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APPROXIMATION OF SOLUTIONS OF RICCATI EQUATIONS*

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Abstract. This paper deals with two interrelated issues. One is an invariant subspace approach to finding solutions for the algebraic Riccati equation for a class of infinite dimensional systems. The second is approximation of the solution of the algebraic Riccati equation by finite dimensional approximants. The theory of exponentially dichotomous operators and bisemigroups is instrumental in our approach.

Key words. Riccati equation, matrix approximation, exponential dichotomy

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1. Introduction.

The goal of this paper is twofold. The first goal is to use the theory of exponentially dichotomous operators and bisemigroups to derive a result from the existence of solutions to an algebraic Riccati equation of the type occurring in LQ-optimal control. This approach allows one to mimic the finite dimensional approach to algebraic Riccati equations; that is, it allows one to use an invariant subspace argument to obtain the extremal solutions to the algebraic Riccati equation. This topic is dealt with in section 2. It is a continuation of earlier work in this direction presented in [18, 19].

The second goal is to use the results obtained in section 2 to discuss finite dimensional approximations of the solutions of the algebraic Riccati equation and of the corresponding closed loop semigroup. Our results in this direction are presented in section 3.

The work on which this paper reports is loosely based on the work done by the first author for his masters thesis, in combination with work on the perturbation of bisemigroup generators of the last two authors [19].

Finite dimensional approximations of solutions of algebraic Riccati equations and of the corresponding closed loop semigroups are the topic of several earlier contributions; see [2, 8, 14, 12, 13, 20]. In comparison with [14] we do not discuss the algebraic Riccati equation coming from H^∞-control theory, but rather confine ourselves to the one stemming from LQ-optimal control. The result we obtain is, in this special case, the same, under slightly different assumptions, but with a completely different, and in our view, more transparent proof. In [20] attention was also focused on the algebraic Riccati equation from LQ-optimal control. The assumptions there are seriously weaker than the ones imposed in previous works. In particular, instead of exponential stability (or exponential stabilizability) in [20] strong stabilizability is assumed. Instead, we consider exponentially dichotomous operators, which allows us to deal with Hamiltonian operators of linear systems that have no spectrum within a strip

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about the imaginary axis and for which one of the off-diagonal operators is compact. However, we obtain stronger results on the closed loop approximants (compare our Theorem 3.4 with Theorem 4.2 in [20]) in return for our stronger assumptions. Again, our methods of proof are quite different from the ones in [20].

Our approach is part of a long tradition of studying stability results for solutions of Riccati equations by performing a stability analysis of certain invariant subspaces of the Hamiltonian operator, while also linking these to stable factorizations of a transfer function [3, 17]. Structural similarities between these interlocking problems and state space approaches to solve convolution equations [5] and stationary transport equations [10] have naturally led to the formal study of exponentially dichotomous operators [4], results on their perturbation [19], and its present stability analysis of Hamiltonian operators of autonomous linear systems.

In [19] we have linked the left and right canonical Wiener–Hopf factorizability of a transfer function built from the Hamiltonian operator

$$\begin{pmatrix} A_0 & -D \\ -Q & -A_1 \end{pmatrix} \tag{1.1}$$

of a linear system to the existence of the stable and anti-stable solution of a Riccati equation, under hardly more than the assumption that $-A_0$ and $-A_1$ generate exponentially decaying C_0-semigroups on a general Banach space. Even though not stated explicitly, stability results for these solutions of the Riccati equations are expected (and thus now conjectured) to hold if the transfer function has a left or right canonical Wiener–Hopf factorization. Using the well-known fact that this is true for positive selfadjoint transfer functions on a Hilbert space, we naturally arrive at the basic outline of the present paper. The stability analysis itself appears to be straightforward.

Our present approach may be viewed as a tool to derive stability results for Riccati equations starting from the Hamiltonian operator, where the derivation of the latter is standard system theory [7, 15]. Many of the existing results (but not all; see [20]) can thus be derived in a transparent way, but the present approach potentially leads to useful applications to delay systems where the underlying spaces are L^1 [11].

When dealing with Hamiltonian operators of the type (1.1) with $D = BR^{-1}B^*$, $Q = C^*C$, and $A_0 = A_1^* = A$, the infinitesimal generator of a C_0-semigroup on a separable Hilbert space $\mathcal{H}$, it is sufficient to require the exponential stabilizability of (A, B) or the exponential detectability of (C, A) to arrive at an exponentially dichotomous operator on $\mathcal{H}\dot{+}\mathcal{H}$ after a similarity implementing state feedback or output injection (e.g., see [7]). Thus for the purpose of this article it is sufficient to deal with Hamiltonian operators that are exponentially dichotomous.

Let us conclude the introduction with some notations and definitions. By $\mathcal{D}(A)$, $\operatorname{Ker} A$, and $\operatorname{Im} A$ we denote the domain, kernel, and range of a linear operator A, respectively, and by $I_{\mathcal{H}}$ the identity operator on a Hilbert space $\mathcal{H}$. By $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{H}_1\dot{+}\mathcal{H}_2$ we denote the orthogonal direct sum of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and by $A \stackrel{\text{def}}{=} A_1\dot{+}A_2$ the linear operator on $\mathcal{H}$ with domain $\{(x_1, x_2) : x_j \in \mathcal{D}(\mathcal{H}_j), j = 1, 2\}$ defined by $A(x_1, x_2) = (A_1x_1, A_2x_2)$.

2. Preliminaries. A closed and densely defined linear operator $-S$ on a Hilbert space $\mathcal{H}$ is called *exponentially dichotomous* [4] if for some bounded projection P commuting with S, the restrictions of S to $\operatorname{Im} P$ and of $-S$ to $\operatorname{Ker} P$ are the infinitesimal generators of exponentially decaying C_0-semigroups. We then define the *bisemigroup* generated by $-S$ as

$$E(t;-S) = \begin{cases} e^{-tS}(I-P), & t>0 \\ -e^{-tS}P, & t<0. \end{cases}$$

Its *separating projection* P is given by $P = -E(0^-;-S) = I_{\mathcal{H}} - E(0^+;-S)$. One easily verifies [4] the existence of $\varepsilon > 0$ such that $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \le \varepsilon\}$ is contained in the resolvent set $\rho(S)$ of S and for every $x \in \mathcal{H}$

$$(\lambda - S)^{-1}x = -\int_{-\infty}^{\infty} e^{\lambda t} E(t;-S)x\,dt, \qquad |\mathrm{Re}\,\lambda| \le \varepsilon. \tag{2.1}$$

As a result, $\|(\lambda - S)^{-1}x\| \to 0$ as $\lambda \to \infty$ in $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \le \varepsilon'\}$ for some $\varepsilon' \in (0,\varepsilon]$.

We have the following perturbation result given also in [19]. We shall give its proof for selfcontainedness.

