ESERCITAZIONE DEL 07/01/2013

① Calcolare il numero di condizionamento in norma 1, 2, ∞ della matrice

$$V = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{\beta} \end{bmatrix}$$

in funzione del parametro reale positivo β.
La matrice data è ortogonale?
Qual è il suo raggio spettrale?

② Calcolare la fattorizzazione $PA = LU$ della matrice

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

e utilizzarla per calcolare il determinante di A e per trovare la soluzione del sistema $Ax = b$ con $b = (7, 7, 3, -5)^T$.

③ Eseguire i primi due passi nella risoluzione con il metodo di Eulero - Cauchy del seguente sistema differenziale

$$\begin{cases} y_1' = 5y_1 + y_1 y_2 \\ y_2' = y_2 \cos(x) \\ y_1(0) = \pi, \; y_2(0) = -\pi \end{cases}$$

utilizzando il passo $h = \frac{\pi}{4}$.

④ Studiare la convergenza del seguente metodo alle differenze finite

$$\eta_{i+2} = \frac{4}{3}\eta_{i+1} - \frac{1}{3}\eta_i + \frac{2}{3} h f(x_{i+2}, \eta_{i+2}).$$

In particolare, dire se il metodo è consistente e stabile, e determinare il suo ordine.

$$\tau(x,h) = \frac{1}{h}\sum_{j=0}^{r} a_j\, y(x+jh) - \sum_{j=0}^{r} b_j\, y'(x+jh)$$

SOLUZIONE

(1)

$$V^T V = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & \sqrt{\beta} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{\beta} \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

La matrice non è ortogonale perché $V^T V \neq I$.

$$\begin{array}{cc} V = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{\beta} \end{bmatrix} & \begin{array}{l} \to 3 \\ \to 2 \\ \to \sqrt{\beta} \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 2 & 3 & \sqrt{\beta} \end{array} & \end{array}$$

$$\|V\|_\infty = \max\{3, 2, \sqrt{\beta}\} = \begin{cases} 3 \text{ se } \sqrt{\beta} < 3 \to \text{se } \beta < 9 \\ \sqrt{\beta} \text{ se } \sqrt{\beta} > 3 \to \text{se } \beta > 9 \end{cases}$$

$$\|V\|_1 = \max\{2, 3, \sqrt{\beta}\} = \begin{cases} 3 \text{ se } \sqrt{\beta} < 3 \to \text{se } \beta < 9 \\ \sqrt{\beta} \text{ se } \sqrt{\beta} > 3 \to \text{se } \beta > 9 \end{cases}$$

Per calcolare le norme $\|V^{-1}\|_\infty$ e $\|V^{-1}\|_1$ calcoliamo V^{-1}: risolviamo tre sistemi lineari

$$Vx = e_i \qquad i = 1, 2, 3$$

ATTENZIONE: possiamo fare direttamente questo passaggio senza applicare l'eliminazione di Gauss perché V è già triangolare superiore!

$$\begin{cases} 2x_1 + x_2 = 1 \rightarrow x_1 = 1/2 \\ 2x_2 = 0 \rightarrow x_2 = 0 \\ \sqrt{\beta} x_3 = 0 \rightarrow x_3 = 0 \end{cases}$$

$$\begin{cases} 2x_1 + x_2 = 0 \rightarrow 2x_1 = -\frac{1}{2} \rightarrow x_1 = -\frac{1}{4} \\ 2x_2 = 1 \rightarrow x_2 = 1/2 \\ \sqrt{\beta} x_3 = 0 \rightarrow x_3 = 0 \end{cases}$$

$$\begin{cases} 2x_1 + x_2 = 0 \rightarrow x_1 = 0 \\ 2x_2 = 0 \rightarrow x_2 = 0 \\ \sqrt{\beta} x_3 = 1 \rightarrow x_3 = \frac{1}{\sqrt{\beta}} \end{cases}$$

