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Sampling expansions and interpolation in unitarily translation invariant reproducing kernel Hilbert spaces

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Sufficient conditions are established in order that, for a fixed infinite set of sampling points on the full line, a function satisfies a sampling theorem on a suitable closed subspace of a unitarily translation invariant reproducing kernel Hilbert space. A number of examples of such reproducing kernel Hilbert spaces and the corresponding sampling expansions are given. Sampling theorems for functions on the half-line are also established in RKHS using Riesz bases in subspaces of $L^2(\mathbb{R}^+)$.

Keywords: nonuniform sampling, unitarily translation invariant subspaces, reproducing kernel spaces, sampling on the half-line, Riesz bases

1. Introduction

The Shannon–Whittaker sampling theorem asserts that any band limited function f in $L^2(\mathbb{R})$ with band width π (i.e., a function whose Fourier transform is supported in $[-\pi, \pi]$) can be uniquely represented in the form

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)},$$

where $\sum_{n=-\infty}^{\infty} |f(n)|^2 < \infty$. For such functions we have

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$$\int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t = \sum_{n=-\infty}^{\infty} |f(n)|^2.$$

Thus there is an isometric one-to-one correspondence between band limited functions f in $L^2(\mathbb{R})$ with band width π and sequences $\{f(n)\}_{n=-\infty}^{\infty}$ in $\ell^2(\mathbb{Z})$.

The Paley–Wiener theorem [25] gives a direct characterization of the space $B_\pi(\mathbb{R})$ of band limited signals whose Fourier transforms have support in $[-\pi, \pi]$, namely, a function in $L^2(\mathbb{R})$ belongs to $B_\pi(\mathbb{R})$ if and only if it is the restriction of an entire function and is of exponential order at most π on the real line. The space $B_\pi(\mathbb{R})$ has several interesting properties:

(i) $B_\pi(\mathbb{R})$ is a reproducing kernel Hilbert space with reproducing kernel

$$k(t,s) = \frac{\sin \pi(t-s)}{\pi(t-s)};$$

(ii) the sequence $\{S_n\}_{n\in\mathbb{N}}$, where $S_n(t) = k(t,n)$, constitutes an orthonormal basis for $L^2(\mathbb{R})$;

(iii) the sequence $\{S_n(t)\}_{n\in\mathbb{N}}$ has the discrete orthogonality property $S_n(m) = \delta_{n,m}$;

(iv) the space $B_\pi(\mathbb{R})$ is closed under differentiation;

(v) the space $B_\pi(\mathbb{R})$ is unitarily translation invariant, i.e., for any $f \in B_\pi(\mathbb{R})$ and $c \in \mathbb{R}$, we have $f(\cdot - c) \in B_\pi(\mathbb{R})$ and $\|f(\cdot - c)\| = \|f\|$.

The Shannon–Whittaker sampling theorem has been generalized in many directions (for some perspectives see [6–8,17,18,20,23,31]). Some of the above properties of the space $B_\pi(\mathbb{R})$ have served as key ingredients in the generalizations and extensions of the Shannon–Whittaker sampling theorem. In particular, it has been shown that the theorem and many of its extensions have an equivalent formulation in appropriate reproducing kernel Hilbert spaces (RKHS). By a reproducing kernel Hilbert space of functions supported on a set S we mean a (complex) Hilbert space H of functions on S, where all evaluation functionals $\xi_t(f) = f(t)$, $f \in H$, for each fixed $t \in S$, are continuous. Then, by the Riesz representation theorem, for each $t \in S$ there exists a unique element $k_t \in H$ such that

$$f(t) = \langle f, k_t \rangle, \quad f \in H,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in H. Let $k(t,u) = \langle k_t, k_u \rangle$ for $t, u \in S$. Then $k(\cdot,\cdot)$ is called the *reproducing kernel* of H. Clearly, $k(\cdot,\cdot)$ is Hermitian and positive definite. For properties of reproducing kernel Hilbert spaces, see [4,5,9,19,22,26,28].

Nashed and Walter [23] have shown that there is a strong affinity between RKHS and sampling expansions. They have derived general sampling theorems for functions in RKHS which are subspaces of $L^2(\mathbb{R})$ closed in the Sobolev space $H^{-1}(\mathbb{R})$. They have also shown that with every sampling expansion one can associate a corresponding RKHS [24]. Applications of RKHS to other recovery problems from partial information are given in [21].

In this paper we introduce an approach to sampling problems, where the unitary translation invariance property of the space plays a pivotal role. The main thrust of this paper is the study of sampling problems on subspaces of a class of unitarily translation invariant RKHS generated from a single function. Within this framework we are able to exploit Gershgorin's theorem and results on Toeplitz matrices. *Shift-invariance* properties, i.e., invariance properties upon translation over integer distances, have recently been used by Aldroubi and Gröchenig [2,3] (see also related references cited therein), but to our knowledge this is the first time that *unitarily translation invariant* spaces have been used explicitly to develop sampling expansions. When the sampling points are equidistant, we also use the symbol function as a tool in studying sampling expansions.

To be specific, we study sampling problems on RKHS with

$$k(t,u) = \int_{-\infty}^{\infty} \phi(x-t)\phi(x-u)\,\mathrm{d}x, \tag{1.1}$$

where $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the Fourier transform of ϕ does not have real zeros. Such functions ϕ come up in the expansion of a general function f in the integer translates of a given function ϕ, both on $\mathbb{R}$ and on $\mathbb{R}^+$, as studied in [15]. The question naturally arises in such settings if f can be reconstructed from its values at countably many, not necessarily equidistant, points. This question is addressed in the present paper.

Given a function f from an RKHS H on an infinite set S, let $\{t_j\}_{j\in J}$ be a sequence of distinct sampling points in S indexed by an infinite subset J of $\mathbb{Z}$. Under mild conditions, it is easy to show that the sampling map $f \mapsto \{f(t_j)\}_{j\in J}$ and the inverse sampling map $\{f(t_j)\}_{j\in J} \mapsto f$ are both continuous. Hence there exist positive constants C_1 and C_2 such that

$$C_1\|f\|_H \leqslant \left[\sum_{j\in J} \left|f(t_j)\right|^2\right]^{1/2} \leqslant C_2\|f\|_H, \quad f \in H. \tag{1.2}$$

In the case of the Shannon–Whittaker sampling theorem, (1.2) reduces to two equalities with $C_1 = C_2 = 1$, where $J = \mathbb{Z}$. The preservation of (1.2) is essential to the derivation of sampling expansions. The requirement that the function belongs to an RKHS allows one to state the sampling inequalities (1.2) in terms of the boundedness and strict positivity of a so-called symbol function when the sampling points are equidistant.

We now describe the organization of this paper. In section 2 we reformulate (mostly known) results in the theoretical framework needed for this paper and augment them by a new property that utilizes unitary translation invariance of the RKHS. In section 3 we derive sampling theorems and interpolation results for signals in unitarily translation invariant RKHS on the line. In section 4 we study sampling problems for RKHS on the positive real line with reproducing kernel

$$k(t,u) = \int_0^{\infty} \phi(x-t)\phi(x-u)\,\mathrm{d}x, \quad t,u \geqslant 0, \tag{1.3}$$

where $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Using the corresponding sampling results on $S = \mathbb{R}$ and a recent result of Goodman et al. [16] on deriving Riesz bases of functions of the half-line by restricting functions on the full line, we finally derive sampling theorems for $S = \mathbb{R}^+$. We conclude this paper with a brief comparison with the results in [23]. Throughout the paper we illustrate the results with the help of four standard examples.