THEOREM 2.1. *Let $-S_0$ be exponentially dichotomous, Γ be a compact operator, and $-S = -S_0 + \Gamma$, where $\mathcal{D}(S) = \mathcal{D}(S_0)$. Suppose the imaginary axis is contained in the resolvent set of S. Then $-S$ is exponentially dichotomous. Moreover, $E(t;-S) - E(t;-S_0)$ is a compact operator, also in the limits as $t \to 0^{\pm}$.*

Proof. There exists $\varepsilon > 0$ such that

$$\int_{-\infty}^{\infty} e^{\varepsilon|t|}\|E(t;-S_0)\|\,dt < \infty. \tag{2.2}$$

Using the resolvent identity

$$(\lambda - S)^{-1} - (\lambda - S_0)^{-1} = -(\lambda - S_0)^{-1}\Gamma(\lambda - S)^{-1}, \qquad |\mathrm{Re}\,\lambda| \le \varepsilon,$$

for some $\varepsilon > 0$, we obtain the convolution integral equation

$$E(t;-S)x - \int_{-\infty}^{\infty} E(t-\tau;-S_0)\Gamma E(\tau;-S)x\,d\tau = E(t;-S_0)x, \tag{2.3}$$

where $x \in \mathcal{H}$ and $0 \ne t \in \mathbb{R}$. In (2.3), the convolution kernel $E(\cdot;-S_0)\Gamma$ is continuous in the norm except for a jump discontinuity in $t = 0$, as a result of the strong continuity (except for the jump) of $E(\cdot;-S_0)$ and the compactness of Γ. Further, (2.2) implies that $e^{\varepsilon|\cdot|}E(\cdot;-S_0)\Gamma$ is Bochner integrable.

The symbol of the convolution integral equation (2.3), which equals $I_{\mathcal{H}} + (\lambda - S_0)^{-1}\Gamma = (\lambda - S_0)^{-1}(\lambda - S)$, tends to $I_{\mathcal{H}}$ in the norm as $\lambda \to \infty$ in the strip $|\mathrm{Re}\,\lambda| \le \varepsilon$, since Γ is compact and $(\lambda - S_0)^{-1}$ tends to zero strongly. Moreover, it is a compact perturbation of the identity which, by definition, only takes invertible values on the imaginary axis. Thus there exists $\varepsilon_0 \in (0,\varepsilon]$ such that the symbol only takes invertible values on the strip $|\mathrm{Re}\,\lambda| \le \varepsilon_0$.

Before proceeding with the proof we now state the Bochner–Phillips theorem [6, 9]:

- Let $\mathcal{A}_0$ be a Banach algebra, $\mathcal{A}$ its natural extension to a Banach algebra with unit element, and $W_{\mathcal{A}_0}$ the Banach algebra of all ordered pairs (A_∞, A), where $A_\infty \in \mathcal{A}$ and A is a Bochner integrable function from $\mathbb{R}$ into $\mathcal{A}_0$, endowed with the norm

$$\|(A_\infty, A)\|_{W_{\mathcal{A}_0}} \stackrel{def}{=} \|A_\infty\|_{\mathcal{A}} + \int_{-\infty}^{\infty} \|A(t)\|_{\mathcal{A}_0}\,dt.$$

Then (A_∞, A) is invertible in $W_{\mathcal{A}_0}$ if and only if A_∞ and all of the Fourier transform values

$$A_\infty + \int_{-\infty}^{\infty} e^{i\lambda t} A(t)\, dt, \qquad \lambda \in \mathbb{R},$$

are invertible elements of $\mathcal{A}$. In that case the inverse (B_∞, B) is given by $B_\infty = (A_\infty)^{-1}$ and

$$B_\infty + \int_{-\infty}^{\infty} e^{i\lambda t} B(t)\, dt = \left[A_\infty + \int_{-\infty}^{\infty} e^{i\lambda t} A(t)\, dt\right]^{-1}, \qquad \lambda \in \mathbb{R}.$$

We now apply this result in two different situations: (i) $\mathcal{A} = \mathcal{A}_0 = L(\mathcal{H})$ is the Banach algebra of bounded linear operators on $\mathcal{H}$, and (ii) $\mathcal{A}_0 = K(\mathcal{H})$ is the Banach algebra of compact operators on $\mathcal{H}$ and $\mathcal{A} = \{\lambda I_{\mathcal{H}} + K : \lambda \in \mathbb{C}, K \in K(\mathcal{H})\}$. We then also use that an element $(A_\infty, A) \in W_{L(\mathcal{H})}$ induces a bounded linear operator on $BC(\mathbb{R}^-; \mathcal{H}) \oplus BC(\mathbb{R}^*; \mathcal{H})$, the bounded continuous functions from $\mathbb{R}$ into $\mathcal{H}$ with a jump discontinuity at $t = 0$, by convolution.

By the Bochner–Phillips theorem, the convolution equation (2.3) has a unique solution $u(\cdot; x) = E(\cdot; -S)x$ with the following properties:

1) $E(\cdot; -S)$ is strongly continuous, except for a jump discontinuity at $t = 0$,
2) $\int_{-\infty}^{\infty} e^{\varepsilon_0 |t|} \|E(t; -S)\|\, dt < \infty$; hence $E(\cdot; -S)$ is exponentially decaying,
3) $E(t; -S) - E(t; -S_0)$ is a compact operator, also in the limits as $t \to 0^{\pm}$, and
4) the identity (2.1) holds.

As result [4], $-S$ is exponentially dichotomous. $\square$

The set $\theta = (A_0, Q, D; \mathcal{H})$ is called a *triple* if $\mathcal{H}$ is a complex Hilbert space, A_0 generates a strongly continuous semigroup on $\mathcal{H}$ of negative exponential type, and Q and D are bounded selfadjoint operators on $\mathcal{H}$. Then obviously $-S_0 = (-A_0) \dot{+} A_0^*$ is exponentially dichotomous on $\mathcal{H} \dot{+} \mathcal{H}$ and $P_0 = I_{\mathcal{H}} \dot{+} 0$ is the separating projection of the corresponding bisemigroup $E(\cdot; -S_0)$. The triple θ is called *semicompact* if D is a compact operator on $\mathcal{H}$, and *compact* if both D and Q are compact operators on $\mathcal{H}$. The triple θ is called *positive semidefinite* if Q and D are positive semidefinite selfadjoint, and *antipodal* if one of Q and D is positive semidefinite selfadjoint and the other is negative semidefinite selfadjoint.

Theorem 2.1 can be used to prove the following more specific result.

THEOREM 2.2. *Let $\theta = (A_0, Q, D; \mathcal{H})$ be a positive semidefinite and semicompact triple. Then the block matrix operator $-S$ defined on $\mathcal{H} \dot{+} \mathcal{H}$ by*

$$S = \begin{bmatrix} A_0 & -D \\ -Q & -A_0^* \end{bmatrix},$$

is exponentially dichotomous.