Quindi:

$$V^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta}} \end{bmatrix} \begin{array}{l} \rightarrow \frac{3}{4} \\ \rightarrow \frac{1}{2} \\ \rightarrow \frac{1}{\sqrt{\beta}} \end{array}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{\sqrt{\beta}} \end{array}$$

$$\|V^{-1}\|_\infty = \max\left\{\frac{3}{4}, \frac{1}{2}, \frac{1}{\sqrt{\beta}}\right\} = \begin{cases} \frac{3}{4} \text{ se } \frac{1}{\sqrt{\beta}} < \frac{3}{4} \rightarrow \text{se } \beta > \frac{16}{9} \\ \frac{1}{\sqrt{\beta}} \text{ se } \frac{1}{\sqrt{\beta}} > \frac{3}{4} \rightarrow \text{se } \beta < \frac{16}{9} \end{cases}$$

$$\|V^{-1}\|_1 = \max\left\{\frac{1}{2}, \frac{3}{4}, \frac{1}{\sqrt{\beta}}\right\} = \begin{cases} 3/4 \text{ se } \frac{1}{\sqrt{\beta}} < \frac{3}{4} \rightarrow \text{se } \beta > \frac{16}{9} \\ \frac{1}{\sqrt{\beta}} \text{ se } \frac{1}{\sqrt{\beta}} > \frac{3}{4} \rightarrow \text{se } \beta < \frac{16}{9} \end{cases}$$

Per calcolare il n° di condizionamento in norma 2 abbiamo bisogno di $V^T V$ (che abbiamo già calcolato)

$$\det(V^TV-\lambda I)=\det\begin{bmatrix}4-\lambda & 2 & 0\\ 2 & 5-\lambda & 0\\ 0 & 0 & \beta-\lambda\end{bmatrix}=$$

$$=(4-\lambda)(5-\lambda)(\beta-\lambda)-2(2\cdot(\beta-\lambda))=$$

$$=(\beta-\lambda)\left[(5-\lambda)(4-\lambda)-4\right]=(\beta-\lambda)(\lambda^2-9\lambda+16)$$

$$\lambda_1=\beta \qquad \lambda_{2/3}=\frac{9\pm\sqrt{81-64}}{2}=\frac{9\pm\sqrt{17}}{2}$$

$$\rho(V^TV)=\begin{cases}\beta & \text{se } \beta>\frac{9+\sqrt{17}}{2}\\ \frac{9+\sqrt{17}}{2} & \text{se } \beta<\frac{9+\sqrt{17}}{2}\end{cases}$$

$$\|V\|_2=\begin{cases}\sqrt{\beta} & \text{se } \beta>\frac{9+\sqrt{17}}{2}\\ \sqrt{\frac{9+\sqrt{17}}{2}} & \text{se } \beta<\frac{9+\sqrt{17}}{2}\end{cases}$$

$$K_2(V)=\begin{cases}\frac{9+\sqrt{17}}{2\beta} & \text{se } \beta<\frac{9-\sqrt{17}}{2}\\ \frac{9+\sqrt{17}}{9-\sqrt{17}} & \text{se } \frac{9-\sqrt{17}}{2}<\beta<\frac{9+\sqrt{17}}{2}\\ \frac{2\beta}{9-\sqrt{17}} & \text{se } \beta>\frac{9+\sqrt{17}}{2}\end{cases}$$

$$K_\infty(V)=\begin{cases}\frac{3}{\sqrt{\beta}} & \text{se } \beta<\frac{16}{9}\\ \frac{9}{4} & \text{se } \frac{16}{9}<\beta<9\\ \frac{3\sqrt{\beta}}{4} & \text{se } \beta>\frac{16}{9}\end{cases}$$

$K_1(V)=K_\infty(V)$ (le condizioni sono le stesse)

$\sigma(V) = \{2, 2, \sqrt{\beta}\}$ (V è triangolare superiore quindi gli autovalori sono gli elementi sulla diagonale)

$\downarrow$

$$\rho(V) = \begin{cases} 2 \text{ se } \sqrt{\beta} < 2 \to \text{se } \beta < 4 \\ \sqrt{\beta} \text{ se } \sqrt{\beta} > 4 \to \text{se } \beta > 4 \end{cases}$$

