$x_0 = 0$ $\quad$ cost. $\quad \dfrac{x}{1!} \quad \dfrac{x^2}{2!} \quad \dfrac{x^3}{3!} \quad \dfrac{x^4}{4!}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + O\left(x^{2k+1}\right)$$

$$\operatorname{sen} x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + O\left(x^{2k+2}\right)$$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1} \rightarrow f'(x) = -(1-x)^{-2} \frac{d}{dx}(1-x) = \frac{1}{(1-x)^2}$$

$$\rightarrow f''(x) = -2(1-x)^{-3} \frac{d}{dx}(1-x) = \frac{2}{(1-x)^3}$$

$$\rightarrow f'''(x) = -6(1-x)^{-4} \frac{d}{dx}(1-x) = \frac{6}{(1-x)^4}$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \longrightarrow f^{(n)}(\emptyset) = n! \rightarrow \frac{f^{(n)}(\emptyset)}{n!} = 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^n + O(x^{n+1})$$

$$S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n$$

$$x S_n(x) = x + x^2 + x^3 + \cdots + x^{n-1} + x^n + x^{n+1}$$

$$(1-x) S_n(x) = 1 - x^{n+1} \longrightarrow S_n(x) = \frac{1 - x^{n+1}}{1-x}$$

$$S_n(1) = n+1$$

$$\lim_{x \to 1} \frac{1 - x^{n+1}}{1-x} = \lim_{x \to 1} \frac{-(n+1)x^n}{-1} = n+1$$

$$\frac{1}{1-x} = S_n(x) + \frac{x^{n+1}}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + O\left(x^{n+1}\right)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + O\left(x^{n+1}\right)$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + O\left(x^{2n+2}\right)$$

$$\operatorname{arctg} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + O\left(x^{2n+3}\right)$$

$$\frac{\pi}{4} = 4 \operatorname{arctg}\left(\frac{1}{5}\right) - \operatorname{arctg}\left(\frac{1}{239}\right)$$

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots + (-1)^n x^n + O(x^{n+1})$$

$$\ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + O(x^{n+2})$$

$$tg(x) \longrightarrow \frac{1}{\cos^2(x)} \rightarrow \frac{+2\,\mathrm{sen}\,x}{\cos^3(x)} \longrightarrow \cdots$$

$$\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}=\lim_{x\to 0}\frac{e^x-1-x}{3x^2}=\lim_{x\to 0}\frac{e^x-1}{6x}$$

$$=\lim_{x\to 0}\frac{e^x}{6}=\frac{1}{6}$$

$$e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+O(x^4)$$

$$\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}=\frac{\frac{1}{6}x^3+O(x^4)}{x^3}=\frac{1}{6}+O(x)\longrightarrow\frac{1}{6}$$

$$f(x) = \frac{\ln(1-2x)}{x} = \frac{g(x)}{x}, \quad g(x) = \ln(1-2x)$$

$$g'(x) = \frac{-2}{1-2x} \longrightarrow g''(x) = -2\,\frac{-1}{(1-2x)^2}(-2) = \frac{-4}{(1-2x)^2}$$

$$g'''(x) = -4 \cdot \frac{-2}{(1-2x)^3}(-2) =$$

$$g^{(n)}(x) = \frac{c_n}{(1-2x)^{n+1}}$$

$$g^{(n+1)}(x) = c_n \cdot \frac{-(n+1)}{(1-2x)^{n+2}}(-2)$$

$$c_{n+1} = 2(n+1)c_n \qquad c_n = -2^n \cdot n!$$

$$g(x) = -\sum_{k=1}^{n} \frac{2^k \cdot k!}{k!} x^k + O(x^{n+1})$$

$$c_1 = -2 \to c_2 = 4\,2 \to c_3 = -8 \cdot 3!$$

$$f(x) = \operatorname{tg} x \rightarrow f'(x) = \frac{1}{\cos^2 x} \rightarrow f''(x) = \frac{-2}{\cos^3 x} \cdot -\operatorname{sen} x = \frac{2 \operatorname{sen} x}{\cos^3 x}$$

$$f'''(x) = 2\,\frac{\cos^3 x \cdot \cos x - \operatorname{sen} x \cdot 3\cos^2 x \cdot (-\operatorname{sen} x)}{\cos^6 x}$$

$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = 0 \qquad f'''(0) = 2$

$$\operatorname{tg} x = x + \frac{2x^3}{3!} + O(x^5) = x + \frac{1}{3}x^3 + O(x^5)$$

$$\operatorname{sen} x = x - \frac{1}{6}x^3 + O(x^5)$$