Proof. Suppose (2.2) is satisfied. Let us define the operator

$$S_Q = \begin{bmatrix} A_0 & 0 \\ -Q & -A_0^* \end{bmatrix}, \qquad \mathcal{D}(S_Q) = \mathcal{D}(S_0).$$

Consider the unique and positive semidefinite solution X of the Lyapunov equation (e.g., [7, (1.12)–(1.13)])

$$A_0^* X + X A_0 = -Q,$$

given by

$$Xx = \int_0^\infty e^{\tau A_0^*} Q e^{-\tau A_0} x \, d\tau, \qquad x \in \mathcal{H}.$$

Note that

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} S_Q \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = S_0.$$

So S_Q and S_0 are similar. Hence $-S_Q$ is exponentially dichotomous, and we obtain

$$E(\cdot; -S_Q) = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} E(\cdot; -S_0) \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}.$$

We also see that the separating projection P_Q of $E(\cdot; -S_Q)$ is given by

$$P_Q = \begin{bmatrix} I & 0 \\ X & 0 \end{bmatrix}.$$

Next, we remark that $S - S_Q$ is a compact operator. Hence by Theorem 2.1, to prove that $-S$ is exponentially dichotomous, it suffices to prove that $-S$ does not have imaginary eigenvalues. Indeed, let λ be an imaginary eigenvalue of $-S$. Then there exist $x \in \mathcal{D}(A_0)$ and $y \in \mathcal{D}(A_0^*)$ such that

$$\begin{aligned} (\lambda + A_0)x - Dy &= 0, \\ -Qx + (\lambda - A_0^*)y &= 0. \end{aligned}$$

Then, since λ is purely imaginary, we have

$$\langle Qx, x\rangle + \langle Dy, y\rangle = \langle(\lambda - A_0^*)y, x\rangle + \langle(\lambda + A_0)x, y\rangle = 0,$$

which implies $Qx = Dy = 0$. But then $(\lambda - A_0)x = (\lambda + A_0^*)y = 0$ for some imaginary λ, and hence $x = y = 0$, as claimed. □

Let

$$P = -E(0^-; -S) = I_{\mathcal{H}\dot{+}\mathcal{H}} - E(0^+; -S)$$

denote the separating projection of $E(\cdot; -S)$. Consider the indefinite scalar product generated by

$$\mathcal{J}_1 = \begin{bmatrix} 0 & -I_{\mathcal{H}} \\ -I_{\mathcal{H}} & 0 \end{bmatrix}$$

on $\mathcal{H}\dot{+}\mathcal{H}$. Since $\mathcal{J}_1 S + S^* \mathcal{J}_1 = 2(Q\dot{+}D)$, the real part $\frac{1}{2}(S + \mathcal{J}_1^{-1} S^* \mathcal{J}_1)$ of S with respect to the indefinite scalar product generated by $\mathcal{J}_1$ is positive semidefinite selfadjoint whenever $\theta = (A_0, Q, D; \mathcal{H})$ is a positive semidefinite triple. Hence, in this case it is clear that $\operatorname{Im} P$ is a $\mathcal{J}_1$-nonpositive and $\operatorname{Ker} P$ is a $\mathcal{J}_1$-nonnegative S-invariant subspace of $\mathcal{H}\dot{+}\mathcal{H}$ (cf. [1, section 3.2]). Also [1], since iS is selfadjoint with respect to the indefinite scalar product generated by

$$\mathcal{J}_2 = \begin{bmatrix} 0 & iI_{\mathcal{H}} \\ -iI_{\mathcal{H}} & 0 \end{bmatrix},$$

it is clear that $\operatorname{Im} P$ and $\operatorname{Ker} P$ are $\mathcal{J}_2$-neutral S-invariant subspaces of $\mathcal{H}\dot{+}\mathcal{H}$ (i.e., on these subspaces the sesquilinear form $(x, y) \mapsto (\mathcal{J}_2 x, y)$ is trivial).

Further, with X as above and $\theta = (A_0, Q, D; \mathcal{H})$ a positive semidefinite triple, we have

$$-S_X \overset{def}{=} \begin{bmatrix} I_{\mathcal{H}} & 0 \\ -X & I_{\mathcal{H}} \end{bmatrix} (-S) \begin{bmatrix} I_{\mathcal{H}} & 0 \\ X & I_{\mathcal{H}} \end{bmatrix} = -S_0 + \begin{bmatrix} I_{\mathcal{H}} \\ -X \end{bmatrix} D \begin{bmatrix} X & I_{\mathcal{H}} \end{bmatrix},$$

which implies that $(A_0 - DX, XDX, -D; \mathcal{H})$ is an antipodal compact triple.

We need the following definitions. Suppose W is a continuous function from the extended imaginary axis $i(\mathbb{R} \cup \{\infty\})$ into $\mathcal{L}(\mathcal{H})$. Then by a *left canonical* (*Wiener–Hopf*) *factorization* of W we mean a representation of W of the form

$$W(\lambda) = W_+(\lambda) W_-(\lambda), \qquad \operatorname{Re} \lambda = 0,$$

in which $W_\pm(\pm\lambda)$ is continuous on the closed right half-plane (the point at ∞ included), is analytic on the open right half-plane, and takes only invertible values for λ in the closed right half-plane (the point at infinity included). Obviously, such an operator function only takes invertible values on the extended imaginary axis. By a *right canonical* (*Wiener–Hopf*) *factorization* we mean a representation of W of the form

$$W(\lambda) = W_-(\lambda) W_+(\lambda), \qquad \operatorname{Re} \lambda = 0,$$

where $W_\pm(\lambda)$ is as above.

THEOREM 2.3. *Let $\theta = (A_0, Q, D; \mathcal{H})$ be a positive semidefinite and semicompact triple. Then we have the following decompositions:*

$$Im\, P \dot{+} Ker\, P_0 = \mathcal{H} \dot{+} \mathcal{H}, \tag{2.4}$$

$$Ker\, P \dot{+} Im\, P_0 = \mathcal{H} \dot{+} \mathcal{H}. \tag{2.5}$$

Proof. Let us introduce the operators

$$V = P_0 P + (I - P_0)(I - P), \tag{2.6}$$

$$V_Q = P_0 P_Q + (I - P_0)(I - P_Q). \tag{2.7}$$

Then

$$V_Q = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}} & 0 \\ X & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{H}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -X & I_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ -X & I_{\mathcal{H}} \end{bmatrix},$$

so that V_Q is invertible. On the other hand, the identity

$$\begin{aligned} &E(t; -S) - \int_{-\infty}^{\infty} E(t - \tau; -S_Q) \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} E(\tau; -S)\, d\tau \\ &= E(t; -S) - \int_{-\infty}^{\infty} E(t - \tau; -S_Q) \begin{bmatrix} 0 & D^{1/2} \\ 0 & 0 \end{bmatrix} \left(E(\tau; -S^*) \begin{bmatrix} 0 & 0 \\ D^{1/2} & 0 \end{bmatrix} \right)^* d\tau \\ &= E(t; -S_Q), \end{aligned}$$

which is analogous to (2.3) and where the integrand is norm continuous in τ, implies that

$$P - P_Q = -\int_{-\infty}^{\infty} E(-\tau; -S_Q) \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} E(\tau; -S)\, d\tau,$$

is compact. Further,

$$V - V_Q = [P_0 - (I - P_0)]\,(P - P_Q)$$

implies that $V - V_Q$ is compact. As a result, V is a Fredholm operator of index zero.