(2)

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$m_{21} = -1$
$m_{31} = 0$
$m_{41} = 0$

$$\to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$m_{32} = -\frac{1}{2}$
$m_{42} = 0$

scambio riga 3 e riga 4 perché $3 > 2$

$$\to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5/3 \end{bmatrix}$$

$m_{43} = -\frac{2}{3}$

$\parallel$

U

$$\det(A) = \frac{\prod_{i=1}^{4} u_{ii}}{(-1)^{\#\text{scambi}}} = \frac{1 \cdot 4(-3)(5/3)}{(-1)^1} = 20$$

Per risolvere il sistema dobbiamo risolvere due sistemi "a cascata":

$$\begin{cases} Ly = Pb \\ Ux = y \end{cases}$$

$$L = \cancel{\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1/2 \\ 0 & 0 \end{bmatrix}} \rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1/2 & -\frac{2}{3} & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 7 \\ 7 \\ 3 \\ -5 \end{bmatrix} \rightarrow Pb = \begin{bmatrix} 7 \\ 7 \\ -5 \\ 3 \end{bmatrix}$$

$$Ly = Pb \rightarrow \begin{cases} y_1 = 7 & \rightarrow y_1 = 7 \\ -y_1 + y_2 = 7 & \rightarrow y_2 = 14 \\ y_3 = -5 & \rightarrow y_3 = -5 \\ -\frac{1}{2}y_2 - \frac{2}{3}y_3 + y_4 = 3 & \rightarrow y_4 = 10 - \frac{10}{3} = \frac{20}{3} \end{cases}$$

$$Ux = y \rightarrow \begin{cases} x_1 + 3x_2 = 7 & \rightarrow x_1 = 1 \\ 4x_2 + 2x_3 = 14 & \rightarrow x_2 = 2 \\ -3x_3 + x_4 = -5 & \rightarrow x_3 = 3 \\ \frac{5}{3}x_4 = \frac{20}{3} & \rightarrow x_4 = 4 \end{cases}$$

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

③ Si ha $f_1(x, y_1, y_2) = 5y_1 + y_1 y_2$

$f_2(x, y_1, y_2) = y_2 \cos(x)$

Passo 0:

$$\begin{cases} x_0 = 0 \\ \eta_1^{(0)} = \pi \\ \eta_2^{(0)} = -\pi \end{cases}$$

$f_1(0, \pi, -\pi) = 5\pi - \pi^2$

$f_2(0, \pi, -\pi) = -\pi \cdot \cos(0) = -\pi$

Passo 1:

$$x_1 = x_0 + h = 0 + \frac{\pi}{4} = \frac{\pi}{4}$$

$$\eta_1^{(1)} = \eta_1^{(0)} + h f_1(x_0, \eta_1^{(0)}, \eta_2^{(0)}) = \pi + \frac{\pi}{4} f_1(0, \pi, -\pi) =$$

$$= \pi + \frac{\pi}{4}(5\pi - \pi^2) = \pi\left(1 + \frac{5}{4}\pi - \frac{1}{4}\pi^2\right)$$

$$\eta_2^{(1)} = \eta_2^{(0)} + h f_2(x_0, \eta_1^{(0)}, \eta_2^{(0)}) = -\pi + \frac{\pi}{4} f_2(0, \pi, -\pi) =$$

$$= -\pi + \frac{\pi}{4}(-\pi) = -\pi\left(1 + \frac{\pi}{4}\right)$$

Passo 2:

$$x_2 = x_1 + h = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\eta_1^{(2)} = \eta_1^{(1)} + h f_1(x_1, \eta_1^{(1)}, \eta_2^{(1)}) = \pi\left(1 + \frac{5}{4}\pi - \frac{1}{4}\pi^2\right) +$$

$$+ \frac{\pi}{4} f_1\left(\frac{\pi}{4}, \pi\left(1 + \frac{5}{4}\pi - \frac{1}{4}\pi^2\right), -\pi\left(1 + \frac{\pi}{4}\right)\right) =$$