2. Theoretical framework

Given a complex Hilbert space H, an infinite sequence $\{f_n\}_{n\in J}$, $J \subset \mathbb{Z}$, of vectors in H is called a *frame* (cf. [10,30]) if there exist positive constants C_1, C_2 such that

$$C_1\|f\|_H \leqslant \left[\sum_{n\in J}\left|\langle f, f_n\rangle\right|^2\right]^{1/2} \leqslant C_2\|f\|_H, \quad f \in H.$$

These inequalities are called the *frame inequalities*. A frame is called an *exact frame* if the removal of any vector from the frame causes it not to be a frame any more. Given a frame, the operator T defined by $Tf = \sum_{n\in J}\langle f, f_n\rangle f_n$ is a bounded linear operator on H. Further, for every $f \in H$ there exists a unique moment sequence $\{a_n\}_{n\in J}$ such that

$$f = \sum_{n\in J} a_n f_n$$

and $\sum_{n\in J}|a_n|^2$ is minimal. A well-known result [10,30] states that a sequence $\{f_n\}_{n\in J}$ in a separable Hilbert space H is an exact (or tight) frame if and only if it is a Riesz basis in H (i.e., if it can be obtained from an orthonormal basis in H by applying a boundedly invertible operator).

The inequalities (1.2) can now be interpreted as frame inequalities, since $\langle f, f_n\rangle = f(t_n)$, where $f_n = k_{t_n}$ for $n \in J$. Given an RKHS H of functions supported on an (infinite) set S and the sequence $\{t_j\}_{j\in J}$ of distinct sampling points with $J \subset \mathbb{Z}$ infinite, the associated sampling problem consists of (1) uniquely reconstructing f belonging to a suitable closed subspace H_0 of H from its values $\{f(t_j)\}_{j\in J}$ at the sampling points, and (2) showing that the frame inequalities (1.2) hold true for $f \in H_0$. Once the direct sampling problem (of finding the subspace H_0 and suitable sampling points for $f \in H_0$) and the inverse sampling problem (of reconstructing $f \in H_0$ from the sampling points, while indicating a suitable RKHS) have been formulated, we introduce a (complex) Hilbert space of sequences $\ell_2(J)$ induced by $\boldsymbol{t} = \{t_j\}_{j\in J}$, the *sampling map* (or *point-evaluation map*)

$$\sigma : H_0 \longrightarrow \ell_2(J), \qquad \sigma f = \left\{f(t_j)\right\}_{j\in J},$$

and the *inverse sampling map* (or *recovery map*)

$$\tau : \ell_2(J) \longrightarrow H_0, \qquad \tau\left\{f(t_j)\right\}_{j\in J} = f.$$

The objective is to select a suitable subspace H_0 and to prove the invertibility and boundedness of the sampling map, the boundedness of the inverse sampling map, and hence the well-posedness of the point evaluation problem.

Letting $k(\cdot,\cdot)$ denote the reproducing kernel of H and $\langle\cdot,\cdot\rangle_H$ its inner product, we define the infinite Gram matrix

$$G_{ij} = \langle k_{t_i}, k_{t_j}\rangle_H = k(t_i, t_j), \quad i, j \in J,$$

which is a positive semi-definite linear operator on $\ell_2(J)$.

We shall make frequent use of the following proposition (see theorem I.9 of [30]). The enumeration of the vectors has been changed from $i, j \in \mathbb{N}$ to $i, j \in \mathbb{Z}$.

Proposition 2.1. Let $\mathcal{X}$ be a separable Hilbert space. Then the following statements are equivalent:

1. The sequence $\{k_{t_j}\}_{j\in\mathbb{Z}}$ forms a Riesz basis in $\mathcal{X}$.
2. The sequence $\{k_{t_j}\}_{j\in\mathbb{Z}}$ is complete in $\mathcal{X}$, and there exist positive constants C_1 and C_2 such that for arbitrary positive integers n, m and arbitrary scalars $c_{-m}, \dots, c_n$ one has
$$C_1^2 \sum_{i=-m}^{n} |c_i|^2 \leqslant \left\| \sum_{i=-m}^{n} c_i k_{t_i} \right\|_{\mathcal{X}}^2 \leqslant C_2^2 \sum_{i=-m}^{n} |c_i|^2.$$
3. The sequence $\{k_{t_j}\}_{j\in\mathbb{Z}}$ is complete in $\mathcal{X}$, and its Gram matrix
$$\left(\langle k_{t_i}, k_{t_j}\rangle\right)_{i,j=-\infty}^{\infty}$$
generates a bounded invertible operator on $\ell^2(\mathbb{Z})$.

We now immediately have the following result.

Corollary 2.2. Let $\boldsymbol{t} = \{t_j\}_{j\in J}$ be infinitely many distinct sampling points in S and let H be a reproducing kernel Hilbert space of functions on S with reproducing kernel $k(\cdot,\cdot)$. Let $k_t(\cdot) = k(t,\cdot)$. Then the following statements are equivalent:

1. There exist positive constants C_1, C_2 such that (1.2) holds for every $f \in H$ and no such relation holds for any proper subset of sampling points.
2. The sequence $\{k_{t_j}\}_{j\in J}$ is a Riesz basis in H.
3. The sequence of functions $\{k_{t_j}\}_{j\in J}$ is complete and the Gram matrix $(k(t_i, t_j))_{i,j\in J}$ is bounded and strictly positive selfadjoint on $\ell^2(J)$.

If any of these conditions hold, we have the sampling expansion

$$f(t) = \sum_{j\in J} f(t_j) \frac{k(t_j, t)}{k(t_j, t_j)}. \tag{2.1}$$

When $S = \mathbb{R}$, the sampling points are equidistant (i.e., $t_j = \alpha j$, $j \in \mathbb{Z}$) and the RKHS H is unitarily translation invariant, the bi-infinite Gram matrix $\{G_{ij}\}_{i,j=-\infty}^{\infty}$ is a Toeplitz matrix (i.e., $G_{ij} = G_{i-j}$, $i, j \in \mathbb{Z}$).

Proposition 2.3. Let $\{\alpha j\}_{j=-\infty}^{\infty}$ be equidistant sampling points in $S = \mathbb{R}$ and let the reproducing kernel Hilbert space H be unitarily translation invariant. Then the following statements are equivalent:

1. There exist positive constants C_1, C_2 such that

$$C_1\|f\|_H \leqslant \left[\sum_{j=-\infty}^{\infty} \left|f(\alpha j)\right|^2\right]^{1/2} \leqslant C_2\|f\|_H, \quad f \in H,$$

   and no such relation holds for any proper subset of sampling points.