Now the operator V satisfies the identities

$$\begin{aligned}\operatorname{Ker} V &= [\operatorname{Im} P \cap \operatorname{Ker} P_0] \dotplus [\operatorname{Ker} P \cap \operatorname{Im} P_0],\\ \operatorname{Im} V &= [\operatorname{Im} P + \operatorname{Ker} P_0] \cap [\operatorname{Ker} P + \operatorname{Im} P_0].\end{aligned}$$

So, in order to establish (2.4) and (2.5) it suffices to prove that

$$\operatorname{Im} P \cap \operatorname{Ker} P_0 = \operatorname{Ker} P \cap \operatorname{Im} P_0 = \{0\}.$$

Indeed, the operator function

$$\begin{aligned} W(\lambda) &= I + \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} (\lambda - S_0)^{-1} \begin{bmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{bmatrix} \\ &= I + \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} \begin{bmatrix} 0 & (\lambda + A_0^*)^{-1} \\ (\lambda - A_0)^{-1} & 0 \end{bmatrix} \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} \end{aligned} \tag{2.8}$$

has the identity operator as its real part for imaginary λ and hence

$$\sup_{\operatorname{Re}\lambda = 0} \|I_{\mathcal{H}} - cW(\lambda)\| < 1$$

for some $c > 0$.

Also, W belongs to the Wiener algebra in the sense that there exists a norm measurable operator function $L(\cdot)$ for which W is equal to I plus the Fourier transform of L, and with L having only compact operators as its values such that

$$\int_{-\infty}^{\infty} e^{\varepsilon |t|} \|L(t)\|\, dt < \infty,$$

because of the norm continuity of $L(t)$ for $t \in \mathbb{R} \setminus \{0\}$ and the exponential decay of $\|L(t)\|$ as $t \to \pm\infty$. As a result [9], cW and hence W has left and right canonical factorizations

$$W(\lambda) = W_-^{(l)}(\lambda) W_+^{(l)}(\lambda) = W_+^{(r)}(\lambda) W_-^{(r)}(\lambda), \qquad |\operatorname{Re}\lambda| \le \varepsilon, \tag{2.9}$$

for some $\varepsilon > 0$, where $W_-^{(l)}(\lambda)$, $W_-^{(r)}(\lambda)$ and their inverses are analytic in the half-plane $\operatorname{Re}\lambda < \varepsilon$ and tend to the identity in the norm as $\lambda \to \infty$ in this half-plane and $W_+^{(l)}(\lambda)$, $W_+^{(r)}(\lambda)$ and their inverses are analytic in the half-plane $\operatorname{Re}\lambda > -\varepsilon$ and tend to the identity in the norm as $\lambda \to \infty$ in this half-plane. Using

$$S = S_0 - \begin{bmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{bmatrix} \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}$$

and

$$W(\lambda)^{-1} = I - \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} (\lambda - S)^{-1} \begin{bmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{bmatrix},$$

we obtain

$$W(\lambda)^{-1}\begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}(\lambda - S_0)^{-1} = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}(\lambda - S)^{-1}. \tag{2.10}$$

Letting $x \in \operatorname{Ker} P_0 \cap \operatorname{Im} P$, we substitute the first of the factorizations (2.9) into (2.10), observe that the left- and right-hand sides of the resulting identity

$$W_-^{(l)}(\lambda)^{-1}\begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}(\lambda - S_0)^{-1}x = W_+^{(l)}(\lambda)\begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}(\lambda - S)^{-1}x$$

are analytic in λ for $\operatorname{Re}\lambda < \varepsilon$ and $\operatorname{Re}\lambda > -\varepsilon$, respectively, apply Liouville's theorem, and obtain

$$\begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}(\lambda - S_0)^{-1}x = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}(\lambda - S)^{-1}x = 0.$$

Next, we employ the equality

$$\begin{aligned} (\lambda - S)^{-1}x - (\lambda - S_0)^{-1}x &= -(\lambda - S)^{-1}\begin{bmatrix} 0 & D \\ Q & 0 \end{bmatrix}(\lambda - S_0)^{-1}x \\ = -(\lambda - S)^{-1}&\begin{bmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{bmatrix}\cdot\begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}(\lambda - S_0)^{-1}x = 0 \end{aligned}$$

to enable the application of Liouville's theorem to the analytic continuation of $(\lambda - S)^{-1}x = (\lambda - S_0)^{-1}x$ and conclude that $x = 0$. As a result, $\operatorname{Ker} P_0 \cap \operatorname{Im} P = \{0\}$, as claimed. In a similar way we prove that $\operatorname{Im} P_0 \cap \operatorname{Ker} P = \{0\}$. □

Theorem 2.3 implies that $(\lambda - S_0)^{-1}(\lambda - S)$ has left and right canonical factorizations (as in (2.9)). Letting

$$\Gamma = \begin{bmatrix} 0 & D \\ Q & 0 \end{bmatrix},$$

these factorizations have the following form ([3, Chapter 1])

$$(\lambda - S_0)^{-1}(\lambda - S) = \left[I + (\lambda - S_0)^{-1}(I - \mathcal{P})\Gamma\right]\left[I + \mathcal{P}(\lambda - S_0)^{-1}\Gamma\right],$$

where

$$\begin{aligned} (I + (\lambda - S_0)^{-1}(I - \mathcal{P})\Gamma)^{-1} &= I - (I - \mathcal{P})(\lambda - S)^{-1}\Gamma, \\ (I + \mathcal{P}(\lambda - S_0)^{-1}\Gamma)^{-1} &= I - (\lambda - S)^{-1}\mathcal{P}\Gamma. \end{aligned}$$

Here $\mathcal{P}$ is either the projection of $\mathcal{H}\dot{+}\mathcal{H}$ onto $\operatorname{Im} P$ along $\operatorname{Ker} P_0$ (for the right canonical factorization) or the projection of $\mathcal{H}\dot{+}\mathcal{H}$ onto $\operatorname{Ker} P$ along $\operatorname{Im} P_0$ (for the left canonical factorization).

The following result has been established in [18] for a positive semidefinite triple, without assuming the compactness of D. As a result, in [18] one does not get the compactness of Π_+, only its boundedness.

THEOREM 2.4. *Let $(A_0, Q, D; \mathcal{H})$ be a positive semidefinite and semicompact triple. Then there exist unique positive semidefinite selfadjoint operators $-\Pi_+$ and Π_- on $\mathcal{H}$, where Π_+ is compact and Π_- is bounded, such that*

(1) *the image and kernel of the separating projection P of $E(\cdot;-S)$ are graph subspaces in the sense that*

$$\mathit{Im}\,P = \mathit{Im}\begin{bmatrix} I_{\mathcal{H}} \\ \Pi_- \end{bmatrix}, \qquad \mathit{Ker}\,P = \mathit{Im}\begin{bmatrix} \Pi_+ \\ I_{\mathcal{H}} \end{bmatrix}, \tag{2.11}$$

(2) Π_- *maps* $\mathcal{D}(A_0)$ *into* $\mathcal{D}(A_0^*)$ *and* Π_+ *maps* $\mathcal{D}(A_0^*)$ *into* $\mathcal{D}(A_0)$,

(3) Π_- *is a solution of the operator Riccati equation*

$$\Pi A_0 x + A_0^* \Pi x + Qx - \Pi D \Pi x = 0, \qquad x \in \mathcal{D}(A_0), \tag{2.12}$$

and Π_+ *is a solution of the operator Riccati equation*

$$A_0 \Pi x + \Pi A_0^* x + \Pi Q \Pi x - Dx = 0, \qquad x \in \mathcal{D}(A_0^*), \tag{2.13}$$

(4) *and* $A_0 - D\Pi_-$ *and* $A_0 + \Pi_+ Q$ *are the infinitesimal generators of exponentially decaying* C_0*-semigroups on* $\mathcal{H}$.