$$= \pi\left(1 + \frac{5}{4}\pi - \frac{1}{4}\pi^2\right) + \frac{\pi}{4}\left[5\pi\left(1 + \frac{5}{4}\pi - \frac{1}{4}\pi^2\right) - \pi^2 \cdot\right.$$

$$\left. \cdot \left(1 + \frac{5}{4}\pi - \frac{1}{4}\pi^2\right)\left(1 + \frac{\pi}{4}\right)\right] = \left(1 + \frac{5}{4}\pi - \pi^2\right)\left(\pi + \frac{5}{4}\pi^2 - \frac{\pi^3}{4}\left(1 + \frac{\pi}{4}\right)\right)$$

$$\eta_2^{(2)} = \eta_2^{(1)} + h f_2(x_1, \eta_1^{(1)}, \eta_2^{(1)}) =$$

$$= -\pi\left(1+\frac{\pi}{4}\right) + \frac{\pi}{4}\left(-\pi\left(1+\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\right) =$$

$$= \left(-\pi\left(1+\frac{\pi}{4}\right)\right)\left(1+\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}\right)$$

4) $\eta_{i+2} = \frac{4}{3}\eta_{i+1} - \frac{1}{3}\eta_i + \frac{2}{3} h f(x_{i+2}, \eta_{i+2})$

$$\eta_{i+2} - \frac{4}{3}\eta_{i+1} + \frac{1}{3}\eta_i = \frac{2}{3} h f(x_{i+2}, \eta_{i+2})$$

$$\boxed{\begin{array}{l} a_0 = \frac{1}{3} \\ a_1 = -\frac{4}{3} \\ a_2 = 1 \end{array}} \qquad \boxed{\begin{array}{l} b_0 = 0 \\ b_1 = 0 \\ b_2 = \frac{2}{3} \end{array}}$$

$$\tau(x,h) = \frac{1}{h}\sum_{j=0}^{r} a_j y(x+jh) - b_j y'(x+jh) =$$

$$= \frac{1}{h}\left\{\frac{1}{3}y(x) - \frac{4}{3}y(x+h) + y(x+2h)\right\} - \frac{2}{3}y'(x+2h) \simeq$$

$$\simeq \frac{1}{h}\left\{\frac{1}{3}y(x) + \left[-\frac{4}{3}y(x) - \frac{4}{3}y'(x)\cdot h - \frac{4}{3}y''(x)\cdot\frac{1}{2}h^2 +\right.\right.$$

$$\left. -\frac{4}{3}y'''(x)\frac{1}{6}h^3 + O(h^4)\right] + \left[y(x) + y'(x)\cdot 2h + \frac{1}{2}y''(x)\cdot 4h^2 +\right.$$

$$\left.\left. + \frac{1}{6}y'''(x)\,8h^3 + O(h^4)\right]\right\} - \frac{2}{3}\left(y'(x) + y''(x)\cdot 2h + y'''(x)\,4h^2\cdot\frac{1}{2} +\right.$$

$$\left. + O(h^3)\right) = \frac{1}{h}\left\{\left(\cancel{\frac{1}{3} - \frac{4}{3} + 1}\right)y(x)\right\} + \left(\cancel{-\frac{4}{3} + 2 - \frac{2}{3}}\right)y'(x) +$$

$$+ \left(\cancel{-\frac{2}{3} + 2 - \frac{4}{3}}\right)y''(x)h + \left(-\frac{2}{9} + \frac{4}{3} - \frac{4}{3}\right)y'''(x)h^2 +$$

$$+ O(h^3) = O(h^2)$$

la formula è convergente del 2^{o} ordine

$$p(w) = w^2 - \frac{4}{3}w + \frac{1}{3} = 0$$

$$w_{1/2} = \frac{4 \pm \sqrt{16-12}}{6} \begin{cases} 1 \\ \frac{1}{3} \end{cases}$$

È stabile (solo una delle radici ha modulo 1).

La formula è quindi convergente