2. The sequence $\{k_{\alpha j}\}_{j=-\infty}^{\infty}$ is a Riesz basis in H.
3. The sequence of functions $\{k_{\alpha j}\}_{j=-\infty}^{\infty}$ is complete and the bi-infinite Toeplitz matrix $(G_{i-j}(\alpha))_{i,j=-\infty}^{\infty}$ defined by $G_{i-j}(\alpha) = \langle k_{\alpha i}, k_{\alpha j}\rangle_H$ is bounded and strictly positive selfadjoint operator on $\ell^2(\mathbb{Z})$.
4. The sequence of functions $\{k_{\alpha j}\}_{j=-\infty}^{\infty}$ is complete and the symbol

$$\widehat{G}(s,\alpha) = \sum_{j=-\infty}^{\infty} s^j G_j(\alpha), \quad |s| = 1,$$

   is positive, essentially bounded, and essentially bounded away from zero.

If any of these conditions hold, we have the sampling expansion

$$f(t) = \sum_{j=-\infty}^{\infty} f(\alpha j)\frac{k(\alpha j, t)}{k(\alpha j, \alpha j)} = \sum_{j=-\infty}^{\infty} f(\alpha j)\frac{k(\alpha j, t)}{G_0(\alpha)}.$$

Proof. The proposition is immediate from corollary 2.2 with the exception of the equivalence of parts 3 and 4. That statement, however, follows from the well-known result ([13], corollary XXIII 2.2) that a bi-infinite Toeplitz matrix is bounded on $\ell^2(\mathbb{Z})$ if and only if its symbol is an L^∞-function, where the statement is being applied to both the Toeplitz matrix and its inverse. □

3. Reproducing kernels of functions on the line and sampling

Suppose H is an RKHS on $\mathbb{R}$ with reproducing kernel $k(\cdot,\cdot)$ that is unitarily translation invariant. Then for all $s, t, u \in \mathbb{R}$ we have

$$k(t+s, u+s) = \langle k_{t+s}, k_{u+s}\rangle = \langle k_t, k_u\rangle = k(t,u),$$

so that $k(\cdot,\cdot)$ is a difference kernel which we rewrite as $\kappa(\cdot)$ satisfying $k(t,u) = \kappa(t-u)$ for all $t, u \in \mathbb{R}$. Moreover, the sesquilinearity of the inner product in H implies that $\kappa(t) = \overline{\kappa(-t)}$.

Let us now discuss specific unitarily translation invariant RKHS.

3.1. Suppose $P(\omega)$ is a positive continuous function of $\omega \in \mathbb{R}$ such that $1/P(\omega)$ belongs to $L^1(\mathbb{R})$. Writing $\hat{f}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} f(t)\,\mathrm{d}t$ for the Fourier transform of f, let H_2^P be the Hilbert space of all measurable functions f on the real line such that

$$\|f\|_{H_2^P} = \left[\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 P(\omega)\,\mathrm{d}\omega\right]^{1/2} < \infty.$$

Further, for every $t \in \mathbb{R}$ we have

$$\left|f(t)\right| \leqslant \frac{1}{\sqrt{2\pi}} \|f\|_{H_2^P} \left[\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{P(\omega)}\right]^{1/2},$$

and hence $f \mapsto f(t)$ is continuous if $1/P(\omega)$ belongs to $L^1(\mathbb{R})$. Thus H_2^P is an RKHS whenever $1/P(\omega)$ belongs to $L^1(\mathbb{R})$. If $k(\cdot,\cdot)$ is the corresponding reproducing kernel, then on one hand we have

$$f(t) = \langle f, k_t\rangle = \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{k}_t(\omega)}P(\omega)\,\mathrm{d}\omega,$$

whereas on the other hand

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} \hat{f}(\omega)\,\mathrm{d}\omega.$$

Hence,

$$\hat{k}_t(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{P(\omega)}, \tag{3.1}$$

and so the reproducing kernel for H_2^P is given by

$$k(t,u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega(t-u)}}{P(\omega)}\,\mathrm{d}\omega.$$

Moreover, the kernel $k(\cdot,\cdot)$ is continuous, while H_2^P is unitarily translation invariant, since $|\hat{f}(\omega)|$ does not change when replacing f by one of its translates. If $P(\omega)$ is a real and even function, then the reproducing kernel $k(\cdot,\cdot)$ is real symmetric.

3.2. Let ϕ be a real function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ whose Fourier transform $\hat{\phi}$ does not have real zeros, and let $\phi_t(u) = \phi(u-t)$. We seek a reproducing kernel Hilbert space H_ϕ on $S = \mathbb{R}$ whose reproducing kernel is given by

$$k_\phi(t,u) = \int_{-\infty}^{\infty} \phi(x-t)\phi(x-u)\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(t-u)}\left|\hat{\phi}(\omega)\right|^2\,\mathrm{d}\omega,$$

where $\hat{\phi}(\omega) = \overline{\hat{\phi}(-\omega)}$. Let $\kappa_\phi(t) = k_\phi(t,0)$. Then

$$k_\phi(t,u) = \kappa_\phi(t-u) = \kappa_\phi\big(|t-u|\big). \tag{3.2}$$

Moreover, since $\phi \in L^1(\mathbb{R})$ we obtain $\kappa_\phi \in L^1(\mathbb{R})$. Further, because $\phi \in L^2(\mathbb{R})$ and translation is a strongly continuous linear operator on $L^2(\mathbb{R})$, we see that κ_ϕ is a bounded continuous function. In fact, as $|\hat{\phi}|^2 \in L^1(\mathbb{R})$, by the Riemann–Lebesgue lemma we have that $\kappa_\phi(t) \to 0$ as $t \to \pm\infty$.

Let us determine the function $P(\omega)$ such that $k_\phi(\cdot,\cdot)$ is the reproducing kernel of the generalized Sobolev space H_2^P. To this end, we seek a positive function $P(\omega)$ of $\omega \in \mathbb{R}$ satisfying $(1/P(\cdot)) \in L^1(\mathbb{R})$ such that $H_\phi = H_2^P$. Then we must solve the equation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t-u)}}{P(\omega)}\, d\omega = \int_{-\infty}^{\infty} \phi(x-t)\phi(x-u)\, dx$$

for $P(\omega)$. Writing $\rho = t-u$ and changing the variable in the right-hand side, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega\rho}}{P(\omega)}\, d\omega = \int_{-\infty}^{\infty} \phi(x)\phi(x+\rho)\, dx,$$

and hence

$$\begin{aligned}\frac{1}{P(\omega)} &= \int_{-\infty}^{\infty} e^{-i\omega\rho}\int_{-\infty}^{\infty} \phi(x)\phi(x+\rho)\, dx\, d\rho \\ &= \int_{-\infty}^{\infty} e^{i\omega x}\phi(x)\int_{-\infty}^{\infty} e^{-i\omega(x+\rho)}\phi(x+\rho)\, d\rho\, dx \\ &= \sqrt{2\pi}\,\hat{\phi}(\omega)\sqrt{2\pi}\,\hat{\phi}(-\omega) = 2\pi\left|\hat{\phi}(\omega)\right|^2.\end{aligned}$$