Proof. According to Theorem 2.3, there exist bounded projections $\mathcal{P}^{(l)}$ and $\mathcal{P}^{(r)}$ on $\mathcal{H}\dot{+}\mathcal{H}$ such that $\mathcal{P}^{(l)}$ projects $\mathcal{H}\dot{+}\mathcal{H}$ onto $\operatorname{Ker} P$ along $\operatorname{Im} P_0$ and $\mathcal{P}^{(r)}$ projects $\mathcal{H}\dot{+}\mathcal{H}$ onto $\operatorname{Im} P$ along $\operatorname{Ker} P_0$. Hence there exist bounded linear operators Π_- and Π_+ on $\mathcal{H}$, so-called angular operators (cf. [3, Chapter 5]), such that

$$\mathcal{P}^{(l)} = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ \Pi_- & 0 \end{bmatrix}, \qquad \mathcal{P}^{(r)} = \begin{bmatrix} 0 & \Pi_+ \\ 0 & I_{\mathcal{H}} \end{bmatrix}. \tag{2.14}$$

As a result, there exist bounded linear operators Π_- and Π_+ on $\mathcal{H}$ such that (2.11) is true.

One easily proves that

$$\mathcal{P}^{(l)} = V^{-1}(I_{\mathcal{H}} - P_0), \qquad \mathcal{P}^{(r)} = V^{-1} P_0,$$

where V is given by (2.6). Since the projections P_0 and $I - P_0$ commute with $(\lambda - S_0)^{-1}$ and P and $I - P$ commute with $(\lambda - S)^{-1}$ whenever $|\operatorname{Re}\lambda| \le \varepsilon$ for some $\varepsilon > 0$, the invertible operator V maps $\mathcal{D}(S_0) = \mathcal{D}(S) = \mathcal{D}(A_0)\dot{+}\mathcal{D}(A_0^*)$ onto itself. Consequently, $\mathcal{P}^{(l)}$ and $\mathcal{P}^{(r)}$ map this domain into itself and hence Π_- maps $\mathcal{D}(A_0)$ into $\mathcal{D}(A_0^*)$ and Π_+ maps $\mathcal{D}(A_0^*)$ into $\mathcal{D}(A_0)$.

The Riccati equations (2.12) and (2.13) follow from the identities

$$S\begin{bmatrix} I_{\mathcal{H}} \\ \Pi_- \end{bmatrix} x = \begin{bmatrix} I_{\mathcal{H}} \\ \Pi_- \end{bmatrix} (A_0 - D\Pi_-)x, \qquad S\begin{bmatrix} \Pi_+ \\ I_{\mathcal{H}} \end{bmatrix} y = \begin{bmatrix} \Pi_+ \\ I_{\mathcal{H}} \end{bmatrix} (-A_0^* - Q\Pi_+)y, \tag{2.15}$$

where $x \in \mathcal{D}(A_0)$ and $y \in \mathcal{D}(A_0^*)$, in the standard way. Furthermore, since S is exponentially dichotomous with separating projection P and

$$\operatorname{Ker} P = \operatorname{Im}\begin{bmatrix} I_{\mathcal{H}} \\ \Pi_- \end{bmatrix}, \qquad \operatorname{Im} P = \operatorname{Im}\begin{bmatrix} \Pi_+ \\ I_{\mathcal{H}} \end{bmatrix},$$

we immediately have part (4) of Theorem 2.4.

Now remark that Π_- and Π_+ are selfadjoint (because of the $\mathcal{J}_2$-neutrality of $\operatorname{Im} P$ and $\operatorname{Ker} P$), while $-\Pi_+$ and Π_- are positive semidefinite (because of the $\mathcal{J}_1$-nonpositivity of $\operatorname{Im} P$ and the $\mathcal{J}_1$-nonnegativity of $\operatorname{Ker} P$).

Finally, from the compactness of $V - V_Q$ and hence from the compactness of

$$V^{-1} - V_Q^{-1} = \begin{bmatrix} I_{\mathcal{H}} & \Pi_+ \\ \Pi_- & I_{\mathcal{H}} \end{bmatrix} - \begin{bmatrix} I_{\mathcal{H}} & 0 \\ X & I_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} 0 & \Pi_+ \\ \Pi_- - X & 0 \end{bmatrix},$$

where V and V_Q are given by (2.6) and (2.7), it follows directly that Π_+ and $\Pi_- - X$ are compact operators. $\square$

In [16] a closely related existence result was obtained under the assumption that the spectrum of the block matrix operator S only consists of algebraically and geometrically simple eigenvalues and does not have finite accumulation points.

3. Approximation. Letting $\mathcal{H}_n$ be a sequence of closed linear subspaces of $\mathcal{H}$, there exist unique operators $\pi_n : \mathcal{H} \to \mathcal{H}_n$ and $\imath_n : \mathcal{H}_n \to \mathcal{H}$ such that $\imath_n \pi_n$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$ and $\pi_n \imath_n$ is the identity operator on $\mathcal{H}_n$. We assume that $\imath_n \pi_n$ tends to $I_{\mathcal{H}}$ in the strong sense.

Starting from a given triple $\theta = (A_0, Q, D; \mathcal{H})$, we define $Q_n = \pi_n Q \imath_n$, $D_n = \pi_n D \imath_n$, which are selfadjoint on $\mathcal{H}_n$ and positive semidefinite whenever Q and D are positive semidefinite. Let A_{0n} be a generator of a strongly continuous semigroup on $\mathcal{H}_n$ of negative exponential type. Then a sequence of triples $\theta_n = (A_{0n}, Q_n, D_n; \mathcal{H}_n)$ is called an *approximant* to the triple θ if the following condition holds: for some $\varepsilon > 0$ we have the approximation

$$\lim_{n\to\infty} e^{\varepsilon|t|} \, \|\hat{\imath}_n E(t; -S_{0n}) \hat{\pi}_n x - E(t; -S_0)x\|_{\mathcal{H}} = 0 \tag{3.1}$$

for every $x \in \mathcal{H}$, uniformly in $t \in \mathbb{R} \setminus \{0\}$. Here $\hat{\pi}_n = \pi_n \dot{+} \pi_n$, $\hat{\imath}_n = \imath_n \dot{+} \imath_n$ and $S_{0n} = A_{0n} \dot{+} (-A_{0n}^*)$ on $\mathcal{H}_n \dot{+} \mathcal{H}_n$. The sequence of triples θ_n is called a *finite dimensional approximant* to θ if it is an approximant to θ and the spaces $\mathcal{H}_n = \pi_n[\mathcal{H}]$ are finite dimensional.

We remark that it is easily seen that $\imath_n Q_n \pi_n$ converges to Q strongly, while $\imath_n D_n \pi_n$ converges to D in norm because of the compactness of D.