Since $\hat{\phi}(\omega) \neq 0$ for $\omega \in \mathbb{R}$, the function

$$P(\omega) = \frac{1}{2\pi}\frac{1}{\hat{\phi}(\omega)\hat{\phi}(-\omega)} = \frac{1}{2\pi}\frac{1}{|\hat{\phi}(\omega)|^2} \tag{3.3}$$

is indeed nonnegative. We have $(1/P) \in L^1(\mathbb{R})$ because $\phi \in L^2(\mathbb{R})$, and P is continuous because $\phi \in L^1(\mathbb{R})$ and $\hat{\phi}(\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Thus, by (3.1),

$$\hat{k}_t(\omega) = \sqrt{2\pi}\left|\hat{\phi}(\omega)\right|^2 e^{i\omega t}.$$

The condition that $\hat{\phi}(\omega) \neq 0$ for every $\omega \in \mathbb{R}$ is sufficient for $k_\phi(\cdot,\cdot)$ to be a reproducing kernel on $S = \mathbb{R}$. Indeed, let $t_1, \ldots, t_n$ be distinct real numbers. Then for every nontrivial n-tuple $(\xi_1, \ldots, \xi_n)$ of complex numbers we have

$$\begin{aligned}\sum_{i,j=1}^{n} k_\phi(t_i, t_j)\xi_i\overline{\xi_j} &= \int_{-\infty}^{\infty} \left|\hat{\phi}(\omega)\right|^2 \sum_{i,j=1}^{n} e^{i(t_i-t_j)\omega}\xi_i\overline{\xi_j}\, d\omega \\ &= \int_{-\infty}^{\infty} \left|\hat{\phi}(\omega)\right|^2 \left|\sum_{i=1}^{n} e^{it_i\omega}\xi_i\right|^2 d\omega > 0,\end{aligned}$$

which proves that $k_\phi(\cdot,\cdot)$ is a reproducing kernel on $S=\mathbb{R}$ if $\hat{\phi}(\omega)\neq 0$ for every $\omega\in\mathbb{R}$. Note that ϕ as above but with compact support in $[-c,c]$ cannot lead to a reproducing kernel on $S=\mathbb{R}$, because in that case $k_\phi(t,u)=0$ whenever $|t-u|>2c$.

3.3. In [15], where the limiting profile arising from orthonormalizing the nonnegative integer translations of a fixed function on the half-line has been studied, the following three examples of the function ϕ were given:

1. $\phi(t)=\mathrm{e}^{-\sigma|t|}$ for some $\sigma>0$. Then $\hat{\phi}(\omega)=(2/\pi)^{1/2}\,\sigma(\sigma^2+\omega^2)^{-1}$, and hence
$$P(\omega)=\frac{(\sigma^2+\omega^2)^2}{4\sigma^2},\qquad k_\phi(t,u)=\kappa_\phi\big(|t-u|\big)=\frac{1+\sigma|t-u|}{\sigma}\,\mathrm{e}^{-\sigma|t-u|}.$$
2. $\phi(t)=\mathrm{e}^{-\sigma^2t^2}$ for some $\sigma>0$. Then $\hat{\phi}(\omega)=\exp(-\omega^2/4\sigma^2)/(\sigma\sqrt{2})$, and hence
$$P(\omega)=\frac{\sigma^2}{\pi}\,\mathrm{e}^{\omega^2/2\sigma^2},\qquad k_\phi(t,u)=\kappa_\phi\big(|t-u|\big)=\sqrt{\frac{\pi}{2\sigma^2}}\,\mathrm{e}^{-(1/2)\sigma^2(t-u)^2}.$$
3. $\phi(t)=1/(\sigma^2+t^2)$ for some $\sigma>0$. Then $\hat{\phi}(\omega)=(\pi/2\sigma^2)^{1/2}\cdot\mathrm{e}^{-\sigma|\omega|}$, and hence
$$P(\omega)=\frac{\sigma^2}{\pi^2}\,\mathrm{e}^{2\sigma|\omega|},\qquad k_\phi(t,u)=\kappa_\phi\big(|t-u|\big)=\frac{2\pi}{\sigma[4\sigma^2+(t-u)^2]}.$$

In addition, we present the following example.

4. $\phi(t)=\mathrm{e}^{-\sigma t}$ for $t>0$ and $\phi(t)=0$ for $t<0$, where $\sigma>0$. Then $\hat{\phi}(\omega)=(2\pi)^{-1/2}\,(\sigma-\mathrm{i}\omega)^{-1}$, and hence
$$P(\omega)=\sigma^2+\omega^2,\qquad k_\phi(t,u)=\kappa_\phi\big(|t-u|\big)=\frac{1}{2\sigma}\,\mathrm{e}^{-\sigma|t-u|}.$$

As we immediately see, each of these functions generates an RKHS H_2^P and a corresponding sampling expansion.

When the sampling points are equidistant (i.e., when $t_j=\alpha j$ for some $\alpha>0$), we define the symbol by

$$\widehat{G}(s,\alpha)=-\kappa_\phi(0)+\sum_{j=0}^{\infty}\big(s^j+s^{-j}\big)\kappa_\phi(\alpha j)=\sum_{j=-\infty}^{\infty}s^j\int_{-\infty}^{\infty}\phi(x)\phi(x+\alpha j)\,\mathrm{d}x,$$

where the series converge uniformly and absolutely in s on the unit circle if the condition

$$\sum_{j=-\infty}^{\infty}\big|\kappa_\phi(\alpha j)\big|<\infty \tag{3.4}$$

is satisfied.

The following result provides conditions on ϕ and the sampling points in order that the Gram matrix $\{\kappa_\phi(|t_i-t_j|)\}_{i,j\in J}$ be bounded on $\ell^2(J)$. In the case of equidistant sampling points, we prove that condition (3.4) holds. Note that the above four examples in 3.3 satisfy these conditions.

Theorem 3.1. Let the distinct sampling points $\{t_j\}_{j\in J}$, with $J\subset\mathbb{Z}$ an infinite set, satisfy $|t_i-t_j|\geqslant\varepsilon>0$ for $i\neq j$ in J. Further, let ϕ have one of the following two properties:

1. $\exists\tau>0$, $\psi\in L^1(\mathbb{R})$: $|\kappa_\phi(t)|\leqslant\psi(t)$ and $\psi(t)$ nonincreasing for $t\geqslant\tau$;
2. $\int_{-\infty}^{\infty}\phi(x)^2(1+x^2)^\gamma\,\mathrm{d}x<\infty$ for some $\gamma>1$.

Then the Gram matrix $\{k_\phi(t_i,t_j)\}_{i,j\in J}$ is bounded on $\ell^2(J)$. In particular, if $t_i=\alpha i$ $(i\in J=\mathbb{Z})$ for some $\alpha>0$, then (3.4) is satisfied.

Proof. Note that

$$\sup_{i\in J}\sum_{j\in J}\bigl|k_\phi(t_i,t_j)\bigr|=\sup_{i\in J}\sum_{j\in J}\bigl|\kappa_\phi\bigl(|t_i-t_j|\bigr)\bigr| \tag{3.5}$$

is an upper bound for the norm of the Gram matrix on $\ell^2(J)$.