THEOREM 3.1. *Let $\theta_n = (A_{0n}, Q_n, D_n; \mathcal{H}_n)$ be a sequence of triples approximant to the positive semidefinite semicompact triple $\theta = (A_0, Q, D; \mathcal{H})$. Put*

$$S_n = \begin{bmatrix} A_{0n} & -D_n \\ -Q_n & -A_{0n}^* \end{bmatrix}.$$

Then

$$\lim_{n\to\infty} \|\hat{\imath}_n E(t; -S_n) \hat{\pi}_n x - E(t; -S)x\|_{\mathcal{H}} = 0 \tag{3.2}$$

for every $x \in \mathcal{H} \dot{+} \mathcal{H}$, uniformly in $t \in \mathbb{R} \setminus \{0\}$.

Proof. Consider the sequence of triples $\theta_n^Q = (A_{0n}, Q_n, 0; \mathcal{H}_n)$ approximant to the positive semidefinite triple $\theta = (A_0, Q, 0; \mathcal{H})$. Put

$$S_n^Q = \begin{bmatrix} A_{0n} & 0 \\ -Q_n & -A_{0n}^* \end{bmatrix}.$$

In analogy with (2.3) we obtain

$$E(t; -S_n^Q)x = E(t; -S_{0n})x + \int_{-\infty}^{\infty} E(t-\tau; -S_{0n}) \Gamma_Q E(\tau; -S_n^Q)x \, d\tau.$$

Because of (3.1), we see that $\|E(t;-S_{0n})\|$ has a finite upper bound which is independent of $t \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. Using dominated convergence, we take the limit under the integral sign and find that for some $\varepsilon > 0$

$$\lim_{n\to\infty} e^{\varepsilon|t|} \left\|\hat{\imath}_n E(t;-S_n^Q)\hat{\pi}_n x - E(t;-S^Q)x\right\|_{\mathcal{H}} = 0 \tag{3.3}$$

for every $x \in \mathcal{H}$, uniformly in $t \in \mathbb{R} \setminus \{0\}$.

Next, in analogy with (2.3) we have

$$E(t;-S_n) - \int_{-\infty}^{\infty} E(t-\tau;-S_n^Q)\Gamma_n^D E(\tau;-S_n)\,d\tau = E(t;-S_n^Q),$$

where $\Gamma_n^D = \begin{pmatrix} 0 & D_n \\ 0 & 0 \end{pmatrix}$. This integral equation implies that

$$\begin{aligned} \hat{\imath}_n E(t;-S_n)\hat{\pi}_n x - \int_{-\infty}^{\infty} \hat{\imath}_n E(t-\tau;-S_n^Q)\Gamma_n^D E(\tau;-S_n)\hat{\pi}_n x\,d\tau \\ = \hat{\imath}_n E(t;-S_n^Q)\hat{\pi}_n x, \end{aligned} \tag{3.4}$$

where $x \in \mathcal{H}\dot{+}\mathcal{H}$. Note that $\Gamma_n^D = \hat{\pi}_n \Gamma_D \hat{\imath}_n$, so that (3.4) can be written in the form

$$\begin{aligned} \hat{\imath}_n E(t;-S_n)\hat{\pi}_n x - \int_{-\infty}^{\infty} \hat{\imath}_n E(t-\tau;-S_n^Q)\hat{\pi}_n \Gamma_D \cdot \hat{\imath}_n E(\tau;-S_n)\hat{\pi}_n x\,d\tau \\ = \hat{\imath}_n E(t;-S_n^Q)\hat{\pi}_n x, \end{aligned}$$

where $x \in \mathcal{H}\dot{+}\mathcal{H}$. Equation (3.3) and the compactness of Γ_D imply that for some $\varepsilon > 0$

$$\lim_{n\to\infty} e^{\varepsilon|t|} \left\|\hat{\imath}_n E(t;-S_n^Q)\hat{\pi}_n \Gamma_D - E(t;-S^Q)\Gamma_D\right\|_{\mathcal{H}} = 0,$$

uniformly in $t \in \mathbb{R} \setminus \{0\}$. Because of the unique solvability of (3.4) on the complex Banach space of bounded continuous $\mathcal{H}_n$-valued functions on the real line with a possible jump discontinuity in $t = 0$, in combination with (3.3), we obtain (3.2) as claimed. □

Let

$$X_n = \int_0^{\infty} e^{\tau A_{0n}^*} Q_n e^{\tau A_{0n}}\,d\tau$$

be the unique solution of the Lyapunov equation

$$A_{0n}^* X_n + X_n A_{0n} = -Q_n.$$

Using dominated convergence one easily proves that, under the hypotheses of Theorem 3.1,

$$\lim_{n\to\infty} \|\imath_n X_n \pi_n x - Xx\| = 0, \qquad x \in \mathcal{H}. \tag{3.5}$$

Similarly, the unique solution

$$Y_n = \int_0^{\infty} e^{\tau A_{0n}} D_n e^{\tau A_{0n}^*}\,d\tau$$

of the Lyapunov equation

$$A_{0n}Y_n + Y_n A_{0n}^* = -D_n$$

has the property that

$$\lim_{n\to\infty} \|\imath_n Y_n \pi_n x - Yx\| = 0, \qquad x \in \mathcal{H},$$

where

$$Y = \int_0^\infty e^{\tau A_0} D\, e^{\tau A_0^*}\, d\tau$$

is the unique solution of the Lyapunov equation

$$A_0 Y + Y A_0^* = -D.$$

Let us now prove the strong stability of Π_- and the stability of Π_+ in the norm if a positive semidefinite and semicompact triple is approximated by a sequence of triples in the sense of the above definition. The obvious way to do so is to study the operator Wiener–Hopf equation

$$u(t;x) - \int_0^\infty E(t-\tau;-S_0)\Gamma u(\tau;x)\, d\tau = E(t;-S_0)x, \tag{3.6}$$

where $x \in \operatorname{Ker} P_0$ and $t > 0$, or the operator Wiener–Hopf equation

$$v(t;x) - \int_{-\infty}^0 E(t-\tau;-S_0)\Gamma v(\tau;x)\, d\tau = E(t;-S_0)x, \tag{3.7}$$

where $x \in \operatorname{Im} P$ and $t < 0$. Unfortunately, their integral kernel $E(\cdot;-S_0)\Gamma$ is, in general, not Bochner integrable. If it were, one would trivially obtain the solutions of (3.6) and (3.7) as follows:

$$u(t;x) = E(t;-S)\mathcal{P}^{(l)}x, \qquad v(t;x) = E(t;-S)\mathcal{P}^{(r)}x.$$

Let us therefore introduce the modified operator convolution kernel

$$\begin{aligned} K(t;-S_0) &= \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} E(t;-S_0) \begin{bmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{bmatrix} \\ &= \begin{cases} \begin{bmatrix} 0 & -Q^{1/2}e^{-tA_0}D^{1/2} \\ 0 & 0 \end{bmatrix}, & t < 0, \\[2ex] \begin{bmatrix} 0 & 0 \\ D^{1/2}e^{tA_0^*}Q^{1/2} & 0 \end{bmatrix}, & t > 0. \end{cases} \end{aligned}$$