When the first condition holds, $0=t_0<t_1<t_2<\cdots$ with $t_{j+1}-t_j\geqslant\varepsilon$ $(j\in J=\mathbb{Z}^+)$ and $N\varepsilon\geqslant\tau$, one gets

$$\begin{aligned}\sum_{j\in\mathbb{Z}^+}\bigl|k_\phi(t_i,t_j)\bigr|&=\sum_{|j-i|<N}\bigl|\kappa_\phi\bigl(|t_i-t_j|\bigr)\bigr|+\sum_{|j-i|\geqslant N}\bigl|\kappa_\phi\bigl(|t_i-t_j|\bigr)\bigr|\\&\leqslant\sum_{|j-i|<N}\bigl|\kappa_\phi\bigl(|t_i-t_j|\bigr)\bigr|+2\sum_{j=N}^{\infty}\psi(j\varepsilon)\\&\leqslant(2N-1)\|\kappa_\phi\|_\infty+\frac{2}{\varepsilon}\int_{(N-1)\varepsilon}^{\infty}\psi(t)\,\mathrm{d}t\\&\leqslant\bigl(2N-1\bigr)\|\kappa_\phi\|_\infty+\frac{2}{\varepsilon}\|\psi\|_1,\end{aligned}$$

which is finite and hence implies (3.5).

When the second condition is satisfied, we have

$$\begin{aligned}\bigl(1+|t|\bigr)^\gamma\bigl|\kappa_\phi(t)\bigr|&\leqslant\int_{-\infty}^{\infty}\bigl(1+|x|\bigr)^\gamma\bigl|\phi(x)\bigr|\cdot\bigl(1+|x+t|\bigr)^\gamma\bigl|\phi(x+t)\bigr|\,\mathrm{d}x\\&\leqslant\int_{-\infty}^{\infty}\bigl(1+|x|\bigr)^{2\gamma}\phi(x)^2\,\mathrm{d}x\leqslant2^\gamma\int_{-\infty}^{\infty}\bigl(1+x^2\bigr)^\gamma\phi(x)^2\,\mathrm{d}x,\end{aligned}$$

which implies (3.5) when $|t_i-t_j|\geqslant\varepsilon$ for $i\neq j$. □

Theorem 3.1 can be generalized by assuming that ϕ belongs to Wiener's amalgam (cf. [11,12]). Such a generalization would require technicalities beyond the scope of ideas and tools used in the proof of theorem 3.1.

Let us now compute the symbols for the four examples above in the case of equidistant sampling points. Since ϕ decays exponentially in the examples 1, 2 and 4 below, the first condition of theorem 3.1 is obviously satisfied for any $\gamma>1$. In example 3 below, the first condition is satisfied for $\gamma=4/3$. The second condition holds for the four examples in section 3.3.

1. For $\phi(t) = \mathrm{e}^{-\sigma|t|}$ for some $\sigma > 0$, we have

$$\widehat{G}(s,\alpha) = \alpha \frac{p(\alpha\sigma) + q(\alpha\sigma)[s + s^{-1}]}{(1 - s\mathrm{e}^{-\alpha\sigma})^2(1 - s^{-1}\mathrm{e}^{-\alpha\sigma})^2},$$

where $p(\beta) = 1/\beta - 4\,\mathrm{e}^{-2\beta} - (1/\beta)\,\mathrm{e}^{-4\beta}$ and $q(\beta) = (1 + 1/\beta)\,\mathrm{e}^{-3\beta} + (1 - 1/\beta)\,\mathrm{e}^{-\beta}$ are positive when $\beta > 0$. Thus $\widehat{G}(s,\alpha)$ is strictly positive on the unit circle for every $\alpha > 0$.

2. For $\phi(t) = \mathrm{e}^{-\sigma^2 t^2}$ with $\sigma > 0$, we have

$$\widehat{G}(s,\alpha) = \sqrt{\frac{\pi}{2\sigma^2}}\left(1 + 2\sum_{j=1}^{\infty} \mathrm{e}^{-\sigma^2\alpha^2 j^2/2}\cos(j\theta)\right) = \sqrt{\frac{\pi}{2\sigma^2}}\,\vartheta_3\left(\frac{1}{2}\theta, \mathrm{e}^{-\sigma^2\alpha^2/2}\right),$$

where $s = \mathrm{e}^{\mathrm{i}\theta}$ and ϑ_3 denotes a Jacobian theta function ([29], section 21.11). Using a product formula ([29], section 21.3) we get

$$\widehat{G}(s,\alpha) = G(\alpha)\sqrt{\frac{\pi}{2\sigma^2}}\prod_{j=1}^{\infty}\left\{\left(1 + \mathrm{e}^{-(j-1/2)\sigma^2\alpha^2}\mathrm{e}^{\mathrm{i}\theta}\right)\left(1 + \mathrm{e}^{-(j-1/2)\sigma^2\alpha^2}\mathrm{e}^{-\mathrm{i}\theta}\right)\right\},$$

where $G(\alpha) = \prod_{j=1}^{\infty}(1 - \mathrm{e}^{-j\sigma^2\alpha^2})$ and $s = \mathrm{e}^{\mathrm{i}\theta}$. Hence, the symbol is strictly positive on the unit circle for every $\alpha > 0$.

3. For $\phi(t) = 1/(\sigma^2 + t^2)$ with $\sigma > 0$, we have

$$\begin{aligned}\widehat{G}(s,\alpha) &= \frac{\pi^2}{\alpha}\frac{2}{\pi\sigma^2}\left(\frac{\alpha}{4\sigma} + \frac{2\sigma}{\alpha}\sum_{j=1}^{\infty}\frac{(-1)^j\cos\{j(\pi-\theta)\}}{j^2 + (2\sigma/\alpha)^2}\right)\\ &= \frac{1}{\sigma^3}\frac{\mathrm{e}^{2(\pi-\theta)\sigma/\alpha} + \mathrm{e}^{-2(\pi-\theta)\sigma/\alpha}}{\mathrm{e}^{2\pi\sigma/\alpha} - \mathrm{e}^{-2\pi\sigma/\alpha}},\end{aligned}$$

where $s = \mathrm{e}^{\mathrm{i}\theta}$ (cf. [29], example 9 to chapter IX), and this is a strictly positive function on the unit circle for every $\alpha > 0$.

4. If $\phi(t) = \mathrm{e}^{-\sigma t}$ for $t > 0$ and $\phi(t) = 0$ for $t < 0$, where $\sigma > 0$, we have

$$\widehat{G}(s,\alpha) = \frac{1}{2\sigma}\frac{1}{(1 - s\mathrm{e}^{-\sigma\alpha})(1 - s^{-1}\mathrm{e}^{-\sigma\alpha})},$$

which is a strictly positive function on the unit circle for every $\alpha > 0$.

We now give sufficient conditions on ϕ and the sampling points for the Gram matrix $\{\kappa_\phi(|t_i - t_j|)\}_{i,j\in J}$ to be bounded below on $\ell^2(J)$ by a positive multiple of the identity. Together with theorem 3.1 we then obtain sufficient conditions on ϕ and the sampling points in order that this Gram matrix be bounded and strictly positive selfadjoint on $\ell^2(J)$ and the frame inequalities (1.2) be satisfied. The four above examples satisfy these conditions. The proof is based on ideas of Schaback ([27], theorem 3.1).

Theorem 3.2. Let $\cdots < t_{-2} < t_{-1} < t_0 = 0 < t_1 < t_2 < \cdots$ be sampling points with $t_{j+1} - t_j \geqslant \varepsilon > 0$ for $j \in \mathbb{Z}$. Let ϕ be a real function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ whose Fourier transform does not have real zeros and for which the function κ_ϕ defined by (3.2) satisfies either of the conditions (i) or (ii) of theorem 3.1. Then the Gram matrix $\{\kappa_\phi(|t_i - t_j|)\}_{i,j\in\mathbb{Z}}$ is bounded and strictly positive selfadjoint on $\ell^2(\mathbb{Z})$. Moreover, if $\mathcal{X}$ denotes the closed linear span of $\{\kappa_\phi(\cdot - t_j)\}_{j=-\infty}^{\infty}$ in H_2^P defined in section 3.1, then

(a) there exist positive constants C_1, C_2 such that the frame inequalities

$$C_1 \|f\|_{H_2^P} \leqslant \left[\sum_{j=-\infty}^{\infty} \left|f(t_j)\right|^2\right]^{1/2} \leqslant C_2 \|f\|_{H_2^P}, \quad f \in \mathcal{X}, \tag{3.6}$$

hold, where $P(\omega)$ is given by (3.3), and

(b) the sampling expansion

$$f(t) = \sum_{j=-\infty}^{\infty} f(t_j) \frac{\kappa_\phi(|t - t_j|)}{\kappa_\phi(0)}, \quad t \in \mathbb{R}, \tag{3.7}$$

is valid for every $f \in \mathcal{X}$.

Proof. For $N \in \mathbb{N}$ and any set of N sampling points and arbitrary complex numbers $c_1, \ldots, c_N$, by Parseval's theorem we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left|\sum_{j=1}^{N} c_j \phi(x - t_j)\right|^2 \mathrm{d}x \\
&\quad = \int_{-\infty}^{\infty} \left|\sum_{j=1}^{N} c_j \mathrm{e}^{\mathrm{i}\omega t_j} \hat{\phi}(\omega)\right|^2 \mathrm{d}\omega \\
&\quad \geqslant \Big(2M \inf_{|\omega|\leqslant 2M} \left|\hat{\phi}(\omega)\right|^2\Big) \frac{1}{4M^2} \int_{-2M}^{2M} \left|\sum_{j=1}^{N} c_j \mathrm{e}^{\mathrm{i}\omega t_j}\right|^2 \big(2M - |\omega|\big)\, \mathrm{d}\omega \\
&\quad \geqslant \Big(2M \inf_{|\omega|\leqslant 2M} \left|\hat{\phi}(\omega)\right|^2\Big) \sum_{i,j=1}^{N} c_i \overline{c_j} A_{ij},
\end{aligned} \tag{3.8}$$

where

$$A_{ij} = \begin{cases} \left(\dfrac{\sin(M(t_i - t_j))}{M(t_i - t_j)}\right)^2 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

We now estimate

$$\sum_{\substack{j=1\\ j\neq i}}^{N} |A_{ij}| \leqslant \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{(M\varepsilon|i-j|)^2} = \frac{1}{(M\varepsilon)^2}\left[\sum_{j=1}^{i-1} \frac{1}{(i-j)^2} + \sum_{j=i+1}^{N} \frac{1}{(j-i)^2}\right]$$

$$\leqslant \frac{2}{(M\varepsilon)^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{3(M\varepsilon)^2}.$$

Using Gershgorin's theorem ([14], theorem 8.1.3), the real symmetric matrix $\{A_{ij}\}_{i,j=1}^{N}$ has all of its eigenvalues in $[\frac{2}{3}, \frac{4}{3}]$ whenever $M\varepsilon \geqslant \pi$. Therefore, for this choice of M the lower bound (3.8) extends to arbitrary subsets of the sampling points and hence the Gram matrix $\{\kappa_\phi(|t_i - t_j|)\}_{i,j\in\mathbb{Z}}$ is strictly positive selfadjoint. Its boundedness follows from theorem 3.1. The frame inequalities (3.6) now follow with the help of proposition 2.1. □

The assumption $\hat{\phi}(\omega) \neq 0$ for $\omega \in \mathbb{R}$ has not been used in the proof of theorem 3.1. Moreover, the proof of theorem 3.2 goes through if $\hat{\phi}(\omega) \neq 0$ for $|\omega| \leqslant 2\pi/\varepsilon$. A close inspection of this proof where the interval $[\frac{2}{3}, \frac{4}{3}]$ is replaced by $[1-\delta, 1+\delta]$ for some $\delta \in (0,1)$, reveals that the conclusion of theorem 3.2 is also true if $\hat{\phi}(\omega) \neq 0$ for $|\omega| \leqslant (2\pi/\varepsilon\sqrt{3})$.

4. Reproducing kernels of functions on the half-line and sampling

In this section we consider sampling expansions for functions in RKHS on $\mathbb{R}^+$. Such RKHS are not unitarily translation invariant, so the techniques of section 3 do not apply immediately. However, it will turn out that the sequence of unilateral translates $\{\phi(\cdot - t_j)\}_{j=0}^{\infty}$ is a Riesz basis of a suitable RKHS on $\mathbb{R}^+$ which can be described explicitly in terms of Hardy spaces, if the sequence of bilateral translates $\{\phi(\cdot - t_j)\}_{j=-\infty}^{\infty}$ is a Riesz basis of H_2^P. A major tool in deriving these results will be the main result of [16] which allows one to derive certain sampling expansions on RKHS of functions on $\mathbb{R}$ from the corresponding results on RKHS of functions on $\mathbb{R}^+$.

Let $0 = t_0 < t_1 < t_2 < \cdots$ be sampling points with $t_{j+1} - t_j \geqslant \varepsilon > 0$ for $j \in \mathbb{Z}^+$. Let ϕ be a real function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ whose Fourier transform does not have real zeros and for which the function κ_ϕ defined by (3.2) satisfies $|\kappa_\phi(t)| \leqslant \psi(t)$ for $t \geqslant \tau$ and $\psi \in L^1(\mathbb{R})$ for some $\tau > 0$. Then, as we have seen in theorem 3.1, the Gram matrix $\{\kappa_\phi(|t_i - t_j|\}_{i,j\in\mathbb{Z}}$ is bounded and strictly positive selfadjoint on $\ell^2(\mathbb{Z})$ and the functions $\{k_{t_j}\}_{j\in\mathbb{Z}}$ form a Riesz basis of some closed subspace of the RKHS H_2^P, where $P(\omega)$ is given by (3.3). The purpose of this section is to prove that the semi-infinite matrix

$$G_{ij} = \int_0^\infty \phi(x - t_i)\phi(x - t_j)\,\mathrm{d}x, \quad i, j \in \mathbb{Z}^+, \tag{4.1}$$

is bounded and strictly positive selfadjoint on $\ell^2(\mathbb{Z}^+)$ or, equivalently, that the functions $\{\phi(\cdot - t_j)\}_{j\in\mathbb{Z}^+}$ on the positive half-line form a Riesz basis of a suitable closed subspace of the RKHS H_2^P defined in section 3.1. Its proof is based on [16], theorem 2.4.