Note that $K(t;-S_0)$ is compact and norm continuous in $t \neq 0$. This integral kernel satisfies

$$\begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} E(t;-S_0)\Gamma = K(t;-S_0) \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix}$$

and leads to operator Wiener–Hopf equations with Bochner integrable convolution kernel and symbol $W(\lambda)$ defined by (2.8). Indeed, these equations are given by

$$w(t;x) - \int_0^\infty K(t-\tau;-S_0)w(\tau;x)\,d\tau = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} E(t;-S_0)x, \tag{3.8}$$

where $x \in \operatorname{Ker} P_0$ and $t > 0$, and by

$$z(t;x) - \int_{-\infty}^0 K(t-\tau;-S_0)z(\tau;x)\,d\tau = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} E(t;-S_0)x, \tag{3.9}$$

where $x \in \operatorname{Im} P$ and $t < 0$. Equations (3.8) and (3.9) are uniquely solvable, because their symbol $W(\lambda)$ has left and right canonical Wiener–Hopf factorizations. Once (3.8) and (3.9) have been solved, we have

$$u(t;x) = E(t;-S_0)x + \int_0^\infty E(t-\tau;-S_0) \begin{bmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{bmatrix} w(\tau;x)d\tau \tag{3.10}$$

for $x \in \operatorname{Ker} P_0$ and $t > 0$, and

$$v(t;x) = E(t;-S_0)x + \int_{-\infty}^0 E(t-\tau;-S_0) \begin{bmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{bmatrix} z(\tau;x)d\tau \tag{3.11}$$

for $x \in \operatorname{Im} P_0$ and $t < 0$. We then finally obtain

$$\mathcal{P}^{(l)}x = u(0^+;x), \qquad \mathcal{P}^{(r)}x = -v(0^-;x),$$

and hence [cf. (2.14)]

$$\Pi_- x = \begin{bmatrix} 0 & I_{\mathcal{H}} \end{bmatrix} u(0^+;x), \quad \Pi_+ x = -\begin{bmatrix} I_{\mathcal{H}} & 0 \end{bmatrix} v(0^-;x). \tag{3.12}$$

THEOREM 3.2. *Let $\theta_n = (A_{0n}, Q_n, D_n; \mathcal{H}_n)$ be a sequence of triples approximant to the positive semidefinite semicompact triple $\theta = (A_0, Q, D; \mathcal{H})$. Then*

$$\lim_{n\to\infty} \|\imath_n \Pi_{-,n} \pi_n x - \Pi_- x\| = 0, \tag{3.13}$$

and

$$\lim_{n\to\infty} \|\imath_n \Pi_{+,n} \pi_n x - \Pi_+ x\| = 0 \tag{3.14}$$

for every $x \in \mathcal{H}$.

Proof. From (3.1), the strong convergence $\imath_n Q_n^{1/2} \pi_n \to Q^{1/2}$ and the compactness of $D^{1/2}$, we obtain for some $\varepsilon > 0$

$$\lim_{n\to\infty} e^{\varepsilon|t|} \|\hat{\imath}_n K(t;-S_{0n})\hat{\pi}_n - K(t)\| = 0,$$

uniformly in $t \in \mathbb{R} \setminus \{0\}$, and hence for some $\varepsilon > 0$

$$\lim_{n\to\infty} \int_{-\infty}^\infty e^{\varepsilon|t|} \|\hat{\imath}_n K(t;-S_{0n})\hat{\pi}_n - K(t)\|\,dt = 0.$$

Thus, using (3.1) and the unique solvability of (3.8), we get for some $\varepsilon > 0$

$$\lim_{n\to\infty} e^{\varepsilon|t|} \|\hat{\imath}_n w_n(t;\hat{\pi}_n x) - w(t;x)\| = 0$$

for every $x \in \mathcal{H}\dot{+}\mathcal{H}$, uniformly in $t \in \mathbb{R}^+$. Similarly, for some $\varepsilon > 0$ we have

$$\lim_{n\to\infty} e^{\varepsilon|t|}\,\|\hat{\imath}_n z_n(t;\hat{\pi}_n x) - z(t;x)\| = 0$$

for every $x \in \mathcal{H}\dot{+}\mathcal{H}$, uniformly in $t \in \mathbb{R}^-$. With the help of (3.1), (3.10), and (3.11), we find for some $\varepsilon > 0$

$$\lim_{n\to\infty} e^{\varepsilon|t|}\,\|\hat{\imath}_n u_n(t;\hat{\pi}_n x) - u(t;x)\| = 0$$

for every $x \in \mathcal{H}\dot{+}\mathcal{H}$, uniformly in $t \in \mathbb{R}^+$, as well as

$$\lim_{n\to\infty} e^{\varepsilon|t|}\,\|\hat{\imath}_n v_n(t;\hat{\pi}_n x) - v(t;x)\| = 0$$

for every $x \in \mathcal{H}\dot{+}\mathcal{H}$, uniformly in $t \in \mathbb{R}^-$. Using (3.12) we then easily obtain (3.13) and (3.14). $\square$

The following result strengthens the convergence properties stated in Theorem 3.2.

THEOREM 3.3. *Let $\theta_n = (A_{0n}, Q_n, D_n; \mathcal{H}_n)$ be a sequence of triples approximant to the positive semidefinite semicompact triple $\theta = (A_0, Q, D; \mathcal{H})$. Then*

$$\lim_{n\to\infty} \|\imath_n(\Pi_{-,n} - X_n)\pi_n - (\Pi_- - X)\| = 0, \tag{3.15}$$

$$\lim_{n\to\infty} \|\imath_n \Pi_{+,n}\pi_n - \Pi_+\| = 0. \tag{3.16}$$

Proof. From (3.8) and $E(t;-S_0) = 0\dot{+}e^{tA_0^*}$ for $t > 0$ it is clear that for every $t > 0$ the right-hand side of (3.8) can be viewed as the result of applying a compact operator to a vector $x \in \mathcal{H}$. Since (3.1) and the compactness of $D^{1/2}$ imply that for some $\varepsilon > 0$

$$\lim_{n\to\infty} \left\|\imath_n D_n^{1/2} e^{tA_{0n}^*}(I - P_{0n})\pi_n - D^{1/2}e^{tA_0^*}(I - P_0)\right\| = 0, \qquad t > 0,$$

we have

$$\lim_{n\to\infty} \left\|\hat{\imath}_n \begin{bmatrix} Q_n^{1/2} & 0 \\ 0 & D_n^{1/2} \end{bmatrix} E(t;-S_{0n})\hat{\pi}_n - \begin{bmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} E(t;-S_0)\right\| = 0,$$

and this allows one to sharpen the derivation of (3.14) and to obtain (3.16) instead.