Theorem 4.1. Let $0 = t_0 < t_1 < t_2 < \cdots$ be sampling points with $t_{j+1} - t_j \geqslant \varepsilon > 0$ for $j \in \mathbb{Z}^+$. Let ϕ be a real function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ whose Fourier transform does not have real zeros, which satisfies either of the conditions (i) and (ii) of theorem 3.1, and which satisfies $\int_0^\infty x|\phi(-x)|^2\,\mathrm{d}x < \infty$ and for which $\operatorname{supp}\phi \cap \mathbb{R}^+$ has positive measure. Then the Gram matrix defined by (4.1) is bounded and strictly positive selfadjoint on $\ell^2(\mathbb{Z}^+)$. Furthermore, if $\mathcal{X}^+$ denotes the closed linear span of $\{k_\phi(\cdot, t_j)\}_{j=0}^\infty$ in H_2^P, then

(a) there exist positive constants C_1, C_2 such that the frame inequalities

$$C_1\|f\|_{H_2^P} \leqslant \left[\sum_{j=0}^\infty |f(t_j)|^2\right]^{1/2} \leqslant C_2\|f\|_{H_2^P}, \quad f \in \mathcal{X}^+, \tag{4.2}$$

hold, and

(b) the sampling expansion

$$f(t) = \sum_{j=0}^\infty f(t_j)\frac{k_\phi(t, t_j)}{k_\phi(t_j, t_j)}, \quad t \geqslant 0, \tag{4.3}$$

holds for every $f \in \mathcal{X}^+$.

Proof. In view of theorem 2.4 of Goodman et al. [16] and theorem 3.2 above, it suffices to prove the following:

(i) The functions $\{\phi(\cdot - t_j)\}_{j\in\mathbb{Z}^+}$ form a Riesz basis of some closed subspace of $L^2(\mathbb{R})$.

(ii) Let $\{a_j\}_{j\in\mathbb{Z}^+}$ belong to $\ell^2(\mathbb{Z}^+)$. Then $\sum_{j\in\mathbb{Z}^+} a_j\phi(x - t_j) = 0$ for almost all $x \in \mathbb{R}^+$ implies $a_j = 0$ for all $j \in \mathbb{Z}^+$.

(iii) We have

$$\sum_{j=1}^\infty \int_{-\infty}^0 |\phi(x - t_j)|^2\,\mathrm{d}x < \infty. \tag{4.4}$$

Part (i) follows directly from theorems 3.1 and 3.2, using corollary 2.2. Condition (ii) is immediate, because of the condition that $\operatorname{supp}\phi \cap \mathbb{R}^+$ has positive measure. In this case the constants $k_\phi(t_j, t_j) = \int_{-t_j}^\infty |\phi(x)|^2\,\mathrm{d}x$ satisfy

$$0 < \int_0^\infty |\phi(x)|^2\,\mathrm{d}x \leqslant k_\phi(t_j, t_j) \leqslant \|\phi\|_2^2 < \infty.$$

Further, to establish (4.4) we estimate

$$\begin{aligned}\sum_{j=1}^\infty \int_{-\infty}^0 |\phi(x - t_j)|^2\,\mathrm{d}x &= \sum_{j=1}^\infty \int_{t_j}^\infty |\phi(-x)|^2\,\mathrm{d}x \leqslant \frac{1}{\varepsilon}\sum_{j=1}^\infty (t_j - t_{j-1})\int_{t_j}^\infty |\phi(-x)|^2\,\mathrm{d}x \\ &\leqslant \frac{1}{\varepsilon}\int_0^\infty \int_t^\infty |\phi(-x)|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\varepsilon}\int_0^\infty x|\phi(-x)|^2\,\mathrm{d}x < \infty,\end{aligned}$$

which settles (4.4). Finally, the positive definiteness of the Gram matrix defined by (4.1) implies that there exists an RKHS of functions on the positive half-line having

$$k(t,u) = \int_0^\infty \phi(x-t)\phi(x-u)\,\mathrm{d}x$$

as its reproducing kernel. □

Let us now derive the reproducing kernel $k(t,u)$ and the sampling expansion (4.3) for the four examples of functions ϕ discussed before.

1. Let $\phi(t) = \mathrm{e}^{-\sigma|t|}$ for some $\sigma > 0$. Then (cf. [15])

$$k(t,u) = \left(\frac{1}{\sigma} + |t-u|\right)\mathrm{e}^{-\sigma|t-u|} - \frac{1}{2\sigma}\,\mathrm{e}^{-\sigma(t+u)}.$$

   Hence the corresponding sampling expansion has the form

$$f(t) = \sum_{j=0}^{\infty} f(t_j)\frac{(1+\sigma|t-t_j|)\,\mathrm{e}^{-\sigma|t-t_j|} - (1/2)\,\mathrm{e}^{-\sigma(t+t_j)}}{1-(1/2)\,\mathrm{e}^{-2\sigma t_j}}.$$

2. Let $\phi(t) = \mathrm{e}^{-\sigma^2 t^2}$ for some $\sigma > 0$. Then (cf. [15])

$$k(t,u) = \frac{1}{2\sigma}\left(\frac{\pi}{2}\right)^{1/2}\mathrm{e}^{-(1/2)\sigma^2(t-u)^2}\operatorname{erfc}\left(\frac{1}{2}\sigma(t+u)\right),$$

   where $\operatorname{erfc}(z) = (2/\sqrt{\pi})\int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t$ (cf. [1], 7.1.2). Hence the corresponding sampling expansion has the form

$$f(t) = \sum_{j=0}^{\infty} f(t_j)\mathrm{e}^{-(1/2)\sigma^2(t-t_j)^2}\frac{\operatorname{erfc}((1/2)\sigma(t+t_j))}{\operatorname{erfc}(\sigma t_j)}.$$

3. Let $\phi(t) = 1/(\sigma^2+t^2)$ for some $\sigma > 0$. Then by elementary though tedious calculations we find

$$k(t,u) = \frac{1}{\sigma[4\sigma^2+(t-u)^2]}\cdot\left[\frac{\sigma}{t-u}\log\frac{\sigma^2+t^2}{\sigma^2+u^2} + \pi + \arctan\frac{t}{\sigma} + \arctan\frac{u}{\sigma}\right].$$

   Hence the corresponding sampling expansion has the form

$$f(t) = \sum_{j=0}^{\infty} f(t_j)\frac{\frac{\sigma}{t-t_j}\log\frac{\sigma^2+t^2}{\sigma^2+t_j^2} + \pi + \arctan\frac{t}{\sigma} + \arctan\frac{t_j}{\sigma}}{\left[1+\left(\frac{t-t_j}{2\sigma}\right)^2\right]\left[\frac{2\sigma t_j}{\sigma^2+t_j^2} + \pi + 2\arctan\frac{t_j}{\sigma}\right]}.$$

4. Let $\phi(t) = \mathrm{e}^{-\sigma t}$ for $t > 0$ and $\phi(t) = 0$ for $t < 0$, where $\sigma > 0$. Then

$$k(t,u) = \frac{1}{2\sigma}\,\mathrm{e}^{-\sigma|t-u|}.$$

Hence the corresponding sampling expansion has the form

$$f(t) = \sum_{j=0}^{\infty} f(t_j)\,\mathrm{e}^{-\sigma|t-t_j|}.$$

5. Concluding remarks

We conclude this paper with a comparison between the sampling results in [23] and those in the present paper.