To prove (3.15), we replace (3.7), (3.9), and (3.11) by

$$v_Q(t;x) - \int_{-\infty}^{0} E(t-\tau;-S_Q)\Gamma_D v_Q(\tau;x)\,d\tau = E(t;-S_Q)x, \tag{3.17}$$

$$z_Q(t;x) - \int_{-\infty}^{0} K(t-\tau;-S_Q) z_Q(\tau;x)\,d\tau = \begin{bmatrix} 0 & 0 \\ 0 & D^{1/2} \end{bmatrix} E(t;-S_Q)x, \tag{3.18}$$

$$v_Q(t;x) = E(t;-S_Q)x + \int_{-\infty}^{0} E(t-\tau;-S_Q) \begin{bmatrix} 0 & D^{1/2} \\ 0 & 0 \end{bmatrix} z_Q(\tau;x)d\tau\,, \tag{3.19}$$

respectively, where $x \in \mathcal{H}$, $\Gamma_D = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$, and the convolution kernel $K(t;-S_Q)$ satisfies

$$K(t;-S_Q)\begin{bmatrix} 0 & 0 \\ 0 & D^{1/2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D^{1/2} \end{bmatrix} E(t;-S_Q).$$

Repeating the proof of (3.16) with the help of (3.17), (3.18), and (3.19) we obtain (3.15). $\square$

It remains to consider the approximation of the C_0-semigroups generated by $A_0 - D\Pi_-$ and $A_0^* + Q\Pi_+$. Indeed, from (2.15) we easily derive the identity

$$S\begin{bmatrix} I_{\mathcal{H}} & \Pi_+ \\ \Pi_- & I_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} A_0 & -D \\ -Q & -A_0^* \end{bmatrix}\begin{bmatrix} I_{\mathcal{H}} & \Pi_+ \\ \Pi_- & I_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} & \Pi_+ \\ \Pi_- & I_{\mathcal{H}} \end{bmatrix}\begin{bmatrix} A_0 - D\Pi_- & 0 \\ 0 & -A_0^* - Q\Pi_+ \end{bmatrix},$$

where $A_0 - D\Pi_-$ and $A_0^* + Q\Pi_+$ both generate exponentially decaying C_0-semigroups on $\mathcal{H}$. Writing down the analogous identity for resolvent operators and applying the inverse Laplace transform, we get

$$\begin{bmatrix} I_{\mathcal{H}} & \Pi_+ \\ \Pi_- & I_{\mathcal{H}} \end{bmatrix}^{-1} E(t;-S)\begin{bmatrix} I_{\mathcal{H}} & \Pi_+ \\ \Pi_- & I_{\mathcal{H}} \end{bmatrix} = \begin{cases} e^{t(A_0-D\Pi_-)}\dot{+}0_{\mathcal{H}}, & t>0, \\ 0_{\mathcal{H}}\dot{+}\left(-e^{-t(A_0^*+Q\Pi_+)}\right), & t<0, \end{cases} \tag{3.20}$$

where $0_{\mathcal{H}}$ denotes the zero operator on $\mathcal{H}$.

THEOREM 3.4. *Let $\theta_n = (A_{0n}, Q_n, D_n; \mathcal{H}_n)$ be a sequence of triples approximant to the positive semidefinite semicompact triple $\theta = (A_0, Q, D; \mathcal{H})$. Then for $t > 0$ we have*

$$\lim_{n\to\infty} \left\| \imath_n e^{t(A_{0n}-D_n\Pi_{-,n})}\pi_n - e^{t(A_0-D\Pi_-)} \right\| = 0, \tag{3.21}$$

$$\lim_{n\to\infty} \left\| \imath_n e^{t(A_{0n}^*+Q_n\Pi_{+,n})}\pi_n - e^{t(A_0^*-Q\Pi_+)} \right\| = 0, \tag{3.22}$$

uniformly in t on compact intervals of either $[0,\infty)$ or $(-\infty,0]$.

Proof. Because of (3.2) and (3.20), it suffices to prove that for each $x \in \mathcal{H}$ we have

$$\lim_{n\to\infty} \|[\hat{\imath}_n M_n \hat{\pi}_n - M]\, x\| = 0, \tag{3.23}$$

$$\lim_{n\to\infty} \left\| \left[\hat{\imath}_n M_n^{-1} \hat{\pi}_n - M^{-1}\right] x \right\| = 0, \tag{3.24}$$

where

$$M = \begin{bmatrix} I_{\mathcal{H}} & \Pi_+ \\ \Pi_- & I_{\mathcal{H}} \end{bmatrix}, \qquad M_n = \begin{bmatrix} I_{\mathcal{H}_n} & \Pi_{+,n} \\ \Pi_{-,n} & I_{\mathcal{H}_n} \end{bmatrix}.$$

Indeed, (3.13), (3.16), and the compactness of the operator Π_+ imply that

$$\lim_{n\to\infty} \|\imath_n(I_{\mathcal{H}_n} - \Pi_{-,n}\Pi_{+,n})\pi_n - (I_{\mathcal{H}} - \Pi_-\Pi_+)\| = 0.$$

Now note that $I_{\mathcal{H}} - \Pi_-\Pi_+$ is invertible, as a result of the existence of the projection P [cf. (2.11)]. Thus

$$\lim_{n\to\infty} \left\|\imath_n(I_{\mathcal{H}_n} - \Pi_{-,n}\Pi_{+,n})^{-1}\pi_n - (I_{\mathcal{H}} - \Pi_-\Pi_+)^{-1}\right\| = 0,$$

and by taking the adjoint

$$\lim_{n\to\infty} \left\|\imath_n(I_{\mathcal{H}_n} - \Pi_{+,n}\Pi_{-,n})^{-1}\pi_n - (I_{\mathcal{H}} - \Pi_+\Pi_-)^{-1}\right\| = 0.$$

We now easily show that

$$M^{-1} = \begin{bmatrix} (I_{\mathcal{H}} - \Pi_+\Pi_-)^{-1} & -(I_{\mathcal{H}} - \Pi_+\Pi_-)^{-1}\Pi_+ \\ -(I_{\mathcal{H}} - \Pi_-\Pi_+)^{-1}\Pi_- & (I_{\mathcal{H}} - \Pi_-\Pi_+)^{-1} \end{bmatrix}. \tag{3.25}$$

Hence from (3.25) and the analogous expression for M_n^{-1}, we now easily derive (3.23) and (3.24). $\square$

4. Conclusions and remarks. In this paper, exponentially dichotomous block matrix operators on $\mathcal{H}\dot{+}\mathcal{H}$ have been studied as additive perturbations of exponentially dichotomous operators of the type $A_0\dot{+}(-A_0^*)$. This allows a considerable range of LQ-optimal control theory applications; for instance, the example in [16] concerning the heat equation could be dealt with in this way. (We did not do so explicitly, because we expect no better results than the ones one can expect for a Hamiltonian that is a Riesz spectral operator, and taking as approximations for $\mathcal{H}_n$ the spaces spanned by the first n vectors in a properly constructed Riesz basis of eigenvectors of the Hamiltonian. The results would be no better than the ones already existing in the literature.)

Also we considered (possibly finite dimensional) approximations. In connection with the latter topic there are still many open questions. Questions that are natural from a numerical analysis point of view come to mind; for instance, how is the speed of convergence in the results described in section 3 tied to the speed of convergence in (3.1) and to the speed of convergence of $\imath_n D\pi_n$ to D? What about Lipschitz estimates and relative error bounds? All these points are open problems, although an analysis of our proofs may provide some answers.

Finally, delay systems defy application of the existing results. In order to be able to deal with applications to delay systems we would need criteria for exponential dichotomy, where the linear operator is not a perturbation of a naturally given exponentially dichotomous operator, and where a Banach space setting is adopted.

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