In [23], the RKHS H_Q is assumed to be a closed subspace of the Sobolev space $H^{-1}(\mathbb{R})$ and is also closed under differentiation but need not have unitary translation invariance properties. Under the conditions that

(i) $\{t_j\}_{j=1}^{\infty}$ is a set of uniqueness of H_Q and $t_j \sim j$ as $j \to \infty$;

(ii) $k(\cdot,\cdot)$ is continuous and

$$\frac{f(t)}{k(t,t)} = \mathrm{O}\big(|t|^{-2}\big), \quad t \to \pm\infty,$$

we have the sampling expansion

$$f(t) = \sum_{j=1}^{\infty} f(t_j)\frac{k(t_j,t)}{k(t_j,t_j)}.$$

Applications are given to Sobolev spaces, the Shannon–Whittaker sampling theorem and Sturm–Liouville transforms. The leading idea of [23] is to imbed the space $B_\pi(\mathbb{R})$ of band limited signals in a closed subspace of $H^{-1}(\mathbb{R})$.

In the present paper, the RKHS H is assumed to be unitarily translation invariant. Starting from a real function $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to represent an extensive class of corresponding reproducing kernels as in (1.1), we arrive at the sampling results of section 3. When modified to encompass reproducing kernels as in (1.3), we derive the sampling results of section 4 on the half-line.

In the present paper some technical assumptions on ϕ are needed to derive theorems 3.1, 3.2 and 4.1. No sampling expansions were obtained for band limited signals. On the other hand, in [23] one requires an asymptotic condition on the sampling points and a growth condition on $f(t)/k(t,t)$ when $f \in H$. The approach in [23] does not lead to sampling expansions in RKHS on $\mathbb{R}^+$.

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References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).

[2] A. Aldroubi and K. Gröchenig, Beurling–Landau-type theorems for non-uniform sampling in shift-invariant spline spaces, J. Fourier Anal. Appl. 6 (2000) 93–103.

[3] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev. 43 (2001) 585–620.

[4] D. Alpay, *The Schur Algorithm, Reproducing Kernel Spaces and System Theory*, SMF/AMS Texts and Monographs, Vol. 5 (Amer. Math. Soc., Providence, RI, 2001).

[5] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950) 337–404.

[6] P.L. Butzer, A survey of the Whittaker–Shannon sampling theorem and some of its extensions, J. Math. Res. Exposition 3 (1983) 185–212.

[7] P.L. Butzer, J.R. Higgins and R.L. Stens, Sampling theory of signal analysis, in: *Development of Mathematics 1950–2000*, ed. J.P. Pier (Birkhäuser, Basel, 2000) pp. 193–234.

[8] P.L. Butzer, W. Splettstösser and R.L. Stens, The sampling theorem and linear predictions in signal analysis, Jahresber. Deutsch. Math.-Verein. 90 (1988) 1–60.

[9] W. Cheney and W. Light, *A Course in Approximation Theory* (Brooks/Cole, 2000).

[10] R.J. Duffin and A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952) 341–366.

[11] H.G. Feichtinger, New results on regular and irregular sampling based on Wiener amalgams, in: *Function Spaces*, ed. K. Jarosz (Marcel Dekker, New York, 1992) pp. 107–121.

[12] H.G. Feichtinger, Wiener amalgams over Euclidean spaces and some of their applications, in: *Function Spaces*, ed. K. Jarosz (Marcel Dekker, New York, 1992) pp. 123–137.

[13] I. Gohberg, S. Goldberg and M.A. Kaashoek, *Classes of Linear Operators*, Vol. II. (Birkhäuser, Basel, 1993).

[14] G.H. Golub and C.F. Van Loan, *Matrix Computations* (John Hopkins Univ. Press, Baltimore, MD, 1983).

[15] T.N.T. Goodman, C.A. Micchelli, G. Rodriguez and S. Seatzu, On the limiting profile arising from orthonormalising shifts of an exponentially decaying function, IMA J. Numer. Anal. 18 (1998) 331–354.

[16] T.N.T. Goodman, C.A. Micchelli and Z. Shen, Riesz bases in subspaces of $L_2(\mathbb{R}_+)$, Construct. Approx. 17 (2000) 39–46.

[17] J.R. Higgins, Five stories about the cardinal series, Bull. Amer. Math. Soc. 12 (1985) 45–89.

[18] J.R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations* (Oxford Univ. Press, Oxford, 1996).

[19] E. Hille, Introduction to general theory of reproducing kernels, Rocky Mountain J. Math. 2 (1972) 321–368.

[20] A.J. Jerri, The Shannon sampling theorem – Its various extensions and applications: A tutorial review, Proc. IEEE 65 (1977) 1565–1596.

[21] M.Z. Nashed, On expansion methods for inverse and recovery problems from partial information, in: *Nonlinear Problems*, eds. T.S. Angell et al. (SIAM, Philadelphia, PA, 1996) pp. 177–189.

[22] M.Z. Nashed and G. Wahba, Generalized inverses in reproducing kernel spaces: An approach to regularization of linear operator equations, SIAM J. Math. Anal. 5 (1974) 974–987.

[23] M.Z. Nashed and G.G. Walter, General sampling theorems for functions in reproducing kernel Hilbert spaces, Math. Control Signals Systems 4 (1991) 363–390.

[24] M.Z. Nashed and G.G. Walter, Reproducing kernel Hilbert spaces from sampling expansions, in: *Mathematical Analysis, Wavelets, and Signal Processing*, eds. M.E.H. Ismail, M.Z. Nashed, A.I. Zayed and A.F. Ghaleb, Cairo, 1994, Contemporary Mathematics, Vol. 190 (Amer. Math. Soc., Providence, RI, 1995) pp. 221–226.

[25] R.E.A.C. Paley and N. Wiener, Fourier transforms in the complex domain, in: *American Mathematical Society Colloquium*, Vol. 19 (Amer. Math. Soc., Providence, RI, 1934).

[26] S. Saitoh, *Theory of Reproducing Kernels and its Applications* (Longman, New York, 1988).
[27] R. Schaback, Error estimates and condition numbers for radial basis function interpolation, Adv. Comput. Math. 3 (1995) 251–264.
[28] H.S. Shapiro, *Topics in Approximation Theory*, Lectures Notes in Mathematics, Vol. 187 (Springer, New York, 1971).
[29] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th ed. (Cambridge Univ. Press, Cambridge, 1927).
[30] R.M. Young, *An Introduction to Nonharmonic Fourier Series*, 2nd ed. (Academic Press, New York, 2000).
[31] A.I. Zayed, *Advances in Shannon's Sampling Theory* (CRC Press, Boca Raton, FL, 1993).